New fractional calculus results involving Srivastava's general class of multivariable polynomials and The $\bar{H}$ - function

V. B. L. CHAURASIA AND V. GILL

Abstract

A significantly large number of earlier works on the subjects of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis. In the present paper, we study and develop an important result involving a fractional differential operator for the product of general multivariable polynomials, general polynomial set and two -functions. The result discussed here can be used to investigate a wide class of new and known results, hitherto scattered in the literature. For the sake of illustration, six interesting special case have also been recorded here of our main findings.

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1. INTRODUCTION

Misra [10] has defined the fractional derivative operators in the following manner:

$$D_x^{\alpha}(x^{\mu-1}) = \frac{d^{\alpha}}{dx^{\alpha}} x^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)} x^{\mu-\alpha-1}, \quad \alpha \neq \mu \tag{1.1}$$

$$D_{k,\alpha,x}(x^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu+k}, \quad \alpha \neq \mu+1 \tag{1.2}$$

$$D_{k,\alpha,x}^{n}(x^{\mu}) = \prod_{r=0}^{n-1} \left[\frac{\Gamma(\mu+1+rk)}{\Gamma(\mu+1+rk-\alpha)} \right] x^{\mu+nk} \tag{1.3}$$

α and k are not necessarily integers.

The general multivariable polynomials is introduced and defined by Srivastava [15] in the following form:

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$$S_{n_1,\ldots,n_k}^{m_1,\ldots,m_k}[y_1,\ldots,y_k]=\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_k=0}^{[n_k/m_k]}\frac{(-n_1)_{m_1s_1}}{s_1!}y_1^{s_1}\cdots\frac{(-n_k)_{m_ks_k}}{s_k!}y_k^{s_k}A[n_1,s_1;\ldots;n_k,s_k],$$

(1.4)

where $n_i=0,1,2,\ldots;m_i(i=1,\ldots,k)$ are non-zero arbitrary positive integer. The coefficients $A[n_1,s_1;\ldots;n_k,s_k]$ being arbitrary constants, real or complex.

Raizada has introduced and studied a generalized polynomial set. The explicit form of this generalized polynomial set [11, p.71, eq. (2.3.4)] is:

$$S_n^{\alpha,\beta,\tau}[x;r,s,q,A,B,m,k,\ell]=B^{qn}x^{\ell(m+n)}(1-\tau x^r)^{sn}\ell^{m+n}$$

$$\sum_{p=0}^{m+n}\sum_{e=0}^{p}\sum_{\delta=0}^{m+n}\sum_{i=0}^{\delta}\frac{(-1)^{\delta}(-p)_e(-\delta)_i(\alpha)_\delta}{p!\,\delta!i!\,e!}\frac{(\alpha-qn)_i}{(1-\alpha-\delta)_i}\left(-\frac{\beta}{\tau}-sn\right)_p$$

$$\left(\frac{\ell+k+re}{\ell}\right)_{m+n}\left(-\frac{\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^{\delta} \tag{1.5}$$

If $\tau\to 0$ and using following result

$$\lim_{|b|\to\infty}(b)_n\left(\frac{z}{n}\right)^n=z^n \tag{1.6}$$

We arrive at the following polynomial set:

$$S_n^{\alpha,\beta,0}[x;r,q,1,0,m,k,\ell]=x^{qn+\ell(m+n)}\ell^{m+n}\sum_{p=0}^{m+n}\sum_{e=0}^{p}\frac{(-p)_e}{p!e!}\left(\frac{\alpha+qn+k+re}{\ell}\right)_{m+n}(\beta x^r)^p$$

(1.7)

The $\bar{H}$ -function occurring in this paper was introduced by Inayat Hussain [7] and studied by Bushman and Srivastava [1]. The $\bar{H}$-function is defined and represented in the following manner:

$$\bar{H}_{P,Q}^{M,N}[z]=\bar{H}_{P,Q}^{M,N}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]$$

$$= \frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty} \bar{\theta}(\xi) z^{\xi} d\xi,\ \omega = \sqrt{-1},\quad (z \neq 0) \tag{1.8}$$

where

$$\bar{\theta}(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=M+1}^{Q} \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)} \tag{1.9}$$

Here a_j (j = 1,…,P) and b_j (j = 1,…,Q) are complex parameter, $\alpha_j \geq 0\ (j = 1,..., P).$

$\beta_j \geq 0 (j = 1,..., Q)$ (not all zero simultaneously) and the exponents a_j (j = 1,…,N) and b_j (j = M+1,…,Q) can take on integer values.

