

Closed trail decompositions on grid graphs

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Abstract

Let $G = G(n, m)$ be a rectangular solid grid graph and $\mathcal{A}(G)$ be a minimum length Eulerian augmentation of G . Let $l_0, \dots, l_t \in \mathbb{N}$ such that $\sum_{i=0}^t l_i = |E(\mathcal{A}(G))|$, where $2(n + m) \leq l_i = 2k_i$. In this paper, we exhibit a constructive procedure providing an edge-disjoint decomposition of $\mathcal{A}(G)$ into closed trails $T_0, \dots, T_t$ such that $|E(T_i)| = l_i$.

Mathematics Subject Classifications (2015). 05C70, 05C85

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1 Introduction

For many real life problems, such as vehicle routing, surveillance of an area, design of telecommunication networks, the underlying mathematical model requires that the edges of a graph G are covered with closed trails such that each edge of the graph is contained in at least one trail, the length of each trail is bounded above by a given number called its capacity and the total length of trails is minimized. The graph obtained by taking the union of capacitated trails will be Eulerian. Thus, the minimum length Eulerian augmentation $\mathcal{A}(G)$ of G gives a strict lower bound to this optimisation problem. It is therefore natural to investigate whether the edges of $\mathcal{A}(G)$ can be decomposed into closed trails of given lengths.

Decompositions of graphs into fixed length trails are difficult to find in general and most known results relate to complete multipartite graphs. Complete graph were investigated in [1], bipartite graphs in [4], [7] and tripartite graphs in [2], [6]. More recently, complete equipartite graphs were discussed in detail in [4], [3]. Other graphs for which this problem was considered closely relate to these graphs and include pseudographs in [5] and regular graphs in [9].

In this paper, we consider edge-disjoint decomposition into fixed length closed trails of $\mathcal{A}(G)$, where G is a rectangular solid grid graph $G = G(n, m)$. These graphs are finite node-induced subgraphs of the infinite two-dimensional grid. As such graphs appear in many interesting applications, for example in

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VLSI-design [8], in chromosome analysis and image segmentation [10], and in planning and designing buildings and cities, it is natural to investigate such decomposition problems. The paper will focus on providing a constructive procedure that provides a decomposition of $\mathcal{A}(G)$ into long closed trails, in the sense that given trail lengths l_i satisfy that $l_i \geq 2(n + m)$. We also assume that $6 \leq n \leq m$, as smaller samples of rectangular grid graphs can be analysed by hand.

The paper is organised as follows. In the next section, we summarise basic definitions and notation used in the paper. Section 3 is devoted to the discussion of decompositions of $\mathcal{A}(G)$ into two trails $T_0 = \Sigma_0$ and Σ_1 of given lengths with the aid of auxiliary graphs. Section 4 gives a constructive procedure that further decomposes Σ_1 into closed trails $T_1, \dots, T_t$ of given lengths. Throughout the paper we use $G(6, 9)$ to illustrate concepts and results.

2 Preliminaries

A graph G is *even*, if the degree of each vertex of G is even. It is well known by Euler's theorem that a connected even graph is Eulerian. A *closed trail of length l* in G is a connected Eulerian subgraph having exactly l edges. We say that a graph G is *decomposable into closed trails of lengths $l_0, l_1, \dots, l_t$* , if G can be written as an edge disjoint union of closed trails $T_0, T_1, \dots, T_t$, satisfying that $|E(T_j)| = l_j$ for each $0 \leq j \leq t$. Following terminology in [5], we say that a sequence of integers $(l_0, l_1, l_2, \dots, l_t)$ is *admissible for G* , if $\sum_{j=0}^t l_j = |E(G)|$. Moreover, if G is decomposable into closed trails $T_0, T_1, \dots, T_t$ satisfying that $|E(T_j)| = l_j$, then we say that the sequence $(l_0, l_1, l_2, \dots, l_t)$ is *realizable in G* .

Let P_n denote a path on $n + 1$ vertices. A rectangular solid grid graph $G = G(n, m)$ is a Cartesian product of P_n and P_m . Without loss of generality, we may assume that $n \leq m$. The vertex set of $V(G)$ is

$$V(G(n, m)) = \{v_{(a,b)} \mid 0 \leq a \leq n, 0 \leq b \leq m, a, b \in \mathbb{Z}\}.$$

Certain cycles of length 4 in G will serve as building blocks in our construction. Let $C_{i,j}$ denote the cycle of length 4 in G with vertex set

$$V(C_{i,j}) = \begin{cases} \{v_{(2i-1,j)}, v_{(2i-1,j-1)}, v_{(2i,j-1)}, v_{(2i,j)}\} & \text{if } j \text{ is odd} \\ \{v_{(2i-1,j-1)}, v_{(2i-1,j)}, v_{(2i-2,j)}, v_{(2i-2,j-1)}\} & \text{if } j \text{ is even.} \end{cases}$$

and we let $\mathcal{S}$ denote the set of these cycles (Figure 1).

It is well known that grid graphs are bipartite and that bipartite graphs do not contain closed trails of odd length. It is also straightforward to see that $|E(T)| \geq 4$ for any trail T in G and hence if T is a closed trail in G then either $|E(T)| = 4k$ or $|E(T)| = 4k + 2$ for some $1 \leq k \in \mathbb{N}$.

Clearly, $G = G(n, m)$ is not an Eulerian graph. A minimum length Eulerian augmentation of G , that can be obtained by considering a minimum cost perfect matching of odd degree vertices, will be denoted by $\mathcal{A}(G)$. It is routine to see that a rectangular solid grid graph $G(n, m)$ has $2n + 2m - 4$ odd degree vertices.

If n and m are odd then duplicating $n + m - 2$ edges is enough to obtain $\mathcal{A}(G)$, and if either n or m or both are even, then $\mathcal{A}(G)$ is obtained by duplicating $m + n$ edges (Figure 1).

