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THE SPACE OF REAL PLACES ON $\mathbb{R}(x, y)$

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Abstract. The space $\mathcal{M}(\mathbb{R}(x, y))$ of real places on $\mathbb{R}(x, y)$ is shown to be path-connected. The possible value groups of these real places are determined and for each one it is shown that the set of real places with that value group is dense in the space. Large collections of subspaces of the space $\mathcal{M}(\mathbb{R}(x, y))$ are constructed such that any two members of such a collection are homeomorphic. A key tool is a homeomorphism between the space of real places on $\mathbb{R}((x))(y)$ and a certain space of sequences related to the "signatures" of [2], which themselves are shown here to be related to the "strict systems of polynomial extensions" of [3].

1. The Main Theorem

For any field F the space of real places on F, i.e., places from F to the field $\mathbb{R}$ of real numbers, will be denoted by $\mathcal{M}(F)$. This space is an important invariant of F for understanding the structure of the reduced Witt ring of quadratic forms over F [4]. It also plays a natural role in real algebraic geometry; given a formally real field F the points of $\mathcal{M}(F)$ correspond to the closed points of the real spectrum of the real holomorphy ring of F [17]. There are many ways of defining the topology on $\mathcal{M}(F)$ (e.g., see [4, 6, 14]); we emphasize that the *Harrison sets* $H(a) = \{\tau \in \mathcal{M}(F) : 0 < \tau(a) < \infty\}$ (for $a \in F$) form a subbasis for the topology [13, Section 2]. It is easy to show that the space of real places on the rational function field $\mathbb{R}(x)$ is a simple closed curve, and the space of real places on an algebraic function field in one

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variable over $\mathbb{R}$ is known to be a (possibly empty) disjoint union of a finite number of simple closed curves [11, page 50]. No analogous result is known for algebraic function fields in two variables over $\mathbb{R}$, or even for the rational function field $\mathbb{R}(x, y)$. It is known that $\mathcal{M}(\mathbb{R}(x, y))$ is compact, Hausdorff and connected [6, page 5], [8, Theorem 2.12], but not whether it contains a disk (and much less whether it contains a torus, a topic of the paper [13]). The main result of this paper is the

THEOREM 1.1. *$\mathcal{M}(\mathbb{R}(x, y))$ is path-connected.*

In Section 2 we show that $\mathcal{M}(\mathbb{R}(x, y))$ has a retraction onto a simple closed curve and that each fiber of this retraction is homeomorphic to $\mathcal{M} := \mathcal{M}(\mathbb{R}((x))(y))$ (the field here is the rational function field over the Laurent series field). Hence in order to prove the theorem above it suffices to show that $\mathcal{M}$ is path-connected. In the next two sections we use ideas going back to MacLane [15] and Brown [2] to show that $\mathcal{M}$ is homeomorphic to a space of sequences (not necessarily of infinite length) $\mathcal{S}$ having a fairly natural topology (e.g., two sequences of infinite length are close together when they have a long finite initial segment in common). That $\mathcal{S}$ (and hence $\mathcal{M}(\mathbb{R}(x, y))$) is path-connected is proven in Section 5. In Section 6 we exploit the homeomorphism of $\mathcal{M}$ and $\mathcal{S}$ to list the possible value groups of elements of $\mathcal{M}(\mathbb{R}(x, y))$ and show that for each of these the set of places with that value group (up to order isomorphism) is dense in $\mathcal{M}(\mathbb{R}(x, y))$. In Section 7 we apply the homeomorphism to study some examples of self-similarity (i.e., of homeomorphic subspaces) in $\mathcal{M}$. For a discussion of self-similarities in a related context (where the maximal but not real closed field $\mathbb{R}((x))$ is replaced by a real closed but not necessarily maximal field), see [12]. Finally the last section is essentially an appendix giving an alternative approach to the bijection $\mathcal{M} \to \mathcal{S}$ of Section 3. The key result of this section describes how the sequence of polynomials associated with an element of $\mathcal{S}$ (see Section 3) gives rise to a sequence which is a strict system of polynomial extensions in the sense of [3]. (Section 3 requires knowledge of concepts that apply to a much more general situation than that in this paper. We hope the last section will be useful for those already familiar with these "strict systems" or with the equivalent "saturated distinguished chains" developed and studied by S.K. Khanduja and N. Popescu and their collaborators [10], [5].)

