



On the Upper Bound of the Third Hankel Determinant for Certain Class of Analytic Functions Related with Exponential Function

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Abstract

In the present paper we introduce a new class of analytic functions f in the open unit disk normalized by $f(0) = f'(0) - 1 = 0$, associated with exponential functions. The aim of the present paper is to investigate the third-order Hankel determinant $H_3(1)$ for this function class and obtain the upper bound of the determinant $H_3(1)$.

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$. Also we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions f in Δ . The familiar coefficient conjecture for the function $f \in \mathcal{S}$ of the form (1) was first presented by the Bieberbach [1] in 1916 and proved by de-Branges [2] in 1985. During 1916-1985 many mathematicians struggled to prove or disprove this conjecture. As result they defined several subfamilies of

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the set $\mathcal{S}$ connected with different image domains. Further, we recall some of them. Let the notations $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ indicate the families of starlike, convex and close-to-convex functions respectively with the following Taylor-Maclaurin series representations:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, z \in \Delta \right\}; \quad (2)$$

$$\mathcal{C} = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, z \in \Delta \right\}; \quad (3)$$

$$\mathcal{K} = \left\{ f \in \mathcal{S} : \frac{f'(z)}{g'(z)} \prec \frac{1+z}{1-z}, \text{ for } g \in \mathcal{C}, z \in \Delta \right\}, \quad (4)$$

where the symbol " $\prec$ " denotes the familiar concept of differential subordination between analytic functions. Now, we recall here the definition of subordination.

Suppose that f and g are two analytic functions in Δ . We say that the function f is subordinate to g and we write $f(z) \prec g(z)$, if there exists a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that (see [14]) $f(z) = g(w(z))$. Thus, $f(z) \prec g(z)$ implies $f(\Delta) \subset g(\Delta)$. In case of univalence of f in Δ , the function f is subordinate to g if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Assume that $\mathcal{P}$ denote the class of analytic functions p normalized by

$$p(z) = 1 + c_1z + c_2z^2 + \dots \quad (5)$$

and satisfying the condition $\Re p(z) > 0$, $z \in \Delta$. It is easy to see that if $p \in \mathcal{P}$, then there exists a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that (see [25])

$$p(z) = \frac{1+w(z)}{1-w(z)}. \quad (6)$$

Padmanabhan and Parvatham introduced in the paper [20] a unified families of starlike and convex functions using familiar notion of convolution with the function $z/(1-z)^a$, for all $a \in \mathbb{R}$. Later on Shanmugam [23] generalized the idea of paper [20] and introduced the set

$$\mathcal{S}_h^*(\phi) = \left\{ f \in \mathcal{A} : \frac{z(f * h)'}{(f * h)} \prec \phi(z), \quad z \in \Delta \right\}, \quad (7)$$

where the symbol " $*$ " stands for the familiar notion of convolution, ϕ is convex and h is a fixed function in $\mathcal{A}$. We obtain the families $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ when taking $z/(1-z)$ and $z/(1-z)^2$ instead of h in $\mathcal{S}_h^*(\phi)$ respectively.

In 1992, Ma and Minda [12] reduced the above restriction to a weaker supposition that ϕ is a function, with $\Re\phi > 0$ in Δ with, whose image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$ and discussed some properties. Here are these classes:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \Delta \right\};$$

$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad z \in \Delta \right\}.$$

The classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ unify various subclasses of starlike $\mathcal{S}^*$ or convex $\mathcal{C}$ functions in Δ . For example, the class $\mathcal{S}^*(\phi)$ generalizes various subfamilies of the set $\mathcal{A}$ as follows:

1. If the function $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \leq B < A \leq 1$, then $\mathcal{S}^*[A, B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the set of Janowski starlike functions defined in [8]. Further, if $A = 1 - 2\alpha$ and $B = -1$ with $0 \leq \alpha < 1$, then we get the set $\mathcal{S}^*(\phi)$ of starlike function of order α .

2. The family $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ was introduced by Sokol and Stankiewicz in [24], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$.

3. For the function $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi)$ leads to the class $\mathcal{S}_{\sin}^*$, introduced in [3].