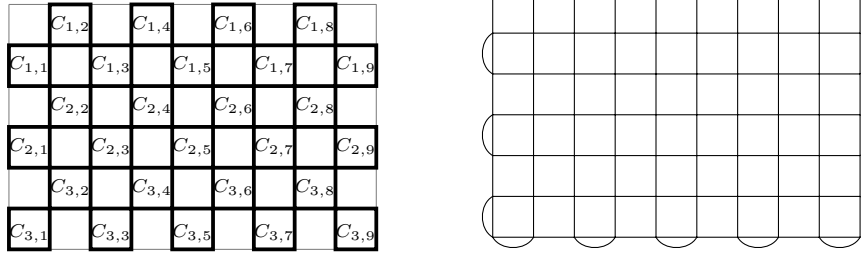


Figure 1: Left: Cycles $C_{i,j}$. Right: Eulerian augmentation of $\mathcal{A}(G)$.

3 Decomposition of $\mathcal{A}(G)$ into two trails

In this section, we introduce auxiliary graphs $\mathcal{H}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{H}_U$ and $\mathcal{H}_{U,J}$ that will provide a specific decomposition of $\mathcal{A}(G)$ into two closed trails T_0 and T_1 of given lengths satisfying that $|E(T_i)| \geq 2(n + m)$, ($i = 0, 1$).

3.1 Auxiliary graph $\mathcal{H}$

Let $\mathcal{H}$ be a graph with vertex set

$$V(\mathcal{H}) = \{c_{i,j} \mid C_{i,j} \in \mathcal{S}\}.$$

We assume that $c_{i,j}$ and $c_{l,t}$ are connected with an edge $e \in E(\mathcal{H})$ precisely when cycles $C_{i,j}$ and $C_{l,t}$ have a common vertex in G (Figure 2). It will be

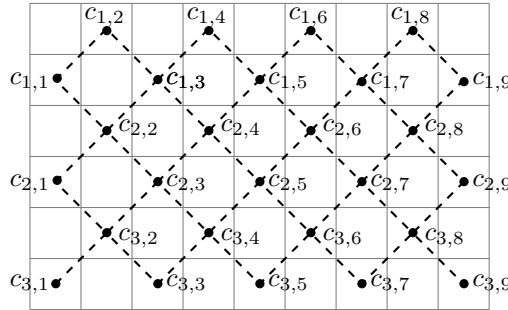


Figure 2: $\mathcal{H}(G)$ superimposed on $G = (6, 9)$.

convenient to refer to vertex columns and let

$$\mathcal{C}_j = \{c_{1,j}, c_{2,j}, \dots\} \quad \text{and} \quad \mathcal{C}_j^r = \{c_{1,j}, c_{2,j}, \dots, c_{r,j}\}$$

denote the j th vertex column and the first r vertices in the j th vertex column in $\mathcal{H}$ respectively. The following lemma is straightforward from the construction of $\mathcal{H}$.

LEMMA 1 *We have that $|\mathcal{C}_j \cup \mathcal{C}_{j+1}| = n$ for all $1 \leq j \leq m-1$. Furthermore, if $|(\mathcal{C}_j \setminus \mathcal{C}_j^r) \cup \mathcal{C}_{j+1} \cup \mathcal{C}_{j+2}^q| \geq n$, then the subgraph induced by $(\mathcal{C}_j \setminus \mathcal{C}_j^r) \cup \mathcal{C}_{j+1} \cup \mathcal{C}_{j+2}^q$ is connected.*

We let $\Gamma(\mathcal{H})$ and $\Gamma(G)$ denote the set of connected subgraphs of $\mathcal{H}$ and G respectively. Define the map

$$\varphi : \Gamma(\mathcal{H}) \rightarrow \Gamma(G), \quad H \mapsto \tilde{H} \quad (1)$$

where $\tilde{H}$ is a subgraph of G induced by the vertex set

$$V(\tilde{H}) = \{v_{i,j} \mid v_{i,j} \in V(C_{t,l}), \quad c_{t,l} \in V(H)\}.$$

Since H is a connected subgraph of $\mathcal{H}$, we have that $\tilde{H}$ is a closed trail in G as it is an edge disjoint union of cycles of length 4 (Figure 3).

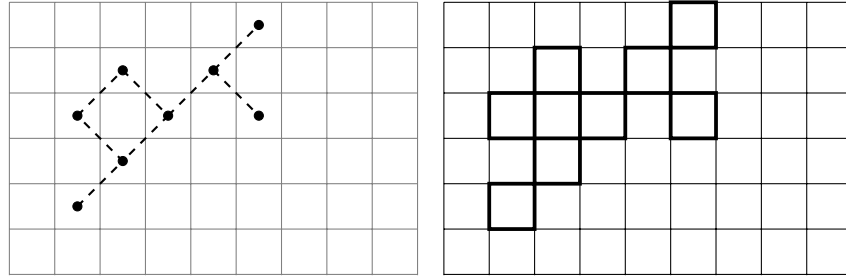


Figure 3: A subgraph H of $\mathcal{H}$ (left) and the corresponding subgraph $\tilde{H}$ of G (right).

Let $\mathcal{H}\varphi = H = T_1$ and let T_0 denote the cycle in G induced by vertices of degree 2 and 3 (Figure 4).

We have the following straightforward result:

LEMMA 2 *Let $l_0 = 2(n+m)$ and $l_1 = |E(\mathcal{A}(G))| - l_0$. Then, (l_0, l_1) is realizable in $\mathcal{A}(G)$. In particular, the closed trails T_0 and T_1 provide an edge disjoint decomposition of $\mathcal{A}(G)$.*

3.2 Auxiliary subgraphs $\mathcal{I}$ and $\mathcal{O}$.

It will be convenient to introduce two subgraphs of $\mathcal{H}$, a so called outer subgraph $\mathcal{O}$ and an inner subgraph $\mathcal{I}$.

Let $\mathcal{I}$ be a subgraph of $\mathcal{H}$ induced by the vertex set

$$V(\mathcal{I}) = \{v \in V(\mathcal{H}) \mid \deg(v) = 4 \text{ and if } uv \in E(\mathcal{H}) \text{ then } \deg(u) = 4\}$$

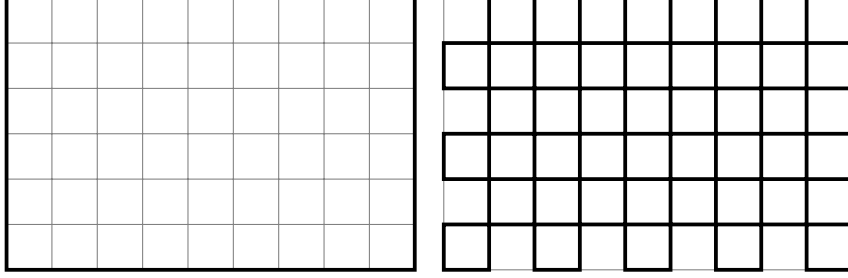


Figure 4: The closed trail T_0 (left) and the closed trail $T_1 = \mathcal{H}\varphi$ (right) providing a decomposition of $\mathcal{A}(G)$.

and let $\mathcal{O}$ denote the subgraph of $\mathcal{H}$ induced by the vertex set $V(\mathcal{H}) \setminus V(\mathcal{I})$ (Figure 5). Clearly, when $n \geq 6$, then $\mathcal{I}$ is a connected subgraph of $\mathcal{H}$ and $|\mathcal{I}| \geq m - 4$.