The following sufficient conditions for the absolute convergence of the defining integral for the $\bar{H}$-function given by Bushman and Srivastava [1]

$$T = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} |A_j \alpha_j| - \sum_{j=M+1}^{Q} |B_j \beta_j| - \sum_{j=N+1}^{P} \alpha_j > 0 \tag{1.10}$$

and $|\arg z| < \frac{1}{2}\pi T$

The following series representation for the $\bar{H}$-function was obtained by Rathie [12]:

$$\bar{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (c_j,\gamma_j;C_j)_{1,N},(c_j,\gamma_j)_{N+1,P} \\ (d_j,\delta_j)_{1,M},(d_j,\delta_j;D_j)_{M+1,Q} \end{matrix} \right. \right] = \sum_{r=0}^{\infty} \sum_{h=1}^{M} \bar{\theta}(\xi_{h,r}) z^{\xi_{h,r}} \tag{1.11}$$

where

$$\bar{\theta}(\xi_{h,r}) = \frac{\prod_{\substack{j=1 \\ j\neq 1}}^{M} \Gamma(d_j - \delta_j \xi_{h,r}) \prod_{j=1}^{N} \{\Gamma(1 - c_j + \gamma_j \xi_{h,r})\}^{C_j}}{\prod_{j=M+1}^{Q} \{\Gamma(1 - d_j + \delta_j \xi_{h,r})\}^{D_j} \prod_{j=N+1}^{P} \Gamma(c_j - \gamma_j \xi_{h,r})} \tag{1.12}$$

$$\frac{(-1)^r}{r!\delta_h} \qquad \text{and} \quad \xi_{h,r} = \frac{d_r + h}{\delta_r}$$

and Binomial expansion is given as

$$(ax^\nu + b)^\lambda = b^\lambda \sum_{\ell=0}^{\lambda} \binom{\lambda}{\ell} \left(\frac{ax^\nu}{b}\right)^\ell \tag{1.13}$$

where $\lambda > 0$.

2. MAIN RESULT

$$D_{k,\alpha,x}^n \left\{ x^\mu (ax^\nu + b)^\lambda S_{N_1,\ldots,N_g}^{M_1,\ldots,M_g} \left[y_1 x^{u_1} (ax^\nu + b)^{\eta_1}, \ldots, y_g x^{u_g} (ax^\nu + b)^{\eta_g} \right] \right.$$

$$\left. S_{n'}^{\alpha',\beta,0} [z_1 x^{u'} (ax^\nu + b)^{\eta'}; s, q, 1, 0, m, k', \ell] \overline{H}_{p_1,q_1}^{m_1,n_1} \left[yx^u (ax^\nu + b)^\eta \middle| \begin{matrix} (c_j,\gamma_j;C_j)_{1,n_1},(c_j,\gamma_j)_{n_1+1,p_1} \\ (d_j,\delta_j)_{1,m_1},(d_j,\delta_j;D_j)_{m_1+1,q_1} \end{matrix} \right] \right\}$$

$$\overline{H}_{P,Q}^{M,N} \left[zx^\lambda \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right]$$

$$= G_1 G_2 \sum_{r=0}^{\infty} \sum_{h=1}^{m_1} \sum_{f=0}^{\infty} \overline{\theta}(\xi_{h,r}) y^{\xi_{h,r}} b^{\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} \eta_i s_i + \eta\xi_{h,r}}$$

$$x^{\mu+\nu f+u'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} u_i s_i + u\xi_{h,r}+nk}$$

$$\frac{(\lambda + \eta'\{qn'+sp+\ell(m+n')\} + \sum_{i=1}^{g} \eta_i s_i + \eta\xi_{h,r})!}{f!(\lambda - f + \eta'\{qn'+sp+\ell(m+n')\} + \sum_{i=1}^{g} \eta_i s_i + \eta\xi_{h,r})!}$$

