DOI: 10.1515/awutm -2015-0015
DE GRUYTER OPEN

Analele Universităţii de Vest,
Timişoara
Seria Matematică – Informatică
LIII, 2, (2015), 73– 79

The fundamental group of the orbit space

Hawete Hattab

Abstract. Let G be a subgroup of the group Homeo(X) of homeomorphisms of a topological space X. Let $\overline{G}$ be the closure of G in Homeo(X). The class of an orbit O of G is the union of all orbits having the same closure as O. We denote by $X/\widetilde{G}$ the space of classes of orbits called the orbit class space. In this paper, we study the fundamental group of the spaces X/G, $X/\overline{G}$ and $X/\widetilde{G}$.

AMS Subject Classification (2000). 54F65 ; 54H20 ; 55P10.
Keywords. Homotopy, Fundamental group, orbit space, orbit class space.

1 Introduction

In algebraic topology one of the basic ways of classifying spaces is by using the notion of homotopy groups $\pi_n(X, x)$ of a space X at $x \in X$. When $n = 1$ it is the fundamental group of X at the point x.

The orbit space is an important tool in the qualitative study of dynamical systems and is a concept of interest in itself. Indeed, this space is an extremely useful way to construct interesting topological spaces. Therefore, an important problem is to compute the fundamental group of the orbit space.

Let X be a topological space and Homeo(X) its group of homeomorphisms. Let G be a subgroup of Homeo(X). The family of orbits $G(x) =$

$\{g(x) : g \in G\}$ by G determines an open equivalence relation on X. The class of an orbit O is the union of all orbits O' having the same closure as O. We define on X a new equivalence relation $\widetilde{G}$ by: $x\widetilde{G}y$ if $\overline{G(x)} = \overline{G(y)}$. This new equivalence relation is open, since for each open subset U of X, the saturated sets of U, by G and by $\widetilde{G}$, are equal.

We denote by X/G the orbit space, $X/\widetilde{G}$ the orbit class space, $q : X \to X/G$ and $p : X \to X/\widetilde{G}$ the canonical projections. The orbit class space is introduced and studied in [9, 14].

There is an extensive literature on the fundamental groups of quotient spaces (one can see for example [1–5, 7, 15–18]). The majority of these papers suppose that the fibers of the quotient map are connected [4, 5, 17] or the initial space is a simplicial complex [1, 7]. Note that, in the case of an orbit space X/G, the fibers of the quotient map (the orbits) are not in general connected. And so, it is interesting to extend these results for orbit space X/G where X is a general topological space. In [2], Armstrong showed the following theorem:

Theorem 1.1. *Let G be a discontinuous*[1] *group of homeomorphisms of a path connected, simply connected, locally compact metric space X, and let H be the normal subgroup of G generated by those elements which have fixed points. Then the fundamental group of the orbit space X/G is isomorphic to the factor group G/H.*

In [3], Armstrong computed the fundamental group of the orbit space of a subgroup G which satisfies three conditions. Let $\overline{G}$ be the closure of G in Homeo(X) and N the smallest normal subgroup of G which contains the path component of the identity of G and all those elements of G which have fixed points.

A. The projection $X \to X/\overline{G}$ has the path lifting property up to homotopy.

B. Given points x, $x' \in X$ plus a neighborhood V of x in X, if $x' \in \overline{G}x$ then $x' \in GV$.

C. The group $\overline{G}/N$ acts discontinuously on X/N.

[1] A group G of homeomorphisms of a topological space X will be called discontinuous if

1. the stabilizer of each point of X is finite, and
2. each point $x \in X$ has a neighborhood U such that any element of G not in the stabilizer of x maps U outside itself (i.e. if $gx \neq x$, then $U \cap gU$ is empty)

.

Armstrong showed that if the condition B is satisfied then X/G and $X/\overline{G}$ have the same homotopy type. In particular if G is a group of isometries of X.

In this paper first, we show that an equicontinuous group of homeomorphisms satisfies also the condition B. Second, we prove that X/G and $X/\widetilde{G}$ have always the same homotopy type. Finally, we show that if X is connected and locally compact and G is an equicontinuous group with a relatively compact orbit then $X/\overline{G} = X/\widetilde{G}$.