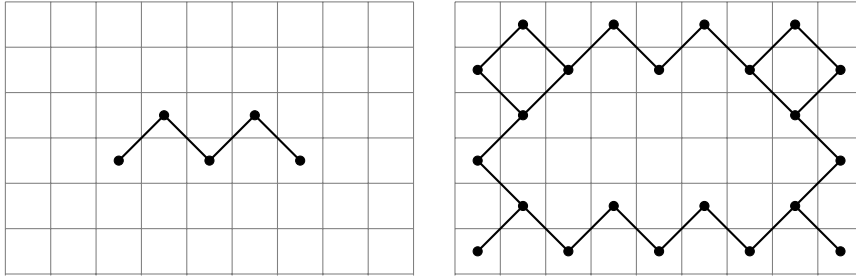


Figure 5: The inner subgraph $\mathcal{I}$ (left) and the outer subgraph $\mathcal{O}$ (right) of $\mathcal{H}$.

For convenience we will also refer to vertex columns of $\mathcal{I}$ and set $\mathcal{I}_j = \mathcal{I} \cap \mathcal{C}_j$ and vertex columns of $\mathcal{O}$ and set $\mathcal{O}_j = \mathcal{O} \cap \mathcal{C}_j$. That is, $\mathcal{O}_j \cup \mathcal{I}_j = \mathcal{C}_j$. In particular, we have that

$$|\mathcal{O}_1 \cup \mathcal{O}_2| = |\mathcal{O}_{m-1} \cup \mathcal{O}_m| = n \quad \text{and} \quad |\mathcal{O}_j \cup \mathcal{O}_{j+1}| = 4 \quad (3 \leq j \leq m-2) \quad (2)$$

It will be also convenient to introduce

$$\mathcal{O}_{outer} = \{c \in \mathcal{O} \mid \deg_{\mathcal{H}}(c) = 2\},$$

where $\deg_{\mathcal{H}}(c)$ is the degree of vertex c in $\mathcal{H}$ and

$$\mathcal{O}_{inner} = \{c \in \mathcal{O} \setminus \mathcal{O}_{outer} \mid \deg_{\mathcal{O}}(c) \neq 1, \deg_{\mathcal{O}}(c) \neq 3\},$$

where $\deg_{\mathcal{O}}(c)$ is the degree of vertex c in $\mathcal{O}$.

3.3 Auxiliary graph $\mathcal{H}_U$.

The aim of the following two subsections is to extend Lemma 2, and provide an edge disjoint decomposition of $\mathcal{A}(G)$ into two trails $\overline{T}_0$ and $\overline{T}_1$ such that

$|E(\overline{T}_0)| > 2(n+m)$. The trail $\overline{T}_0$ will be determined by the auxiliary graph $\mathcal{H}_U$ or $\mathcal{H}_{U,j}$.

Let $U \subseteq \mathcal{O}_{inner}$ and consider the graph generated by $V(\mathcal{H}) \setminus U$. Add the following edges that connect certain vertices in $\mathcal{O}_{outer}$:

$$E_U = \{c_1 c_2 \mid c_1, c_2 \in \mathcal{O}_{outer}, \exists d \in U \text{ such that } c_1 d, d c_2 \in E(\mathcal{O})\}.$$

The graph obtained will be denoted by $\mathcal{H}_U$ (Figure 6). Paths in $\mathcal{H}_U$, whose edges belong to E_U will be denoted by

$$P(E_U) = \{c_1 e_1 c_2 e_2 \dots e_{k-1} c_k \mid e_i \in E_U (1 \leq i \leq k-1)\}.$$

DEFINITION 1 We say that $\pi \in P(E_U)$ is a full path, if there does not exist a path $\rho \in P(E_U)$ such that $E(\pi) \subset E(\rho)$. The set of full paths of $\mathcal{H}_U$ will be denoted by $FP(E_U)$.

Example. The graph $\mathcal{H}_U$ in Figure 6 has two full paths, one being the full path connecting $c_{1,2}$ and $c_{1,6}$ and the other being the full path connecting $c_{3,2}$ and $c_{3,4}$.

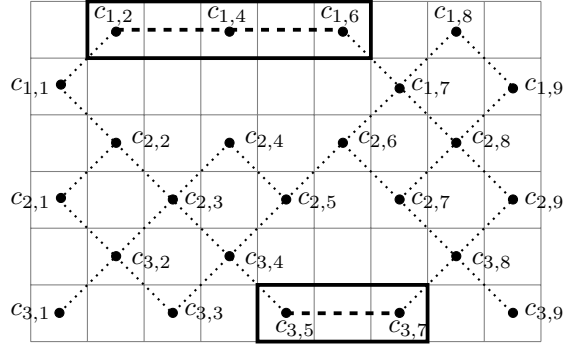


Figure 6: The graph $\mathcal{H}_U$, where $U = \{c_{1,3}, c_{1,5}, c_{3,6}\}$ superimposed on $G(6,9)$. Bold dashed edges belong to E_U . Subgraphs $\bar{\pi}_1 = R(c_{1,2}, c_{1,6})$ and $\bar{\pi}_2 = R(c_{3,5}, c_{3,7})$ of G are highlighted with bold edges.

Each full path π with end vertices c and d of $\mathcal{H}_U$ determines a unique simple cycle $\bar{\pi} = R(c, d)$ in G . Indeed, if for example $c = c_{1,i}, d = c_{1,j} \in V(\mathcal{H})$, then $R(c, d)$ is the unique cycle in G such that $|E(C_{1,i}) \cap E(R(c, d))| = |E(C_{1,j}) \cap E(R(c, d))| = 3$ (Figure 6).