$$\left(\frac{a}{b}\right)^{f} \overline{H}^{M,\,N+n}_{P+n,Q+n}\left[zx^{\lambda} \left| \begin{matrix} (-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-\sum_{i=1}^{g} u_i s_i - u\,\xi_{h,r},\lambda:1)_{t=0,n-1} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right.$$

$$\left. \left. \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (\alpha-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-\sum_{i=1}^{g} u_i s_i - u\xi_{h,r},\lambda:1)_{t=0,n-1} \end{matrix} \right. \right], \qquad (2.1)$$

where

$$G_1 = \sum_{s_1=0}^{[N_1/M_1]} \cdots \sum_{s_g=0}^{[N_g/M_g]} \frac{(-N_1)_{M_1 s_1}}{s_1!} \cdots \frac{(-N_g)_{M_g s_g}}{s_g!} B[N_1,s_1;\ldots;N_g,s_g]\, y_1^{s_1} \cdots y_g^{s_g}$$

$$G_2 = \sum_{p=0}^{m+n'} \sum_{e=0}^{p} \frac{(-p)_e}{p!e!}\, \ell^{(m+n')} \left(\frac{\alpha'+qn'+k'+se}{\ell}\right)_{m+n'} \beta^p\, z_1^{qn'+sp+\ell(m+n')}$$

and the sufficient conditions of validity of (2.1) are as follows:

(i) The quantities $u_j, \eta_j, u', \eta' (j = 1,\ldots, g)$ are all positive.

(ii) $\nu : N_i = 0,1,2,\ldots,; M_i (i = 1,\ldots, g)$ are non-zero arbitrary positive integer.

The coefficients B $[N_1,s_1 ;\ldots; N_g,s_g]$ being arbitrary constant, real or complex.

(iii) $$T = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} |A_j \alpha_j| - \sum_{j=M+1}^{Q} |B_j \beta_j| - \sum_{j=N+1}^{P} \alpha_j > 0$$

(iv) $$|\arg z| < \frac{1}{2} T \pi$$

(v) Re (μ) > 0

(vi) $\alpha < 0$

(vii) $$\lambda + \eta'\{qn'+sp+\ell(m+n')\} + \sum_{i=1}^{g} \eta_i s_i + \eta \xi_{h,r} > 0$$

PROOF. To establish (2.1), we first use series representation (1.4), (1.7) and (1.11) for $S_{N_1,\ldots,N_g}^{M_1,\ldots,M_g}[x_1,\ldots,x_g], S_n^{\alpha,\beta,0}$ and $\overline{H}$ -function respectively and contour integral representation (1.8) for $\overline{H}$-function occurring in the left hand side of (2.1), it takes the following form (say Δ).

$$\Delta = D_{k,\alpha,x}^{\eta}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda}\sum_{s_1=0}^{[N_1/M_1]}\cdots\sum_{s_g=0}^{[N_g/M_g]}\frac{(-N_1)_{M_1s_1}}{s_1!},\ldots,\frac{(-N_g)_{M_gs_g}}{s_g!}\right.$$

$$B[N_1,s_1;\ldots;N_g,s_g]y_1^{s_1},\ldots,y_g^{s_g}x^{u_1s_1+\ldots+u_gs_g}(ax^{\nu}+b)^{\eta_1s_1+\ldots+\eta_gs_g}$$

$$\{z_1x^{u'}(ax^{\nu}+b)^{\eta'}\}^{qn'+\ell(m+n')}\ell^{(m+n')}\sum_{p=0}^{m+n'}\sum_{e=0}^{p}\frac{(-p)_e}{p!\,e!}\left(\frac{\alpha'+qn'+k'+se}{\ell}\right)_{m+n'}$$

$$\left.\{\beta z_1^{s}x^{u's}(ax^{\nu}+b)^{\eta's}\}^{p}\sum_{r=0}^{\infty}\sum_{h=1}^{m_1}\overline{\theta}(\xi_{h,r})y^{\xi_{h,r}}x^{u\xi_{h,r}}(ax^{\nu}+b)^{\eta\xi_{h,r}}\frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty}\overline{\theta}(\xi)z^{\xi}x^{\lambda\xi}d\xi\right.$$