The main result of [3] is the following theorem:

Theorem 1.2. *If X is simply connected and if conditions A, B and C are satisfied, then $\pi_1(X/G)$ is isomorphic to $\overline{G}/N$.*

2 Equicontinuous group action

In [10], El Kacimi et al. studied some qualitative properties of an equicontinuous group of homeomorphisms of a metric space. They showed the following theorem:

Theorem 2.1. [10] *Let G be an equicontinuous group of homeomorphisms of a metric space X. We have the following properties:*

1. *Every orbit O is minimal ($\overline{O}$ is a minimal set).*
2. *If every orbit is closed, then the orbit space X/G is Hausdorff.*
3. *The orbit class space $X/\widetilde{G}$ is always Hausdorff.*
4. *If X is compact, the closure $\overline{G}$ of G in Homeo(X) is a compact subgroup of homeomorphisms of X, and for all $x \in X$, we have $\overline{G}(x) = \overline{G(x)}$.*

We start by showing the following theorem:

Theorem 2.2. *Let G be an equicontinuous group of homeomorphisms of a connected locally compact metric space X. If G has a relatively compact orbit, then the closure $\overline{G}$ of G in Homeo(X) is a compact subgroup of homeomorphisms of X, and for all $x \in X$, we have $\overline{G}(x) = \overline{G(x)}$.*

To prove Theorem 2.2 we need the following lemma:

Lemma 2.3. [12] *Let G be an equicontinuous group of homeomorphisms of a connected locally compact metric space X. If G has a relatively compact orbit, then every orbit is relatively compact.*

Proof. of Theorem 2.2. Since G is equicontinuous and, according to Lemma 2.3, the closure of each orbit is compact, by applying Ascoli Theorem [8] the closure $\overline{G}$ of G is a compact subgroup of homeomorphisms de X.

We show now that, for all $x \in X$, we have $\overline{G}(x) = \overline{G(x)}$.

Let y be in $\overline{G}(x)$, then there exists $g \in \overline{G}$ such that $g(x) = y$. Since there exists a sequence (g_n) of element of G which converges to g, $(g_n(x))$ converges to $g(x) = y$ and so $y \in \overline{G(x)}$.

Conversely, if $y \in \overline{G(x)}$, then there exists a sequence (g_n) of G such that $(g_n(x))$ converges to y. Since $\overline{G}$ is compact, there exists a subsequence (g_{n_p}) of $\overline{G}$ which converges to $h \in \overline{G}$ thus $(g_{n_p}(x))$ converges to $h(x)$ which implies that $y = h(x) \in \overline{G}(x)$. □

By applying Theorem 2.2, we obtain the following corollary:

Corollary 2.4. *Let G be an equicontinuous group of homeomorphisms of a connected locally compact metric space X. If G has a relatively compact orbit, then the orbit class space $X/\widetilde{G}$ of G coincides with the orbit space $X/\overline{G}$ of $\overline{G}$.*

3 Fundamental group of orbit spaces

Recall that a continuous map $f : X \to Y$ between two topological spaces is called *a quasi-homeomorphism* if the map which assigns to each open set $V \subset Y$ the open set $f^{-1}(V)$ is a bijective map [6].

According to [14], we have the following lemma:

Lemma 3.1. *The map $\varphi : X/G \to X/\widetilde{G}$ which associates to each orbit its class is a surjective quasi-homeomorphism.*

Lemma 3.2. *Let $f : X \to Y$ be a quasi-homeomorphism. For every $x, y \in X$, we have the following implication: $f(x) = f(y) \Rightarrow \overline{\{x\}} = \overline{\{y\}}$. If moreover X is a T_0-space, then f is an embedding ($f : X \to f(X)$ is a homeomorphism).*

Proof. If $f(x) = f(y)$, then $\overline{\{f(x)\}} = \overline{\{f(y)\}}$. Since f is a closed map, then $f(\overline{\{x\}}) = f(\overline{\{y\}})$. Using the fact that every locally closed subset of X is f-saturated, we obtain $\overline{\{x\}} = f^{-1}(f(\overline{\{x\}})) = f^{-1}(f(\overline{\{y\}})) = \overline{\{y\}}$ and so $\overline{\{x\}} = \overline{\{y\}}$. If moreover X is a T_0-space, then $x = y$ and so f is an injective map. The fact that f is continuous and open, implies that f is an embedding. □

Recall that, two spaces X and Y are *homotopy equivalent*, or of the *same homotopy type*, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

Lemma 3.3. [13] *If $f : X \to Y$ is a surjective quasi-homeomorphism, then f is a homotopy equivalence.*

We can now prove the following theorem:

Theorem 3.4. *The orbit space and the orbit class space are homotopy equivalent.*

Proof. By applying Lemmas 3.3 and 3.1, we obtain Theorem 3.4. □

Recall that a continuous map $f : X \to Y$ between topological spaces is said to be a *weak homotopy equivalence* if it induces isomorphisms in all homotopy groups. Homotopy equivalences are weak homotopy equivalences, and the Whitehead Theorem [11, Theorem 4.5] claims that any weak homotopy equivalence between CW-complexes is a homotopy equivalence. However, in general, these two concepts differ. And so all homotopy groups the orbit space and the orbit class space are isomorphic.