DEFINITION 2 We say that H is a full subgraph of $\mathcal{H}_U$, if it is connected and for all $v \in E(H)$ such that $v \in V(\pi)$ for some $\pi \in FP(E_U)$, we have that $E(\pi) \subseteq E(H)$. The set of full subgraphs of $\mathcal{H}$ is denoted by $\Gamma(\mathcal{H}_U)$.

We observe, that if $H \in \Gamma(\mathcal{H}_U)$, then H is an edge disjoint union of full paths $\pi_1, \pi_2, \dots, \pi_i$ and connected subgraphs $H_1, H_2, \dots, H_j$ of $\mathcal{H}_U \setminus FP(E_U)$. Define

$$\psi : \Gamma(\mathcal{H}_U) \rightarrow \Gamma(G), H \mapsto (\bar{\pi}_1 \cup \dots \cup \bar{\pi}_i) \cup (H_1 \cup \dots \cup H_j)\varphi.$$

Clearly, ψ assigns to full subgraphs of $\Gamma(\mathcal{H}_U)$, closed trails in G (see Figure 7).

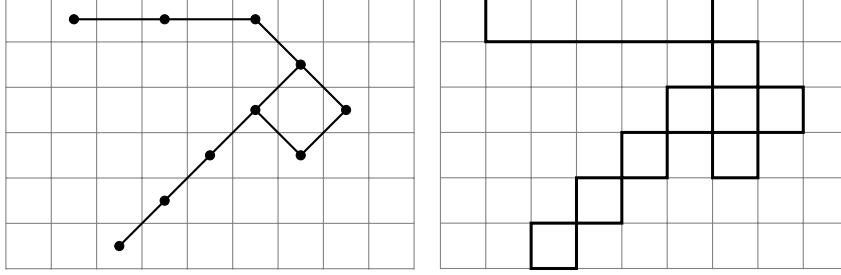


Figure 7: A full subgraph H of $\mathcal{H}_U$ superimposed on $G(6,9)$ (left) and the corresponding Eulerian subgraph $\hat{H}$ of G (right).

Let $C_U = \bar{T}_0$ denote the cycle in G induced by the edge set $E(\mathcal{A}(G)) \setminus E(\mathcal{H}_U)\psi$ and let $\bar{T}_1 = \mathcal{H}_U\psi$ (Figure 8).

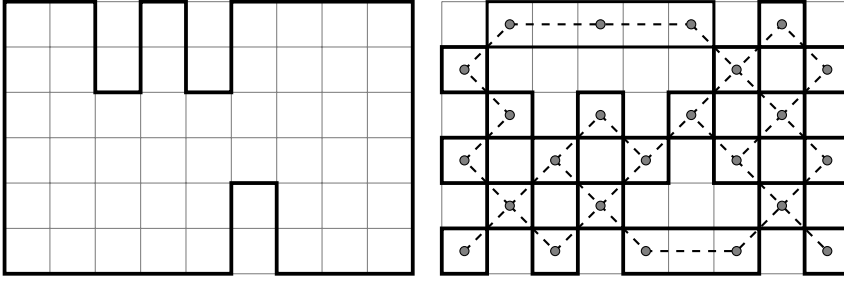


Figure 8: The cycle $\bar{T}_0 = C_U$ (left) and the graph $\bar{T}_1 = \mathcal{H}_U\psi$ (right), where $U = \{c_{1,3}, c_{1,5}, c_{3,6}\}$. The closed trails C_U and $\mathcal{H}_U\psi$ provide an edge disjoint decomposition of $\mathcal{A}(G)$.

It follows from our construction, that

$$|E(\bar{T}_0)| = 2(m+n) + 4|U| \quad \text{and} \quad |E(\bar{T}_1)| = |E(\mathcal{A}(G))| - |E(\bar{T}_0)| = 4l.$$

Extending Lemma 2, we now obtain the following immediate result:

LEMMA 3 *Let $k \leq \max\{|U| : U \subseteq \mathcal{O}_{inner}\}$. Let $l_0 = 2(n+m) + 4k$ and $l_1 = |\mathcal{A}(G)| - l_0$. Then, (l_0, l_1) is realizable in $\mathcal{A}(G)$. In particular, if $U \subseteq \mathcal{O}_{inner}$ with $|U| = k$, then $\bar{T}_0 = C_U$ and the closed trail $\bar{T}_1 = \mathcal{H}_U\psi$ provide an edge disjoint decomposition of $\mathcal{A}(G)$.*

3.4 Auxiliary graph $\mathcal{H}_{U,J}$.

Finally we, introduce the auxiliary subgraph $\mathcal{H}_{U,J}$, where $U \subseteq \mathcal{O}_{inner}$ and $J \subseteq \mathcal{I}$. Extending Lemma 3, this graph will provide an edge disjoint decomposition of

$\mathcal{A}(G)$ into two closed trails Σ_0 and Σ_1 , satisfying that $|E(\Sigma_i)| = 2(m+n) + 4k$ where $0 \leq k \leq \max\{|U| : U \subseteq \mathcal{O}_{inner}\} + |\mathcal{I}|$. We consider the following cases:

Case 1: If $k = 0$, then we let $U = J = \emptyset$ and set $\mathcal{H}_{U,J} = \mathcal{H}$. We let

$$\Sigma_0 = T_0 \quad \text{and} \quad \Sigma_1 = T_1 = \mathcal{H}\varphi.$$

By Lemma 2, $|E(\Sigma_0)| = 2(m+n)$ and Σ_0 and Σ_1 provide an edge disjoint decomposition of $\mathcal{A}(G)$.

Case 2: If $k = 1$, then let $\mathcal{H}_{U,J} = \mathcal{H}_U$, where $U = \{c_{1,3}\}$ and $J = \emptyset$. Let

$$\Sigma_0 = \bar{T}_0 = C_U \quad \text{and} \quad \Sigma_1 = \bar{T}_1 = \mathcal{H}_U\psi.$$

By Lemma 3, $|E(\Sigma_0)| = 2(m+n) + 4$ and Σ_0 and Σ_1 provide an edge disjoint decomposition of $\mathcal{A}(G)$ (Figure (9)).