(2.2)

interchanging the order of summation and integration (which is permissible under the condition stated)

$$\Delta=\sum_{s_1=0}^{[N_1/M_1]}\cdots\sum_{s_g=0}^{[N_g/M_g]}\frac{(-N_1)_{M_1s_1}}{s_1!}\cdots\frac{(-N_g)_{M_gs_g}}{s_g!}B[N_1,s_1;\ldots;N_g,s_g]\,y_1^{s_1}\ldots y_g^{s_g}$$

$$\sum_{p=0}^{m+n'}\sum_{e=0}^{p}\frac{(-p)_e}{p!e!}\ell^{(m+n')}\left(\frac{\alpha'+qn'+k'+se}{\ell}\right)_{m+n'}\beta^{p}z_1^{qn'+sp+\ell(m+n')}$$

$$\sum_{r=0}^{\infty}\sum_{h=1}^{m_1}\overline{\theta}(\xi_{h,r})y^{\xi_{h,r}}\frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty}\overline{\theta}(\xi)z^{\xi}D_{k,\alpha,x}^{n}$$

$$\left\{ x^{\mu+u'\{qn'+\ell(m+n')+sp\}+\sum_{i=1}^{g} u_i s_i+u\,\xi_{h,r}+\lambda\xi} \; (ax^{\nu}+b)^{\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} \eta_i s_i+\eta\zeta_{h,r}} \right\} d\xi$$

(2.3)

now using the Binomial expansion given by (1.13), we get with the help of equation (1.3)

$$\Delta = G_1 G_2 \sum_{r=0}^{\infty} \sum_{h=1}^{m_1} \overline{\theta}(\xi_{h,r}) y^{\xi_{h,r}} b^{\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} \eta_i s_i+\eta\xi_{h,r}}$$

$$\sum_{f=0}^{\infty} \frac{(\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} \eta_i s_i+\eta\xi_{h,r})!}{f!\,(\lambda-f+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} \eta_i s_i+\eta\xi_{h,r})!}$$

$$\left(\frac{a}{b}\right)^{f} x^{\mu+\nu f+u'\{qn'+\ell(m+n')+sp\}+\sum_{i=1}^{g} u_i s_i+u\,\xi_{h,r}+\eta k} \left\{ \frac{1}{2\pi\omega} \int_{-\omega\infty}^{\omega\infty} \overline{\theta}(\xi)(x^{\lambda}z)^{\xi} \right.$$

$$\left. \prod_{t=0}^{n-1} \frac{\Gamma[1+tk+\mu+\nu f+u'\{qn'+\ell(m+n')+sp\}+\sum_{i=1}^{g} u_i s_i+u\xi_{h,r}+\lambda\xi]}{\Gamma(1+\mu+\nu f+tk-\alpha+u'\{qn'+\ell(m+n')+sp\}+\sum_{i=1}^{g} u_i s_i+u\xi_{h,r}+\lambda\xi]} \right] d\xi',$$

further, interpreting the above result with the help of (1.9), we get required result (2.1) after a little simplification. Thus completes the proof.

3. SPECIAL CASES

(i) Taking g = 1 in main result, the general multivariable polynomials reduce to the general class of polynomials given by [16] and consequently, we obtain the following result:

$$D^{n}_{k,\alpha,x}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda}\, S^{M_1}_{N_1}\left[y_1 x^{u_1}(ax^{\nu}+b)^{\eta_1}\right]\right.$$

$$\left. S^{\alpha',\beta,0}_{n'}[z_1 x^{u'}(ax^{\nu}+b)^{\eta'}; s,q,1,0,m,k',\ell]\,\overline{H}^{m_1,n_1}_{p_1,q_1}\left[yx^{u}(ax^{\nu}+b)^{\eta}\,\middle|\,\begin{matrix}(c_j,\gamma_j;C_j)_{1,n_1},(c_j,\gamma_j)_{n_1+1,p_1}\\(d_j,\delta_j)_{1,m_1},(d_j,\delta_j;D_j)_{m_1+1,q_1}\end{matrix}\right]\right.$$

$$\left.\overline{H}^{M,N}_{P,Q}\left[zx^{\lambda}\,\middle|\,\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right\}$$