4. The family $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ was introduced by Mediratta et al. in [13] given as:

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec e^z, \quad z \in \Delta \right\}, \quad (8)$$

or equivalently

$$\mathcal{S}_e^* = \left\{ f \in \mathcal{S} : \left| \log \frac{zf'(z)}{f(z)} \right| \prec e^z, \quad z \in \Delta \right\}. \quad (9)$$

By using Alexander type relation, we also recall [13] by the following set:

$$\mathcal{C}_e = \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec e^z, \quad z \in \Delta \right\}.$$

The above mentioned families $\mathcal{S}_e^*$ and $\mathcal{C}_e$ are symmetric about the real axis.

In [15], Noonan and Thomas studied the q^{th} Hankel determinants $H_q(n)$ of functions $f \in \mathcal{A}$ of the form (1) for $q \geq 1$ and $n \geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1). \quad (10)$$

In particular

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \quad (a_1 = 1).$$

Since $f \in \mathcal{S}$, $a_1 = 1$, thus

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

The concept of Hankel determinant is very useful in the theory of singularities [4] and in the study of power series with integral coefficients. The Hankel determinant $H_q(n)$ have been investigated by several authors to study its rate of growth as $n \rightarrow \infty$ and to determine bounds on it for specific values of q and n . For example, Pommerenke [22] proved that the Hankel determinants of univalent functions satisfy $|H_q(n)| < kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, ($n = 1, 2, \dots, q = 2, 3, \dots$) where $\beta > 1/1400$ and k depends only on q . Note that the Hankel determinant $H_2(1) = a_3 - a_2^2$ is related to the well-known Fekete-Szegő functional [7] for univalent functions. Although we know many sharp bounds of $H_2(2)$ and significantly less sharp bounds of $H_3(1)$ for some proper subfamilies of $\mathcal{S}$, the sharp results for the whole class $\mathcal{S}$ are not known. Moreover, we are even unable to formulate a reasonable conjecture about it. Ehrenborg studied Hankel determinant of the exponential polynomials [6] and Noor studied Hankel determinant for Bazilevic functions in [18] and for functions with bounded boundary rotations in [17] and [16]; also for close-to-convex functions in [19]. Until now, very few researches have studied the above determinants for the function class, subordinate to e^z . Thus, in this paper, we aim to investigate the third-order Hankel determinant $H_3(1)$ for a certain class defined below, which is associated with exponential function and obtain the upper bound of the determinant. To derive our results, we shall need the following results.

2 Preliminary results

Some preliminary results required in the following section are now listed.

Lemma 1. ([5]) If $p \in \mathcal{P}$ and has the form (5) then

$$|c_n| \leq 2, \quad n = 1, 2, \dots \quad (11)$$

and the inequality is sharp.

Lemma 2. ([21], [9]) If $p \in \mathcal{P}$ and has the form (5) then

$$|c_{n+k} - \mu c_n c_k| < 2 \quad \text{for } 0 \leq \mu \leq 1; \quad (12)$$

$$|c_m c_n - c_k c_l| \leq 4 \quad \text{for } m + n = k + l; \quad (13)$$

$$|c_{n+2k} - \mu c_n c_k^2| \leq 2(1 + 2\mu) \quad \text{for } \mu < -\frac{1}{2}; \quad (14)$$

$$\left| c_2 - \frac{c_1^2}{2} \right| < 2 - \frac{|c_1|^2}{2}; \quad (15)$$

and for the complex number λ , we have

$$c_2 - \lambda c_1^2 \leq 2 \max\{1, |2\lambda - 1|\}. \quad (16)$$

For the inequalities (12), (13), (14), (15) see [21] and (16) is given in [9].

Lemma 3. ([10], [11]) If the function $p \in \mathcal{P}$ is given by (5), then exists some x, z with $|x| \leq 1, |z| \leq 1$ such that

$$2c_2 = c_1^2 + x(4 - c_1^2); \quad (17)$$

$$4c_3 = c_1^3 + 2c_1 x(4 - c_1^2) - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \quad (18)$$

3 Main results

Definition 1. A function $f \in \mathcal{S}$ is said to be in the class SC_α^* , $\alpha \in [0, 1]$, if satisfies the following condition:

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec e^z.$$

Remark. For $\alpha = 0$, the family $SC_0^* := \mathcal{S}_e^* = \mathcal{S}^*(e^z)$ was introduced by Mediratta et al. in [13] and for $\alpha = 1$, we reobtain the set $SC_1^* := C_e$.