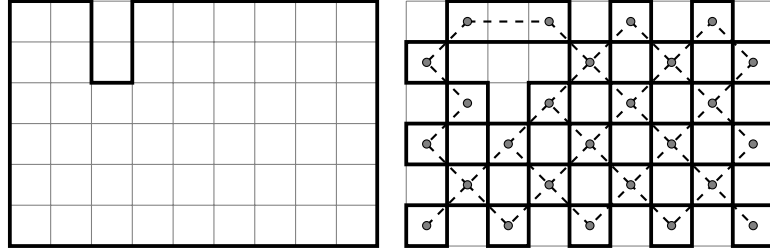


Figure 9: Case 2: Left: $\Sigma_0 = C_U$. Right: The graph $\mathcal{H}_U$ with $U = \{c_{1,3}\}$ superimposed on $G(6,9)$ determining the closed trail $\Sigma_1 = \mathcal{H}_U\psi$.

Case 3: Assume that $1 < k \leq |V(\mathcal{I})|$, where $k = (n-4)z_0 + z_1$ for some $z_0, z_1 \in \mathbb{N}^0$ with $z_1 < n-4$. We let $U = \{c_{1,3}\}$ and $J \subseteq \mathcal{I}$ such that $|J| = k-1$. In particular, if $z_0 = 0$, we assume that J is the path with $k-1$ vertices in $\mathcal{I}_3 \cup \mathcal{I}_4$ starting at vertex $c_{4,2}$. If $z_0 \geq 1$, then we choose

$$J = \mathcal{I}_3 \cup \mathcal{I}_4 \cup \dots \cup \mathcal{I}_{2z_0+2} \cup K,$$

where $K \subseteq \mathcal{I}_{2z_0+3}^r$ or $K \subseteq \mathcal{I}_{2z_0+3} \cup \mathcal{I}_{2z_0+4}^r$.

Let $\mathcal{H}_{U,J}$ denote the subgraph of $\mathcal{H}_U$ induced by the vertex set $V(\mathcal{H}_U) \setminus \mathcal{J}$. That is,

$$V(\mathcal{H}_{U,J}) = \left(\left(\bigcup_{j=1}^{2z_0+2} \mathcal{O}_j \right) \setminus \{c_{1,3}\} \right) \cup \left((C_{2z_0+3} \cup C_{2z_0+4}) \setminus K \right) \cup \bigcup_{j=2z_0+5}^m C_j.$$

Let

$$\Sigma_0 = C_U \cup J\psi \quad \text{and} \quad \Sigma_1 = \mathcal{H}_{U,J}\psi.$$

Clearly, $|E(\Sigma_0)| = 2(m+n) + 4 + 4|J| = 2(m+n) + 4k$ and by our construction, both Σ_0 and Σ_1 are closed trails in G that provide an edge disjoint decomposition of $\mathcal{A}(G)$ (Figure (10)).

It will be useful to consider the following edge disjoint decomposition of Σ_1 into two closed trails $\Sigma_{1,1}$ and $\Sigma_{1,2}$. To define $\Sigma_{1,1}$, consider the subgraph $\mathcal{H}_{U,J}^1$ of $\mathcal{H}_{U,J}$ with vertex set

$$V(\mathcal{H}_{U,J}^1) = \left(\left(\bigcup_{j=1}^{2z_0+2} \mathcal{O}_j \right) \setminus \{c_{1,3}\} \right) \cup \left((C_{2z_0+3} \cup C_{2z_0+4}) \setminus K \right) \cup L,$$

where $L = \{c_{1,2z_0+5}, c_{2,2z_0+5}, \dots\} \subseteq C_{2z_0+5} \cup C_{2z_0+6}$ such that $|L| = z_1$. We let $\mathcal{H}_{U,J}^2$ be the subgraph with vertex set $V(\mathcal{H}_{U,J}^2) = V(\mathcal{H}_{U,J}) \setminus V(\mathcal{H}_{U,J}^1)$. Clearly, $\Sigma_{1,1} = \mathcal{H}_{U,J}^1\psi$ and $\Sigma_{1,2} = \mathcal{H}_{U,J}^2\psi$ provide a decomposition of Σ_1 . Using (2) and Lemma 1 we observe that

$$|V(\mathcal{H}_{U,J}^1)| = n + 1 + 2 + 4(z_0 - 1) + (n - z_1) + z_1 = 2n + 4z_0 - 1 \quad (3)$$

and so $|E(\Sigma_{1,1})| = 8n + 16z_0 - 4$. It follows that

$$2|E(\Sigma_0)| = 4(n + m) + 8 + 8[(n - 4)z_0 + z_1] \leq |E(\Sigma_{1,1})| \quad (4)$$

holds, since $6 \leq n \leq m$ by assumption.

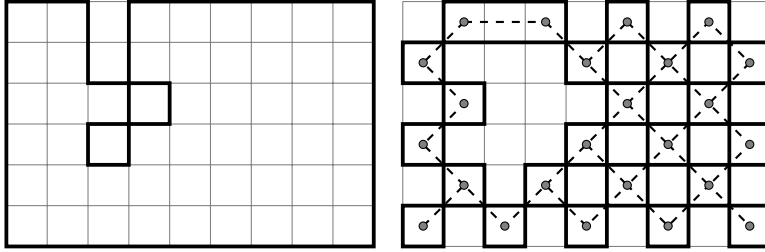


Figure 10: Case 3: Left: $\Sigma_0 = C_U \cup J\psi$, where $U = \{c_{1,3}\}$ and $J = \{c_{2,4}, c_{3,2}\}$. Right: The graph $\mathcal{H}_{U,J}$ superimposed on $G(6,9)$ determining the closed trail $\Sigma_1 = \mathcal{H}_{U,J}\psi$.

Case 4: If $|V(\mathcal{I})| < k$, then let $U \subseteq \mathcal{O}_{inner}$ satisfying that $|U| + |\mathcal{I}| = k$ and we let $J = \mathcal{I}$. Let $\mathcal{H}_{U,J}$ denote the subgraph of $\mathcal{H}_U$ induced by the vertex set $V(\mathcal{H}_U) \setminus \mathcal{I}$. We let

$$\Sigma_0 = C_U \cup \mathcal{I}\psi \quad \text{and} \quad \Sigma_1 = \mathcal{H}_{U,J}\psi.$$

We have that $|E(\Sigma_0)| = 2(m + n) + 4(|U| + |\mathcal{I}|) = 2(m + n) + 4k$ and that Σ_0 and Σ_1 provide an edge disjoint decomposition of $\mathcal{A}(G)$ (Figure (11)).