$$= G_2 G_3 \sum_{r=0}^{\infty}\sum_{h=1}^{m_1}\sum_{f=0}^{\infty}\overline{\theta}(\xi_{h,r})\, y^{\xi_{h,r}}\, b^{\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\eta_1 s_1+\eta\xi_{h,r}}$$

$$x^{\mu+\nu f+u'\{qn'+sp+\ell(m+n')\}+u_1 s_1+u\xi_{h,r}+nk}$$

$$\frac{(\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\eta_1 s_1+\eta\xi_{h,r})!}{f!(\lambda-f+\eta'\{qn'+sp+\ell(m+n')\}+\eta_1 s_1+\eta\xi_{h,r})!}$$

$$\left(\frac{a}{b}\right)^{f}\overline{H}^{M,\,N+n}_{P+n,Q+n}\left[zx^{\lambda}\,\middle|\,\begin{matrix}(-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-u_1 s_1-u\,\xi_{h,r},\lambda:1)_{t=0,n-1}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(\alpha-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-u_1 s_1-u\xi_{h,r},\lambda:1)_{t=0,n-1}\end{matrix}\right] \qquad (3.1)$$

where

$$G_3 = \sum_{s_1=0}^{[N_1/M_1]}\frac{(-N_1)_{M_1 s_1}}{s_1!}B[N_1,s_1]\,y_1^{s_1},$$

the condition of validity of the above result easily follow from (2.1).

(ii) If we take $C_j = 1 (j = 1,..., n_1)$, $D_j = 1 (j = m_1 + 1,..., q_1)$ and $A_j = 1 (j = 1,..., N)$, $B_j = 1$ $(j = M+1,\ldots,\theta)$ in main result, $\bar{H}$-function reduces to the well known Fox H – function given by [4]. Then, main result takes the following form after a little simplification:

$$D^{n}_{k,\alpha,x}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda} S^{M_1,\ldots,M_g}_{N_1,\ldots,N_g}\left[y_1 x^{u_1}(ax^{\nu}+b)^{\eta_1},\ldots, y_g\, x^{u_g}(ax^{\nu}+b)^{\eta_g}\right]\right.$$

$$\left. S^{\alpha',\beta,0}_{n'}[z_1 x^{u'}(ax^{\nu}+b)^{\eta'}; s,q,1,0,m,k',\ell] H^{m_1,n_1}_{p_1,q_1}\left[yx^{u}(ax^{\nu}+b)^{\eta}\middle| \begin{matrix}(c_j,\gamma_j)_{1,n_1},(c_j,\gamma_j)_{n_1+1,p_1}\\ (d_j,\delta_j)_{1,m_1},(d_j,\delta_j)_{m_1+1,q_1}\end{matrix}\right]\right.$$

$$H^{M,N}_{P,Q}\left[zx^{\lambda}\middle| \begin{matrix}(a_j,\alpha_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q}\end{matrix}\right]$$

$$= G_1 G_2 \sum_{r=0}^{\infty}\sum_{h=1}^{m_1}\sum_{f=0}^{\infty} \phi(\eta_r) y^{\eta_r} b^{\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g}\eta_i s_i+\eta\eta_r}$$

$$x^{\mu+\nu f+u'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g}u_i s_i+u\eta_r+nk}$$

$$\frac{(\lambda+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g}\eta_i s_i+\eta\eta_r)!}{f!(\lambda-f+\eta'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g}\eta_i s_i+\eta\eta_r)!}$$

$$\left(\frac{a}{b}\right)^{f} H^{M,N+n}_{P+n,Q+n}\left[zx^{\lambda}\middle| \begin{matrix}(-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-\sum_{i=1}^{g}u_i s_i-u\eta_r,\lambda:1)_{t=0,n-1}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q}\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (\alpha-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-\sum_{i=1}^{g}u_i s_i-u\eta_r,\lambda;1)_{t=0,n-1}\end{matrix}\right], \tag{3.2}$$

where

$$\phi(\eta_r)=\frac{\prod_{\substack{j=1\\ j\neq h}}^{m_1}\Gamma(d_j-\delta_j\eta_r)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\eta_r)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-d_j+\delta_j\eta_r)\prod_{j=n_1+1}^{p_1}\Gamma(c_j-\gamma_j\eta_r)}$$

and

$$\eta_r=\frac{d_h+r}{\delta_h}$$

the condition of validity of above result can easily obtain from the result (2.1).