Theorem 1. If the function $f \in SC_\alpha^*$, where f is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbb{C}$ then we have

$$|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\alpha)}. \quad (19)$$

Proof. Because $f \in SC_{\alpha}^*$, from the definition of subordination, we know that exists a Schwartz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ such that

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) = e^{w(z)}.$$

But

$$\begin{aligned} (1 - \alpha) \frac{zf'(z)}{f(z)} &= \\ &= (1 - \alpha) [1 + a_2z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 + \dots]. \end{aligned} \quad (20)$$

$$\begin{aligned} \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) &= \alpha \left(1 + \frac{\sum_{n=2}^{\infty} na_n(n-1)z^{n-1}}{1 + \sum_{n=2}^{\infty} na_nz^{n-1}} \right) = \\ &= \alpha [1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 + \dots]. \end{aligned} \quad (21)$$

From the relations (20) and (21) we obtain

$$\begin{aligned} (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) &= \\ &= 1 + za_2(1 + \alpha) + z^2[2a_3(1 + 2\alpha) - a_2^2(1 + 3\alpha)] + \\ &+ z^3[a_2^3(1 + 7\alpha) - 3a_2a_3(1 + 5\alpha) + 3a_4(1 + 3\alpha)] + \dots \end{aligned} \quad (22)$$

We define a function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots,$$

$p(z) \in \mathcal{P}$ and

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + c_3z^3 + \dots}.$$

But

$$\begin{aligned} e^{w(z)} &= 1 + w(z) + \frac{[w(z)]^2}{2!} + \frac{[w(z)]^3}{3!} + \dots = \\ &= 1 + \frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + c_3z^3 + \dots} + \frac{1}{2} \left(\frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + c_3z^3 + \dots} \right)^2 + \\ &+ \frac{1}{6} \left(\frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + c_3z^3 + \dots} \right)^3 + \dots = \end{aligned} \quad (23)$$

$$\begin{aligned}
 &= 1 + \frac{1}{2} (c_1 z + c_2 z^2 + \dots) \left[1 - \frac{c_1 z}{2} + \left(\frac{c_1^2}{4} - \frac{c_2}{2} \right) z^2 - \left(\frac{c_1^3}{8} - \frac{c_1 c_2}{2} + \frac{c_3}{2} \right) z^3 + \dots \right] \\
 &\quad + \frac{1}{2} (c_1 z + c_2 z^2 + \dots)^2 \left[1 - \frac{c_1 z}{2} + \left(\frac{c_1^2}{4} - \frac{c_2}{2} \right) z^2 - \left(\frac{c_1^3}{8} - \frac{c_1 c_2}{2} + \frac{c_3}{2} \right) z^3 + \dots \right]^2 \\
 &\quad + \frac{1}{48} (c_1 z + c_2 z^2 + \dots)^3 \left[1 - \frac{c_1 z}{2} + \left(\frac{c_1^2}{4} - \frac{c_2}{2} \right) z^2 - \left(\frac{c_1^3}{8} - \frac{c_1 c_2}{2} + \frac{c_3}{2} \right) z^3 + \dots \right]^3 + \dots \\
 &= 1 + \frac{1}{2} c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{8} \right) z^2 + \left(\frac{c_1^3}{48} - \frac{c_1 c_2}{4} + \frac{c_3}{2} \right) z^3 + \dots
 \end{aligned}$$

On comparing the coefficients of z , z^2 and z^3 between the equations (22) and (23) we obtain

$$a_2 = \frac{c_1}{2(1+\alpha)}; \quad (24)$$

$$a_3 = \frac{c_2}{4(1+2\alpha)} + \frac{c_1^2(1+4\alpha-\alpha^2)}{16(1+2\alpha)(1+\alpha)^2}; \quad (25)$$

It can be written,

$$|a_3 - a_2^2| = \left| \frac{c_2}{4(1+2\alpha)} - \frac{c_1^2(\alpha+3)}{16(1+2\alpha)(1+\alpha)} \right|.$$

Using Lemma 3, we thus know that

$$|a_3 - a_2^2| = \left| \frac{x(4-c_1^2)}{8(1+2\alpha)} - \frac{c_1^2(1-\alpha)}{16(1+2\alpha)(1+\alpha)} \right|.$$