We summarise the above in the following lemma:

LEMMA 4 *Assume that the sequence (l_0, l_1) is admissible for $\mathcal{A}(G)$, where $l_0 = 2(m + n) + 4k$ such that $0 \leq k \leq \max\{|U| : U \subseteq \mathcal{O}_{inner}\} + |\mathcal{I}|$. Then (l_0, l_1) is realizable in $\mathcal{A}(G)$. In particular, if U and $J \subseteq \mathcal{I}$ are determined as above, then $\Sigma_0 = C_U \cup J\psi$ and the closed trail $\Sigma_1 = \mathcal{H}_{U,J}\psi$ provide an edge disjoint decomposition of $\mathcal{A}(G)$.*

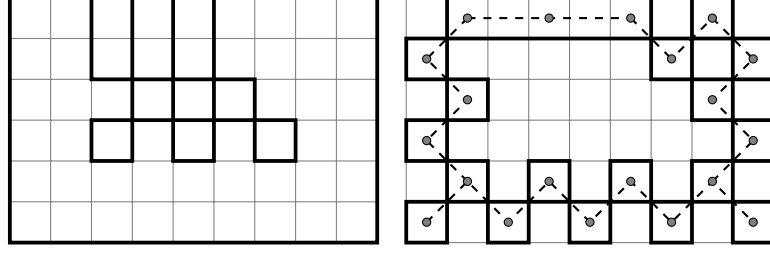


Figure 11: Case 4: Left: $\Sigma_0 = C_U \cup J\psi$, where $U = \{c_{1,3}, c_{1,5}\}$ and $J = \mathcal{I}$. Right: The graph $\mathcal{H}_{U,J}$ superimposed on $G(6,9)$ determining the closed trail $\Sigma_1 = H_{U,J}\psi$ (bold edges).

4 Decomposition of $\mathcal{A}(G)$ into closed trails

Building on the construction introduced in the Section 3, we first provide a decomposition of $\mathcal{A}(G)$ into closed trails $T_0, T_1, \dots, T_t$ satisfying that $|E(T_i)| = 4k_i \geq 2(m+n)$. Then, with the help of a technical step introduced in Lemma 6 we exhibit a procedure that exchanges edges of trails having a common vertex in G that will lead to the main result where trail lengths are either $l_i = 4k_i$ or $l_i = 4k_i + 2$.

LEMMA 5 *Let $(l_0, l_1, \dots, l_t)$ be an admissible sequence for $\mathcal{A}(G)$ satisfying that $l_0 = 2(n+m) + 4k_0$, ($k_0 \geq 0$) and that $l_i = 4k_i$ for $i \geq 1$. Then, there exists an edge-disjoint decomposition of $\mathcal{A}(G)$ into closed trails $T_0, T_1, \dots, T_t$ satisfying that $|E(T_i)| = l_i$.*

Proof. By Lemma 3, $\mathcal{A}(G)$ can be decomposed into two closed trails Σ_0 and Σ_1 satisfying that $|E(\Sigma_0)| = l_0$ and $|E(\Sigma_1)| = |E(\mathcal{A}(G))| - l_0$. Consider the subgraph $\mathcal{H}_{U,J}$ of $\mathcal{H}_U$ determining Σ_1 . We construct connected subgraphs $\mathcal{H}_1, \dots, \mathcal{H}_t$ of $\mathcal{H}_{U,J}$ satisfying that $|V(\mathcal{H}_i)| = k_i$ and let $T_i = \mathcal{H}_i\psi$. It will then follow that T_i is a closed trail for all $1 \leq i \leq t$ satisfying that $|E(T_i)| = 4k_i = l_i$. We will follow the cases introduced in Section 3.4. Without loss of generality, we may assume that $l_0 \leq l_i = 4k_i$ for all $1 \leq i \leq t$.

Case 1: If $k_0 = 0$, then $\mathcal{H}_{U,J} = \mathcal{H}$. Let

$$V(\mathcal{H}_1) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_{j_1-1} \cup \mathcal{C}_{j_1}^{r_1},$$

where j_1 and r_1 is chosen such that $|V(\mathcal{H}_1)| = k_1$ holds. For $1 \leq p$, we let

$$V(\mathcal{H}_{p+1}) = (\mathcal{C}_{j_p} \setminus \mathcal{C}_{j_p}^{r_p}) \cup \mathcal{C}_{j_p+1} \cup \dots \cup \mathcal{C}_{j_{p+1}} \cup \mathcal{C}_{j_{p+1}}^{r_{p+1}},$$

where j_p and r_p is chosen such that $|V(\mathcal{H}_{p+1})| = k_{p+1}$ holds. Since $\mathcal{C}_j \cup \mathcal{C}_{j+1}$ induces a connected subgraph in $\mathcal{H}$ and since $|\mathcal{C}_j \cup \mathcal{C}_{j+1}| = n \leq k_j$ for all $1 \leq j \leq t$, we have that $\mathcal{H}_p$ is a connected subgraph of $\mathcal{H}$ for all $1 \leq p \leq t$ by Lemma 1. Let $H_j\varphi = T_j$. Then, it follows that $T_1, T_2, \dots, T_t$ provides an edge-disjoint decomposition of $\mathcal{A}(G)$ and that $|E(T_j)| = l_j$ holds (Figure 12).