2. The reduction to $\mathcal{M}$

In this section we prove the

PROPOSITION 2.1. *$\mathcal{M}(\mathbb{R}(x, y))$ is path-connected if $\mathcal{M}$ is path-connected.*

We will prove the proposition by showing that $\mathcal{M}(\mathbb{R}(x,y))$ has a retraction onto a simple closed curve, each fiber of which is assumed to be path-connected since it is a continuous image of $\mathcal{M}$.

Let $\mathrm{res}\colon \mathcal{M}(\mathbb{R}(x,y)) \to \mathcal{M}(\mathbb{R}(x))$ be induced by the restriction map. Next, composition of any place in $\mathcal{M}(\mathbb{R}(x))$ with the y-adic place $\pi_y\colon \mathbb{R}(x,y) \to \mathbb{R}(x) \cup \{\infty\}$ yields a place in $\mathcal{M}(\mathbb{R}(x,y))$. Thus we have a natural injective map "comp" from $\mathcal{M}(\mathbb{R}(x))$ onto a subset C of $\mathcal{M}(\mathbb{R}(x,y))$. Both res and comp are continuous (for any $f \in \mathbb{R}(x,y)$, $\mathrm{comp}^{-1}(H(f) \cap C)$ is empty if $\pi_y(f) = \infty$ and is $H(\pi_y(f))$ otherwise). The composition of these two maps

$$\mathrm{comp} \circ \mathrm{res}\colon \mathcal{M}(\mathbb{R}(x,y)) \to C$$

is a retraction and for each $\tau \in C$, the fiber $(\mathrm{comp} \circ \mathrm{res})^{-1}(\tau)$ is the set of all $\delta \in \mathcal{M}(\mathbb{R}(x,y))$ with $\mathrm{res}(\delta) = \mathrm{res}(\tau)$. Since comp is a homeomorphism, the subspace C is a simple closed curve, and hence is path-connected. Now consider any $\tau \in C$. There exists $t \in \mathbb{R}(x)$ with $\tau(t) = 0$ and $\mathbb{R}(x) = \mathbb{R}(t)$. Restriction of mappings gives a continuous map from $\mathcal{M}(\mathbb{R}((t))(y))$ into the fiber $(\mathrm{comp} \circ \mathrm{res})^{-1}(\tau)$. Hence the Proposition above follows from the

LEMMA 2.2. *Every real place τ on $\mathbb{R}(x,y)$ with $\tau(x) = 0$ extends to a real place of $\mathbb{R}((x))(y)$.*

We will show in Remark 3.3 below that each real place τ as in the previous lemma has a unique extension to a real place of $\mathbb{R}((x))(y)$.

We now prove the lemma.

PROOF. The valued field $(\mathbb{R}(x,y),\tau)$ has a maximal immediate extension (F,τ_F). The valued field $\mathbb{R}((x))$ (which induces the same place on $\mathbb{R}(x)$ as τ) has a maximal extension (E,τ_E) to a valued field with the same residue class field (namely, $\mathbb{R}$) and the same value group as τ. Then by the proof of [9, Theorem 7] we have an analytic (i.e., place-preserving) isomorphism $\Upsilon\colon F \to E$ fixing $\mathbb{R}(x)$. (The statement of Theorem 7 of [9] assumes that the residue class fields of E and F are closed under the taking of roots. However, the proof only needs that for each element b of the residue class field, either b or $-b$ has roots of all orders, and this is true for our residue class fields $\mathbb{R}$.) There is a canonical place $\delta\colon \mathbb{R}((x))(z) \to E \cup \{\infty\}$ from the rational function field $\mathbb{R}((x))(z)$ to E taking z to $\Upsilon(y)$. (If $\Upsilon(y)$ is transcendental over $\mathbb{R}((x))$, then δ is a homomorphism of course.) The composition of δ with the canonical homomorphism $\psi\colon \mathbb{R}(x,y) \to \mathbb{R}((x))(z)$ taking x to x and y to z is a homomorphism and hence is just the restriction of Υ to $\mathbb{R}(x,y)$. Since $\tau_E\Upsilon = \tau_F$, we have $\tau_E\delta\psi = \tau$. Identifying y with z (so ψ becomes an inclusion), we have that $\tau_E\delta$ is an extension of τ to a real place on $\mathbb{R}((x))(y)$. □