Letting $|x| = t \in [0, 1]$, $c_1 = c \in [0, 2]$ and applying the triangle inequality, the above equation reduces to

$$|a_3 - a_2^2| \leq \frac{t(4-c^2)}{8(1+2\alpha)} + \frac{c^2(1-\alpha)}{16(1+2\alpha)(1+\alpha)}.$$

Suppose that

$$F(c, t) := \frac{t(4-c^2)}{8(1+2\alpha)} + \frac{c^2(1-\alpha)}{16(1+2\alpha)(1+\alpha)},$$

then we get

$$\frac{\partial F}{\partial t} = \frac{4-c^2}{8(1+2\alpha)} \geq 0,$$

which shows that $F(c, t)$ is an increasing function on the closed interval $[0, 1]$ about t . Therefore the function $F(c, t)$ can get the maximum value at $t = 1$, that is

$$\max F(c, t) = F(c, 1) = \frac{4 - c^2}{8(1 + 2\alpha)} + \frac{c^2(1 - \alpha)}{16(1 + 2\alpha)(1 + \alpha)}.$$

Next, let

$$G(c) = \frac{4 - c^2}{8(1 + 2\alpha)} + \frac{c^2(1 - \alpha)}{16(1 + 2\alpha)(1 + \alpha)} = \frac{1}{2(1 + 2\alpha)} - \frac{c^2(1 + 3\alpha)}{16(1 + 2\alpha)(1 + \alpha)}.$$

The function $G(c)$ has a maximum value at $c = 0$, which is

$$|a_3 - a_2^2| \leq G(0) = \frac{1}{2(1 + 2\alpha)}$$

and the proof is done. $\square$

Theorem 2. If the function $f \in SC_\alpha^*$, where f is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbb{C}$ then we have

$$|a_2 a_3 - a_4| \leq \frac{(4 - \tilde{c}^2)\tilde{c}}{24} + \frac{3(4 - \tilde{c}^2)\tilde{c}}{8} + \frac{4 - \tilde{c}^2}{12} + \frac{\tilde{c}^3 \varepsilon(\rho)}{72} \quad (26)$$

where

$$\tilde{c} = -\frac{9\sqrt{\frac{499312}{3} - \frac{23840}{3}\sqrt{298}} - 108}{298\sqrt{298} - 6236} \in [0, 2]$$

and $\varepsilon(\rho) = -2\alpha^3 - 14\alpha^2 + 17\alpha + 5$, $\rho = \frac{1}{6}\sqrt{2}\sqrt{149} - \frac{7}{3}$.

Proof. Knowing that

$$\begin{aligned} a_4 &= \frac{c_3}{6(1 + 3\alpha)} - \frac{c_1 c_2 (4\alpha^2 - 9\alpha - 1)}{24(1 + 3\alpha)(1 + 2\alpha)(1 + \alpha)} + \\ &+ \frac{c_1^3 (4\alpha^4 - 31\alpha^3 + 21\alpha^2 - 17\alpha - 1)}{288(1 + \alpha)^3(1 + 2\alpha)(1 + 3\alpha)}; \end{aligned} \quad (27)$$

we have

$$\begin{aligned} |a_2 a_3 - a_4| &= \\ &\left| \frac{c_1 c_2 (2\alpha^2 + 1)}{12(1 + 3\alpha)(1 + 2\alpha)(1 + \alpha)} - \frac{c_3}{6(1 + 3\alpha)} - \frac{c_1^3 (2\alpha^4 - 2\alpha^3 - 39\alpha^2 - 40\alpha - 5)}{144(1 + \alpha)^3(1 + 2\alpha)(1 + 3\alpha)} \right|. \end{aligned}$$

Again, by applying Lemma 3, we get

$$|a_2 a_3 - a_4| = \left| \frac{c_1^3 (-2\alpha^4 - 16\alpha^3 + 3\alpha^2 + 22\alpha + 5)}{144(1+\alpha)^3(1+2\alpha)(1+3\alpha)} + \frac{(4-c_1^2)c_1 x^2}{24(1+3\alpha)} - \frac{(4-c_1^2)(1-|x|^2)z}{12(1+3\alpha)} - \frac{c_1 x(4-c_1^2)(2\alpha^2+6\alpha+1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} \right|.$$