Case 2: If $k_0 = 1$, then $\mathcal{H}_{U,J} = \mathcal{H}_U$, where $U = \{c_{1,3}\}$. Let

$$V(\mathcal{H}_1) = \left(\{c_{1,4}\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \dots \cup \mathcal{C}_{j_1-1} \cup \mathcal{C}_{j_1}^{r_1} \right) \setminus \{c_{1,3}\}$$

where j_1 and r_1 is chosen such that $|V(\mathcal{H}_1)| = k_1$ holds. Since $k_0 \leq k_1$ by assumption, $\{c_{1,4}\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \subseteq V(\mathcal{H}_1)$ and so $\mathcal{H}_1$ is a full subgraph of $\mathcal{H}_{U,J}$. For $1 \leq p$, we let

$$V(\mathcal{H}_{p+1}) = \left((\mathcal{C}_{j_p} \setminus \mathcal{C}_{j_p}^{r_p}) \cup \mathcal{C}_{j_p+1} \cup \dots \cup \mathcal{C}_{j_{p+1}} \cup \mathcal{C}_{j_{p+1}}^{r_{p+1}} \right) \setminus \{c_{1,3}\}$$

where j_p and r_p is chosen such that $|V(\mathcal{H}_{p+1})| = k_{p+1}$ holds. By Lemma 1, we have that $\mathcal{H}_p$ is a connected subgraph of $\mathcal{H}_U$ for all $1 \leq p \leq t$. Let $H_j\psi = T_j$. Then, it follows that $T_1, T_2, \dots, T_t$ provides an edge-disjoint decomposition of $\mathcal{A}(G)$ and that $|E(T_j)| = l_j$ holds (Figure 12).

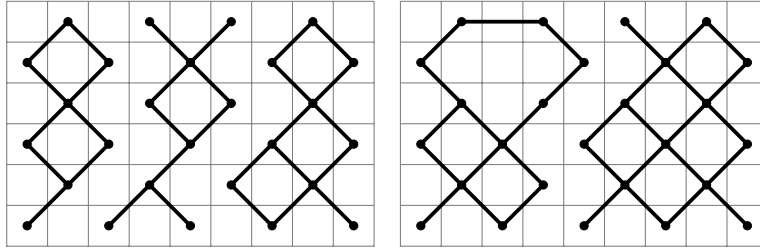


Figure 12: Left: Subgraphs $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ determining a decomposition of $\Sigma_1 = \mathcal{H}\psi$ (Case 1). Right: Subgraphs $\mathcal{H}_1, \mathcal{H}_2$ determining a decomposition of $\Sigma_1 = \mathcal{H}_U\psi$ (Case 2).

Case 3: Assume that $1 < k \leq |V(\mathcal{I})|$, where $k = (n-4)z_0 + z_1$ for some $z_0, z_1 \in \mathbb{N}^0$ with $z_1 < n-4$. We let $U = \{c_{1,3}\}$ and $J \subseteq \mathcal{I}$ such that $|J| = k-1$ and consider $\mathcal{H}_{U,J}$ as determined in Section 3.4. Let

$$V(\mathcal{H}_1) = \left(\{c_{1,4}\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \dots \cup \mathcal{C}_{j_1-1} \cup \mathcal{C}_{j_1}^{r_1} \right) \setminus \left(\{c_{1,3}\} \cup V(J) \right)$$

where j_1 and r_1 is chosen such that $|V(\mathcal{H}_1)| = k_1$ holds. Since $l_0 \leq l_1$ by assumption, $\{c_{1,4}\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \subseteq V(\mathcal{H}_1)$ hold, and hence, $\mathcal{H}_1$ is a full subgraph of $\mathcal{H}_{U,J}$. Observe that the subgraph generated by the vertex set $V(\mathcal{H}_{U,J}^1) \setminus V(\mathcal{H}_1)$ remains connected. Let

$$V(\mathcal{H}_2) = \left((\mathcal{C}_{j_1} \setminus \mathcal{C}_{j_1}^{r_1}) \cup \mathcal{C}_{j_1+1} \cup \dots \cup \mathcal{C}_{j_2-1} \cup \mathcal{C}_{j_2}^{r_2} \right) \setminus \left(\{c_{1,3}\} \cup V(J) \right),$$

where j_2 and r_2 is chosen such that $|V(\mathcal{H}_2)| = l_2$ holds. Since $l_0 \leq l_1 = 4k_1$ and $l_0 \leq l_2 = 4k_2$, we obtain using (4) that

$$|E(\mathcal{H}_{U,J})| < 2l_0 \leq |E(\mathcal{H}_1\psi)| + |E(\mathcal{H}_2\psi)|.$$

Hence, $|V(\mathcal{H}_{U,J}^1)| \leq |V(\mathcal{H}_2)| \cup |V(\mathcal{H}_1)|$ and so $V(\mathcal{H}_{U,J}^1) \setminus V(\mathcal{H}_1) \subseteq V(\mathcal{H}_2)$. We may hence deduce that $\mathcal{H}_2$ is a connected subgraph of $\mathcal{H}_{U,J}$. For $3 \leq p$, we let

$$V(\mathcal{H}_{p+1}) = (\mathcal{C}_{j_p} \setminus \mathcal{C}_{j_p}^{r_p}) \cup \mathcal{C}_{j_{p+1}} \cup \dots \cup \mathcal{C}_{j_{p+1}} \cup \mathcal{C}_{j_{p+1}}^{r_{p+1}}$$

where j_p and r_p is chosen such that $|V(\mathcal{H}_{p+1})| = l_{p+1}$ holds. By Lemma 1 we have that $\mathcal{H}_p$ is a connected subgraph of $\mathcal{H}_{U,J}$ for all $1 \leq p \leq t$. Let $H_j\psi = T_j$. The closed trails $T_1, T_2, \dots, T_p$ provide an edge decomposition of $\mathcal{A}(G)$ such that $|E(T_j)| = l_j$. (Figure 13.)

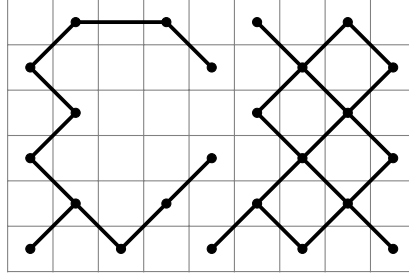


Figure 13: Subgraphs $\mathcal{H}_1, \mathcal{H}_2$ determining a decomposition of $\Sigma_1 = \mathcal{H}_{U,J}\psi$, where $U = \{c_{1,3}\}, J = \{c_{2,4}, c_{3,2}\}$ (Case 3).