Proposition 3.5. *If the group G satisfies the condition B, then the spaces X/G, $X/\overline{G}$ and $X/\widetilde{G}$ are homotopy equivalent.*

Proof. By applying [3, Proposition 1] and Theorem 3.4, we obtain Proposition 3.5. □

Proposition 3.6. *If G is an equicontinuous group, then X/G and $X/\overline{G}$ have the same homotopy type.*

Lemma 3.7. *Let x and y be two elements of X, $Gx \subset \overline{Gy}$ if and only if every open invariant set containing x contains y.*

Proof. Let U be an open invariant set containing x. If $Gx \subset \overline{Gy}$ then $\overline{Gy} \cap U \neq \emptyset$; and so $Gy \cap U \neq \emptyset$ which implies that $y \in U$.

Conversely, if x is not in $\overline{Gy}$ then x is in $X - \overline{Gy}$, which is an open invariant subset and so $y \in X - \overline{Gy}$ which is impossible. □

Proof. Proposition 3.6. By applying [3, Proposition 1], it suffices to check condition B for an equicontinuous group. Let V be a neighborhood of x in X. If $x' \in \overline{G}x$, then $x' \in \overline{Gx}$. By the minimality of each orbit we obtain $\overline{Gx} = \overline{Gx'}$ and so $x \in \overline{Gx'}$. Since $x \in GV$,by applying Lemma 3.7, we have $x' \in GV$. □

Theorem 3.8. *If X is simply connected and if conditions A, B and C are satisfied, then $\pi_1(X/G)$, $\pi_1(X/\overline{G})$ and $\pi_1(X/\widetilde{G})$ are isomorphic to $\overline{G}/N$.*

Proof. By applying Proposition 3.5 and Theorem 1.2 we obtain Theorem 3.8. □

References

[1] **M. A. Armstrong**, On the fundamental group of an orbit space, *Mathematical Proceedings of the Cambridge Philosophical Society*, **61**, (1965), 639-646

[2] **M. A. Armstrong**, The fundamental group of the orbit space of a discontinuous group, *Mathematical Proceedings of the Cambridge Philosophical Society*, **64**, (1968), 299-301

[3] **M. A. Armstrong**, Calculating the fundamental group of an orbit space, *Proc. Amer. Math. Soc.*, **84**, (1982), 267-271

[4] **J. S. Calcut, R. E. Gompf, and J. D. McCarthyc**, On fundamental groups of quotient spaces, *Topology and its Applications*, **159**, (2012), 322330

[5] **J.S. Calcut, R.E. Gompf, and J.D. McCarthy**, Quotient maps with connected fibers and the fundamental group, (2009)

[6] **A. Grothendieck and J. Dieudonné**, Éléments de Géométrie Algébrique, *Die Grundlehren der mathematischen Wissenschaften*, **166**, (1971)

[7] **E. D. Farjoun**, Fundamental group of homotopy colimits, *Advances in Mathematics*, **182**, (2004), 127

[8] **M. Krzysztof**, *Analysis Part II Integration, Distributions, Holomorphic Functions, Tensor and Harmonic Analysis*, PWNReidel, WarszawaDordrecht, 1980

[9] **C. Bonatti, H. Hattab, and E. Salhi**, Quasi-orbits spaces associated to T_0-spaces, *Fund. Math.*, **211**, (2011), 267-291

[10] **A. El Kacimi, H. Hattab, and E. Salhi**, Remarque sur certains groupes d'homéomorphismes d'espaces métriques, *JP Jour. Geometry and Topology*, **4**, (2004), 225-242

[11] **A. Hatcher**, *Algebraic Topology*, Cambridge University Press, 2002

[12] **H. Hattab**, Flows of locally finite graphs

[13] **H. Hattab**, A model of quotient spaces

[14] **H. Hattab and E. Salhi**, Groups of homeomorphisms and spectral topology, *Topology Proceedings*, **28**, (2004), 503-526

[15] **C. Kosniowski**, *A First Course in Algebraic Topology*, 1980

[16] **F. Rhodes**, On the Fundamental Group of a Transformation Group, *Proc. London Math. Soc.*, **3-16**, (1966), 635-650

[17] **S. Smale**, A Vietoris mapping theorem for homotopy, *Proc. Amer. Math. Soc.*, **8**, (1957), 604610

[18] **B. de Smit**, The fundamental group of the Hawaiian earring is not free, *Internat. J. Algebra Comput.*, **2**, (1992), 3337

Hawete Hattab

Laboratory of Dynamical Systems and Combinatorics,
Sfax University Tunisia
Umm Alqura University,
Department of Mathematics. KSA

E-mail: hattab.hawete@yahoo.fr

Received: 4.11.2015

Accepted: 22.01.2016