Assume that $|x| = t \in [0, 1]$, $c_1 = c \in [0, 2]$. Then, using the triangle inequality, we deduce that

$$|a_2 a_3 - a_4| \leq \frac{(4-c^2)ct^2}{24(1+3\alpha)} + \frac{(4-c^2)ct(2\alpha^2+6\alpha+1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} + \frac{(4-c^2)}{12(1+3\alpha)} + \frac{c^3(-2\alpha^4-16\alpha^3+3\alpha^2+22\alpha+5)}{144(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.$$

Setting

$$F(c, t) := \frac{(4-c^2)ct^2}{24(1+3\alpha)} + \frac{(4-c^2)ct(2\alpha^2+6\alpha+1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} + \frac{(4-c^2)}{12(1+3\alpha)} + \frac{c^3(-2\alpha^4-16\alpha^3+3\alpha^2+22\alpha+5)}{144(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.$$

Hence, we have

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)ct}{12(1+3\alpha)} + \frac{(4-c^2)c(2\alpha^2+6\alpha+1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} \geq 0,$$

namely, $F(c, t)$ is an increasing function on the closed interval $[0, 1]$ about t . This implies that the maximum value of $F(c, t)$ occurs at $t = 1$, which is

$$\max F(c, t) = F(c, 1) = \frac{(4-c^2)c}{24(1+3\alpha)} + \frac{(4-c^2)c(2\alpha^2+6\alpha+1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} + \frac{(4-c^2)}{12(1+3\alpha)} + \frac{c^3(-2\alpha^4-16\alpha^3+3\alpha^2+22\alpha+5)}{144(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.$$

Then

$$\begin{aligned} \max F(c, t) &\leq \frac{(4-c^2)c}{24} + \frac{(4-c^2)c(2\alpha^2+6\alpha+1)}{24} \\ &\quad + \frac{4-c^2}{12} + \frac{c^3(\alpha+1)\varepsilon(\alpha)}{144} \end{aligned}$$

$$\leq \frac{(4-c^2)c}{24} + \frac{9(4-c^2)c}{24} + \frac{4-c^2}{12} + \frac{2c^3\varepsilon(\rho)}{144}$$

where $\varepsilon(\alpha) = -2\alpha^3 - 14\alpha^2 + 17\alpha + 5$ which is a positive function, $\varepsilon(\alpha) \leq \varepsilon(\rho)$ and $\rho = \frac{1}{6}\sqrt{2}\sqrt{149} - \frac{7}{3}$.

Now, we define

$$E(c) = \frac{(4-c^2)c}{24} + \frac{9(4-c^2)c}{24} + \frac{4-c^2}{12} + \frac{2c^3\varepsilon(\rho)}{144}.$$

Equation $E'(c) = 0$ which is $3(-30 + \varepsilon(\rho))c^2 - 12c + 120 = 0$ implies

$$c^* = \frac{9\sqrt{\frac{499312}{3} - \frac{23840}{3}\sqrt{298}} + 108}{298\sqrt{298} - 6236} \notin [0, 2],$$

$$\tilde{c} = -\frac{9\sqrt{\frac{499312}{3} - \frac{23840}{3}\sqrt{298}} - 108}{298\sqrt{298} - 6236} \in [0, 2].$$

The function $E(c)$ is an increasing function on the closed interval $[0, \tilde{c}]$ and also an decreasing function on the interval $[\tilde{c}, 2]$ and the maximum value of $E(c)$ occurs at $c = \tilde{c}$. Thus $|a_2a_3 - a_4| \leq \max_{c \in [0, 2]} E(c) = E(\tilde{c}) \approx 1.74$ and the proof of the Theorem 2 is completed. $\square$

Theorem 3. *If the function $f \in SC_\alpha^*$, where f is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbb{C}$ then we have*

$$|a_2a_4 - a_3^2| \leq 2. \quad (28)$$

Proof. Suppose that $f \in SC_\alpha^*$, then from the equation (25) we have

$$|a_2a_4 - a_3^2| = \left| \frac{c_1c_3}{12(1+\alpha)(1+3\alpha)} - \frac{c_2^2}{16(1+2\alpha)^2} - \frac{c_1^2c_2}{96} \cdot \frac{7\alpha^3 + 5\alpha^2 - \alpha + 1}{(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)} \right. \\ \left. + \frac{c_1^4(5\alpha^5 - 25\alpha^4 - 262\alpha^3 - 394\alpha^2 - 175\alpha - 13)}{2304(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \right|.$$