Case 4: This is a redundant case as Σ_1 cannot be decomposed into trails $T_1, \dots, T_t$ such that $|E(T_i)| = l_i \geq l_0$. To verify this, we show that $|E(\Sigma_1)| < 2l_0$. Consider $\mathcal{H}_{U,J}$ as determined in Case 4 in Section 3.4. Note that

$$l_0 = |E(\Sigma_0)| = 2(n+m) + 4(|\mathcal{I}| + |\mathcal{U}|) \quad \text{and} \quad |E(\Sigma_1)| = 4(2n+2m-8) - 4|\mathcal{U}|.$$

Using that $|\mathcal{U}| \geq 1, |\mathcal{I}| > m-4$ and $m \geq n$, it can be now readily verified that $|E(\Sigma_1)| < 2l_0 \leq l_1 + l_2$. $\square$

In the next lemma, we show that the construction introduced in Lemma 5 can be modified to accommodate trail lengths $l = 4k + 2$.

LEMMA 6 *Consider the trail decomposition $T_0, T_1, \dots, T_i, T_{i+1}, \dots, T_t$ of $\mathcal{A}(G)$ as determined in Lemma 5. Then, there exists a trail decomposition*

$$T_0, T_1, \dots, \bar{T}_i, \bar{T}_{i+1}, \dots, T_t$$

such that $|E(\bar{T}_i)| = 4k_i + 2$ and $|E(\bar{T}_{i+1})| = 4k_i - 2$.

Proof. Let $T_i = \mathcal{H}_i\varphi$ and $T_{i+1} = \mathcal{H}_{i+1}\varphi$. Considering cases 1, 2 and case 3 when $i \geq 2$, there exist vertices $d_1, d_2, d_3, d_4 \in V(\mathcal{H}_{U,J})$ and a vertex column C_j such that either of the following two cases occur:

- (a) $d_1 \in V(\mathcal{H}_i) \cap C_j, d_2, d_3 \in V(\mathcal{H}_{i+1}) \cap C_{j+1}$ and $d_4 \in V(\mathcal{H}_{i+1}) \cap C_{j+1}$
- (b) $d_1 \in V(\mathcal{H}_i) \cap C_j, d_2, d_3 \in V(\mathcal{H}_i) \cap C_{j+1}$ and $d_4 \in V(\mathcal{H}_{i+1}) \cap C_{j+1}$.

We consider (a), as (b) is similar. To obtain $\bar{T}_i$ and $\bar{T}_{i+1}$ from T_i and T_{i+1} we exchange certain edges of $d_1\psi, d_2\psi, d_3\psi$ and $d_4\psi$ as depicted in Figure 14 that lengthens T_i by two edges and shortens T_{i+1} by two edges as desired.

In case 3, if $i = 1$ and neither (a) nor (b) hold, then $V(T_1) \cap V(J\varphi) \neq 0$, and $V(T_2) \cap V(J\varphi) \neq 0$. We construct $\bar{T}_1$ and $\bar{T}_2$ in two steps. First, by exchanging edges between T_1 and Σ_0 , we lengthen T_1 by two edges to obtain $\bar{T}_1$ and shorten Σ_0 by two edges and obtain $\bar{\Sigma}_0$. Then, we lengthen $\bar{\Sigma}_0$ by two edges to obtain $\bar{\Sigma}_0$ so that $|E(\bar{\Sigma}_0)| = l_0$ and shorten T_2 by two edges to obtain $\bar{T}_2$. In both steps, we apply a similar technique as described above. Since $V(T_1) \cap V(J\varphi) \neq 0$, there exist $d_1 = c_{1,j} \in V(\mathcal{H}_i)$ $d_2 = c_{1,j-1}, d_3 = c_{2,j}, d_4 = c_{2,j+1} \in J$. Exchange edges between subgraphs $d_1\varphi, d_2\varphi, d_3\varphi, d_4\varphi$ as depicted in Figure 15. As observed in case 3 in Section 3, $V(\mathcal{H}_{U,J}) \setminus V(\mathcal{H}_1) \subseteq V(\mathcal{H}_2)$ and therefore there exist vertices $\bar{d}_1 \in J, \bar{d}_2, \bar{d}_3, \bar{d}_4 \in V(\mathcal{H}_2)$ as in case (a). The edge exchanging technique depicted in Figure 14 can be applied to lengthen $J\varphi$ by two edges and shorten T_2 by two edges to obtain $\bar{T}_2$.



Figure 14: Left: Subgraph of T_i (bold edges) and T_{i+1} (dashed edges). Right: Swapping edges to obtain $\bar{T}_i$ (bold edges) and $\bar{T}_{i+1}$ (dashed edges) from T_i and T_{i+1} .



Figure 15: Left: Subgraph of $J\varphi \subseteq \Sigma_0$ (bold edges) and T_1 (dashed edges). Right: Swapping edges to obtain $\bar{\Sigma}_0$ (bold edges) and $\bar{T}_1$ (dashed edges).

□

Our main result is the following proposition.

PROPOSITION 1 *Let $(l_0, l_1, \dots, l_t)$ be an admissible sequence for $\mathcal{A}(G)$ satisfying that $l_0 = 2(n + m) + 4k_0$, ($k_0 \geq 0$), $l_0 \leq l_i$ and $l_i = 4k_i$ or $l_i = 4k_i + 2$ for $i \geq 1$. Then, there exists an edge-disjoint decomposition of $\mathcal{A}(G)$ into closed trails $T_0, T_1, \dots, T_t$ satisfying that $|E(T_i)| = l_i$.*

Proof. Let $S_1 = \{l_i \mid l_i = 4k_i\}$ and $S_2 = \{l_i \mid l_i = 4k_i + 2\}$. Without loss of generality, we may assume that $l_i \in S_2$ precisely when $i \geq j$ for some $1 \leq j$. Since $4|V(\mathcal{H}_{U,J})|$, we have that $|S_2|$ is even. Lemma 5 can be applied

to obtain a closed trail decomposition $T_0, T_1, \dots, T_{j-1}, \bar{T}_j, \bar{T}_{j+1}, \dots, \bar{T}_{t-1}, \bar{T}_t$ of $\mathcal{A}(G)$ satisfying that $T_i = 4k_i$ for $i < j$ and

$$|\bar{T}_j| = l_j + 2, |\bar{T}_{j+1}| = l_{j+1} - 2, \dots, |\bar{T}_{t-1}| = l_{t-1} - 2, |\bar{T}_t| = l_t + 2.$$

To obtain the desired trail decomposition, we apply Lemma 6 to shorten $\bar{T}_{j+2z}$ by two edges and lengthen $\bar{T}_{j+2z+1}$ by two edges, where $z \geq 0$. $\square$

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