3. The bijection $\mathcal{M} \to \mathcal{S}$

Let

$$\Gamma = (\{\pm 1\} \times 0 \times 0) \cup (0 \times \mathbb{R} \times 0) \cup (0 \times \mathbb{Q} \times \{\pm 1\}),$$

which is a subset of the lexicographically ordered group $\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$. (Here, $\mathbb{R}$ will denote a fixed copy of the field of real numbers.) We will often identify $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ with $0 \times \mathbb{Z} \times 0$, $0 \times \mathbb{Q} \times 0$, and $0 \times \mathbb{R} \times 0$, respectively, when discussing elements and subsets of $\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$. For $q \in \mathbb{Q}$, it is also convenient to write $q^- = (0, q, -1)$ and $q^+ = (0, q, 1)$, so that in the ordered group $\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$ we have $q^- < q < q^+$. Finally we will write $-\infty < \mathbb{Z} \times \mathbb{R} \times \mathbb{Z} < \infty$.

We let ω denote the first infinite ordinal, and let $\omega + 1$ denote its successor. A *presignature* is a sequence $S = \langle (q_i, \theta_i) \rangle_{0 \le i < n}$ (usually abbreviated $\langle q_i, \theta_i \rangle_{i<n}$) of elements of $\Gamma \times \mathbb{R}$ where $0 \le n \le \omega + 1$ such that for all $i < n$ we have

$$\theta_i = 0 \Leftrightarrow q_i \notin \mathbb{Q} \qquad \text{and} \qquad \theta_i = 0 \Rightarrow i + 1 = n.$$

The *degree* of S is $(\Gamma_S : \mathbb{Z})$ where for $m \le n$ we set $\Gamma_m = \mathbb{Z} + \sum_{i<m} \mathbb{Z} q_i$ and set $\Gamma_S = \Gamma_n$; the *length* of S is n. We call S a *signature* if $(\Gamma_\omega : \mathbb{Z}) < \infty$ when $n = \omega + 1$ and if the sequence $(q_i/(\Gamma_i : \mathbb{Z}))_{i<n}$ is strictly increasing. It should be noted that these definitions are different from those of [2]. We also write $S \rhd T$ if T is a presignature which is an initial segment (not necessarily proper and possibly empty) of S. Finally, we let $\mathcal{S}$ denote the set of signatures of infinite degree.

We now describe a bijection from $\mathcal{M}$ to $\mathcal{S}$. We first indicate how, given a signature $S = \langle q_i, \theta_i \rangle_{i<n}$, to construct *its associated sequence* $g = \langle g_i \rangle_{i<n'}$ of polynomials in $\mathbb{R}((x))[y]$ where $n' = n$ if S has infinite degree and $n' = n + 1$ otherwise. The case when S has infinite degree (so $n' = n$) applies directly to the bijection Φ of Theorem 3.1(A) below, and the case when S has finite degree (where $n' = n + 1$) is critical for the application of part (C) of Theorem 3.1, which plays a fundamental role in many of our arguments. First set $e_i = e_i^S = (\Gamma_{i+1} : \Gamma_i)$ for all $i < n$, and for each $m \le n$,

$$J_m = J_m^S = \{\sigma \in \bigoplus_{0 \le i < m} \mathbb{Z} : 0 \le \sigma(i) < e_i \quad \forall i < m\}.$$

If $g_0, \cdots, g_{m-1}$ have been defined then for each $\sigma \in J_m$ we set

$$g^\sigma = g_0^{\sigma(0)} \cdots g_{m-1}^{\sigma(m-1)}.$$

If $k < m$ we will identify J_k with the subset of J_m consisting of elements σ with $\sigma(i) = 0$ for all $i \geq k$; this identification does not effect the meaning of g^σ.

Let us set $g_0 = y$; for $0 < m+1 < n'$ inductively define

$$g_{m+1} = g_m^{e_m} + \theta_m x^{s_m} g^{\sigma_m} \tag{3.1}$$

where the exponents $s_m \in \mathbb{Z}$ and $\sigma_m \in J_m$ are the (unique!) ones with

$$e_m q_m = s_m + \sum_{i<m} \sigma_m(i) q_i.$$

(Note that g_{m+1} is independent of the choice of (q_{m+1}, θ_{m+1}).) This defines the polynomials g_i except in the case that $n = \omega + 1$ and $i = \omega$. In this case $(\Gamma_\omega : \mathbb{Z}) < \infty$, so there exists a least m with $e_i = 1$ for all $i \geq m$. Then for all $k \geq m$, g_{k+1} has the form $g_{k+1} = g_k + \theta_k x^{s_k} g^{\sigma_k}$ where for all k we have $\sigma_k \in J_m$. We then let g_ω be the limit of the g_k as $k \to \infty$, i.e.,

$$g_\omega = g_m + \sum_{k \geq m} \theta_k x^{s_k} g^{\sigma_k} = g_m + \sum_{\sigma \in J_m} (\sum \theta_k x^{s_k}) g^\sigma$$

where for each $\sigma \in J_m$, the inside sum on the right above is over all k with $\sigma_k = \sigma$ and $m \leq k < \omega$. (Since S is a signature, the sequence $(q_i)_{i \geq m}$ is strictly increasing, so the sequence of integers s_k in each inside sum above is strictly increasing; thus the inside sum is in $\mathbb{R}((x))$.)