(iii) Letting $u \to 0$, $\eta \to 0$, M = 1, N = P and Q = Q+1 in (2.1), the $\overline{H}$-function occurring therein breaks up into the generalized Wright hypergeometric function ${}_P\overline{\psi}_Q(.)$ given by [1]. Then, (2.1) takes the following form after a little simplification which is also believed to be new :

$$D^n_{k,\alpha,x}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda}S^{M_1,\ldots,M_g}_{N_1,\ldots,N_g}\left[y_1x^{u_1}(ax^{\nu}+b)^{\eta_1},\ldots,y_g\,x^{u_g}(ax^{\nu}+b)^{\eta_g}\right]\right.$$

$$\left.S^{\alpha',\beta,0}_{n'}[z_1x^{u'}(ax^{\nu}+b)^{\eta'};s,q,1,0,m,k',\ell]\;{}_P\overline{\psi}_Q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,P};\\ (b_j,\beta_j;B_j)_{1,Q};\end{matrix}zx^{\lambda}\right]\right\}$$

$$=G_1G_2G_4\,\overline{H}^{1,P+n}_{P+n,Q+n+1}\left[-z\,x^{\lambda}\left|\begin{matrix}(-\mu-tk-\nu f-u'\{qn'+sp+\ell(m+n)'\}-\sum_{i=1}^{g}u_i s_i,\lambda;1)_{t=0,n-1},\\ (0,1),(1-b_j,\beta_j;B_j)_{1,Q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-a_j,\alpha_j;A_j)_{1,P} \\ (\alpha-\mu-tk-\nu f-u'\{qn'+sp+\ell(m+n')\}-\sum_{i=1}^{g} u_i s_i,\lambda;1)_{t=0,n-1}\end{matrix}\right], \tag{3.3}$$

where

$$G_4=\sum_{f=0}^{\infty} b^{\lambda+\eta'\{qn'+\ell(m+n')+sp\}+\sum_{i=1}^{g}\eta_i s_i} x^{\mu+\nu f+u'\{qn'+sp+\ell(m+n')\}+\sum_{i=1}^{g} u_i s_i+nk}$$

$$\cdot \frac{(\lambda+\eta'\{qn'+sp+\ell(m+n)'\}+\sum_{i=1}^{g}\eta_i s_i)!}{f!(\lambda-f+\eta'\{qn'+\ell(m+n')+sp\}+\sum_{i=1}^{s}\eta_i s_i)!}\left(\frac{a}{b}\right)^f,$$

and the conditions of validity of (3.3) easily follows from those given in (2.1).

(iv) Reducing the first $\bar{H}$-function of unity in (2.1) and on giving suitable values to the parameters involved therein second $\bar{H}$-function, then The $\bar{H}$-function reduces to generalized Riemann zeta function $\phi(z,p,\delta)$ [3,p.27, eq.(11)] and [7, p.4127, eq. (27)], the (2.1) yields the following interesting result:

$$D^n_{k,\alpha,x}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda} S^{M_1,\ldots,M_g}_{N_1,\ldots,N_g}\left[y_1 x^{u_1}(ax^{\nu}+b)^{\eta_1},\ldots,y_g x^{u_g}(ax^{\nu}+b)^{\eta_g}\right]\right.$$

$$\left. S^{\alpha',\beta,0}_{n'}[z_1 x^{u'}(ax^{\nu}+b)^{\eta'};s,q,1,0,m,k',\ell]\phi(zx^{\lambda},p,\delta)\right\}=G_1 G_2 G_4$$

$$\cdot \bar{H}^{1,2+n}_{2+n,2+n}\left[-zx^{\lambda}\left|\begin{matrix}(-\mu-tk-\nu f-u'\{qn'+\ell(m+n')+sp\}-\sum_{i=1}^{g} u_i s_i,\lambda;1)_{t=0,n-1}, \\ (0,1),(-\delta,1;f),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(0,1;1),(1-\delta,1;\rho)\\ (\alpha-\mu-tk-\nu f-u'\{qn'+sp+\ell(m+n')\}-\sum_{i=1}^{g} u_i s_i,\lambda:1)_{t=0,n-1}\end{matrix}\right]. \tag{3.4}$$

The conditions of validity of the above result can be easily followed directly from those given with main result.