In view of Lemma 3, we thus obtain

$$|a_2a_4 - a_3^2| = \left| \frac{c_1^4(5\alpha^5 + 47\alpha^4 - 46\alpha^3 - 178\alpha^2 - 103\alpha - 13)}{2304(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \right. \\ \left. + \frac{c_1^2x(4-c_1^2)(7\alpha^3 + 17\alpha^2 + 11\alpha + 1)}{192(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)} - \frac{(4-c_1^2)c_1^2x^2}{48(1+\alpha)(1+3\alpha)} \right|$$

$$+ \frac{c_1 (4 - c_1^2) (1 - |x|^2) z}{24 (1 + \alpha) (1 + 3\alpha)} - \frac{x^2 (4 - c_1^2)^2}{64 (1 + 2\alpha)^2} \Big|.$$

Also let $|x| = t \in [0, 1]$, $c_1 = c \in [0, 2]$. Then, using the triangle inequality we get

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{c^4 |5\alpha^5 + 47\alpha^4 - 46\alpha^3 - 178\alpha^2 - 103\alpha - 13|}{2304 (1 + \alpha)^4 (1 + 2\alpha)^2 (1 + 3\alpha)} \\ &+ \frac{c^2 t (4 - c^2) (7\alpha^3 + 17\alpha^2 + 11\alpha + 1)}{192 (1 + \alpha)^2 (1 + 2\alpha)^2 (1 + 3\alpha)} + \frac{(4 - c^2) c^2 t^2}{48 (1 + \alpha) (1 + 3\alpha)} \\ &+ \frac{c (4 - c^2)}{24 (1 + \alpha) (1 + 3\alpha)} + \frac{t^2 (4 - c^2)^2}{64 (1 + 2\alpha)^2}. \end{aligned}$$

Knowing that $\max_{\alpha \in [0, 1]} |5\alpha^5 + 47\alpha^4 - 46\alpha^3 - 178\alpha^2 - 103\alpha - 13| = 288$ the above inequality can be rewritten

$$|a_2 a_4 - a_3^2| \leq \frac{288c^4}{2304} + \frac{36c^2 (4 - c^2)}{192} + \frac{(4 - c^2) c^2}{48} + \frac{c (4 - c^2)}{24} + \frac{(4 - c^2)^2}{64}$$

or equivalent

$$|a_2 a_4 - a_3^2| \leq \frac{-39c^4 - 24c^3 + 408c^2 + 96c + 144}{576}.$$

Next, let

$$G(c) := \frac{-39c^4 - 24c^3 + 408c^2 + 96c + 144}{576}.$$

Now it easily to derive that $G(c)$ is an increasing function on the interval $[0, 2]$ therefore we have a maximum value at $c = 2$, also which is $|a_2 a_4 - a_3^2| \leq G(c) = 2$. The proof of the Theorem 3 is thus completed. $\square$

Theorem 4. *If the function $f \in SC_\alpha^*$, then we have*

$$|H_3(1)| \leq 18,001. \quad (29)$$

Proof. In order to establish the upper bound for $H_3(1)$ we proceed to compute certain inequalities. Using the form of a_5 posted below

$$\begin{aligned} a_5 &= \frac{c_1 c_3 (1 + 7\alpha)}{12 (1 + \alpha) (1 + 3\alpha) (1 + 4\alpha)} - \frac{c_1^2 c_2 (-188\alpha^5 + 184\alpha^4 + 340\alpha^3 - 2\alpha^2 - 44\alpha - 2)}{192 (1 + 4\alpha) (1 + 3\alpha) (1 + 2\alpha)^2 (1 + \alpha)^2} \\ &+ \frac{c_1^4 (484\alpha^7 - 2328\alpha^6 + 520\alpha^5 + 616\alpha^4 - 1424\alpha^3 - 308\alpha^2 + 132\alpha + 4)}{4608 (1 + 4\alpha) (1 + \alpha)^4 (1 + 2\alpha)^2 (1 + 3\alpha)} \\ &+ \frac{c_2^2 4\alpha (1 - \alpha)}{32 (1 + 4\alpha) (1 + 2\alpha)^2} + \frac{1}{8 (1 + 4\alpha)} \left(c_4 - \frac{1}{2} c_1 c_3 \right). \end{aligned} \quad (30)$$

and the equalities from Lemma 3 we get

$$\begin{aligned} a_5 = & \left(\frac{M(\alpha)}{4} + \frac{N(\alpha)}{2} + P(\alpha) + \frac{Q(\alpha)}{4} \right) c_1^4 \\ & + \left(\frac{M(\alpha)}{2} + \frac{N(\alpha)}{2} + \frac{Q(\alpha)}{2} \right) x c_1^2 (4 - c_1^2) - \frac{M(\alpha)}{4} x^2 c_1^2 (4 - c_1^2) \\ & + \frac{M(\alpha)}{2} c_1 (4 - c_1^2) (1 - |x|^2) z + \frac{Q(\alpha)}{4} x^2 (4 - c_1^2)^2 + \frac{1}{8(1+4\alpha)} \left(c_4 - \frac{1}{2} c_1 c_3 \right) \end{aligned}$$

where $M(\alpha) = \frac{1+7\alpha}{12(1+\alpha)(1+3\alpha)(1+4\alpha)}$, $N(\alpha) = \frac{-188\alpha^5+184\alpha^4+340\alpha^3-2\alpha^2-44\alpha-2}{192(1+4\alpha)(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}$,
 $P(\alpha) = \frac{484\alpha^7-2328\alpha^6+520\alpha^5+616\alpha^4-1424\alpha^3-308\alpha^2+132\alpha+4}{4608(1+4\alpha)(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}$ and
 $Q(\alpha) = \frac{4\alpha(1-\alpha)}{32(1+4\alpha)(1+2\alpha)^2}$. According to inequality (12)

$$\left| c_4 - \frac{1}{2} c_1 c_3 \right| \leq 2.$$

Let $|x| = t \in [0, 1]$, $c_1 = c \in [0, 2]$. By making use of the triangle inequality and the maximum values of the functions $M(\alpha)$, $N(\alpha)$, $P(\alpha)$ and $Q(\alpha)$ on the interval $[0, 1]$ for the argument α we have

$$|a_5| \leq \frac{547}{384} c^4 + \frac{211}{192} c^2 (4 - c^2) + \frac{1}{6} c^2 (4 - c^2) + \frac{1}{3} c (4 - c^2) + \frac{1}{128} (4 - c^2)^2 + \frac{1}{4}$$

or equivalent

$$|a_5| \leq \frac{1}{384} (64c^4 - 128c^3 + 1920c^2 + 512c + 144) \leq \frac{8848}{384}.$$

since the function $\varphi(c) = 64c^4 - 128c^3 + 1920c^2 + 512c + 144$ gets its maximum at $c = 2$.

For the coefficient a_3 , we deduce using triangle inequality

$$|a_3| \leq \frac{|c_2|}{4(1+2\alpha)} + \frac{|c_1|^2 |1+4\alpha-\alpha^2|}{16(1+2\alpha)(1+\alpha)^2} \leq \frac{2}{4} + \frac{4 \cdot 4}{16} = 1, 5.$$

It follows the upper bound for the coefficient a_4 .

$$\begin{aligned} |a_4| = & \left| \frac{c_3}{6(1+3\alpha)} - \frac{c_1 c_2 (4\alpha^2 - 9\alpha - 1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} + \frac{c_1^3 (4\alpha^4 - 31\alpha^3 + 21\alpha^2 - 17\alpha - 1)}{288(1+\alpha)^3(1+2\alpha)(1+3\alpha)} \right| \\ & \leq \frac{2}{6} + \frac{2 \cdot 2}{24} |4\alpha^2 - 9\alpha - 1| + \frac{1}{128} \cdot 8 \cdot |4\alpha^4 - 31\alpha^3 + 21\alpha^2 - 17\alpha - 1| \leq 2. \end{aligned}$$

Because

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

by applying the triangle inequality, we obtain the Hankel determinant of order three

$$|H_3(1)| \leq |a_3| \cdot |a_2a_4 - a_3^2| + |a_4| \cdot |a_4 - a_2a_3| + |a_5| \cdot |a_3 - a_2^2|. \quad (31)$$

Next, substituting relations (26), (28), (19) in (31) we get the inequality (29). Thus, the proof is completed. $\square$

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