(v) If we take u = 0, η = 0, M = 1, N = P = 0 and Q = 2 in (2.1), the $\bar{H}$-function occurring therein breaks up into the generalized Wright-Bessel function $\bar{J}_{\xi}^{a',b'}(z)$ given by [6]. Then, (2.1) takes the following form after a little simplification:

$$D_{k,\alpha,x}^{n}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda} S_{N_1,\ldots,N_g}^{M_1,\ldots,M_g}\left[y_1 x^{u_1}(ax^{\nu}+b)^{\eta_1},\ldots,y_g x^{u_g}(ax^{\nu}+b)^{\eta_g}\right]\right.$$

$$\left. S_{n'}^{\alpha',\beta,0}[z_1 x^{u'}(ax^{\nu}+b)^{\eta'};s,q,1,0,m,k',\ell]\bar{J}_{\xi}^{a',b'}[-z x^{\lambda}]\right\}=G_1 G_2 G_4$$

$$.\bar{H}_{n,2+n}^{1,n}\left[zx^{\lambda}\left|\begin{matrix}(-\mu-tk-\nu f-u'\{qn'+sp+\ell(m+n')\}-\sum_{i=1}^{g} u_i s_i,\lambda:1)_{t=0,n-1}\\ (0,1),(-\xi,a';b'),(\alpha-\mu-tk-\nu f-u'\{qn'+sp+\ell(m+n')\}-\sum_{i=1}^{g} u_i s_i,\lambda:1)_{t=0,n-1}\end{matrix}\right.\right] \tag{3.5}$$

The conditions of validity of the above result easily follow from (2.1).

(vi) Reducing the first $\bar{H}$-function to unity in (2.1) and giving suitable values to the parameters involved therein the second $\bar{H}$-function, then $\bar{H}$-function reduces to the Mittag-Leffler function [8, p.65, eq.(2.9.98)] and we easily get after little simplification the following new and interesting result:

$$D_{k,\alpha,x}^{n}\left\{x^{\mu}(ax^{\nu}+b)^{\lambda} S_{N_1,\ldots,N_g}^{M_1,\ldots,M_g}\left[y_1 x^{u_1}(ax^{\nu}+b)^{\eta_1},\ldots,y_g x^{u_g}(ax^{\nu}+b)^{\eta_g}\right]\right.$$

$$\left. S_{n'}^{\alpha',\beta,0}[z_1 x^{u'}(ax^{\nu}+b)^{\eta'};s,q,1,0,m,k',\ell]E_{M,N}[zx^{\lambda}]\right\}=G_1 G_2 G_4$$

$$\cdot\bar{H}_{1+n,n+2}^{1,1+n}\left[-zx^{\lambda}\left|\begin{matrix}(-\mu-tk-vf-u'\{qn'+sp+\ell(m+n')\}-\sum_{i=1}^{g}u_i s_i,\lambda:1)_{t=0,n-1},(0,1;1)\\(0,1),(1-N,M;1),(\alpha-\mu-tk-vf-u'\{qn'+sp+\ell(m+n')\}-\sum_{i=1}^{g}u_i s_i,\lambda:1)_{t=0,n-1}\end{matrix}\right.\right] \tag{3.6}$$

valid under the same conditions as given with (1.2).

The importance of our result lies in their manifold generality. In view of the generality of $\bar{H}$-function and multivariable general class of polynomials on specializing the various parameters, we can obtained from our main result, several fractional calculus result containing remarkably wide variety of useful functions, which expressible in terms of H-functions, G-functions, Mittag-Leffler functions and their various special cases. Thus the main result presented in this article would at once yield a very large number of results containing a large variety of simpler special functions occurring scientific and technological fields.

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V. B. L. Chaurasia
Department of Mathematics, University of Rajasthan,
Jaipur-302004, Rajasthan, India.
E-mail: drvblc@yahoo.com

Vinod Gill
Department of Mathematics, Arya Institute of Engineering and Technology,
Kukas, Jaipur-302028, Rajasthan, India.
E-mail: vinod.gill08@gmail.com