We claim that for each $\tau \in \mathcal{M}$ there is a unique valuation v_τ associated with τ with value group in the lexicographic product $\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$, with $v_\tau(x) = (0,1,0)$ and such that if the value group of v_τ is not contained in $0 \times \mathbb{Q} \times 0$ then $v_\tau(f) \in \Gamma$ for any monic polynomial f of minimal degree (in y) with $v_\tau(f) \notin 0 \times \mathbb{Q} \times 0$. (Note that for such a polynomial, the value $v_\tau(f)$ is independent of the choice of f and

$$v_\tau(\sum h_i f^i) = \min_i v_\tau(h_i f^i) \tag{3.2}$$

for all polynomials h_i with degree less than that of f.) The place τ has rational rank one or two (there is no homomorphism from $\mathbb{R}(x,y)$ into $\mathbb{R}$). If the rational rank is one, then there is a valuation v associated with τ mapping into $0 \times \mathbb{Q} \times 0$ with $v(x) = (0,1,0)$ (note that multiplication of $\mathbb{Q}$ by a positive element of $\mathbb{Q}$ is an order isomorphism). Then v is our unique required valuation v_τ (with the obvious notation, we have

$$v(h) = (0, a/b, 0) \Leftrightarrow v(h^b/x^a) = 0, \tag{3.3}$$

a condition independent of the choice of v). Next suppose that τ has rational rank two. If it has rank one, then there is a valuation v associated with τ mapping into $0 \times \mathbb{R} \times 0$ with $v(x) = (0, 1, 0)$. This is our required valuation v_τ. (If v' were another valuation associated with τ mapping into $0 \times \mathbb{R} \times 0$ with $v'(x) = (0, 1, 0)$ and $v(f) > v'(f)$, then for some integers s and t we would have $sv(f) > (0, t, 0) > sv'(f)$, so f^s/x^t would map to 0 and to ∞ under τ, a contradiction. The uniqueness of v now follows from the observations (3.2) and (3.3).) Finally suppose τ is not rank one. First suppose that there is a valuation v associated with τ mapping into $0 \times \mathbb{Q} \times \mathbb{Q}$ with $v(x)$ of the form $(0, a, b)$ where $a > 0$. Then we may assume that $v(x) = (0, 1, 0)$ since the vector space isomorphism taking $(0, a, b)$ to $(0, 1, 0)$ and fixing $(0, 0, 1)$ is an order isomorphism of $0 \times \mathbb{Q} \times \mathbb{Q}$. The element $v(f)$ has the form $(0, q, r)$ where $r \neq 0$ and (by composing v with the order isomorphism taking $(0, q, r)$ to $(0, q, r/|r|)$ and fixing $(0, 1, 0)$) we may assume that $v(f)$ has the form $(0, q, \pm 1) \in \Gamma$. In our last case there is a v associated with τ mapping into $\mathbb{Q} \times \mathbb{Q} \times 0$ with $v(x)$ of the form $(0, a, 0)$. Again without loss of generality $a = 1$ and $v(f)$ is $(\pm 1, 0, 0) \in \Gamma$, as required. The proofs of uniqueness in the rank two case are closely analogous to the proofs in the rank one case.

We now present the bijection $\Phi\colon \mathcal{M} \to \mathcal{S}$, together with some of its properties. Given $S \in \mathcal{S}$ we will use the notation introduced above when defining the associated sequence g of S.

THEOREM 3.1.

(A) *There is a unique bijection $\Phi\colon \mathcal{M} \to \mathcal{S}$ such that if $\tau \in \mathcal{M}$ and $S = \langle q_i, \theta_i \rangle_{i<n} \in \mathcal{S}$ then $\Phi(\tau) = S$ if and only if $v_\tau(g_i) = q_i$ for all $i < n$, where $(g_i)_{i<n}$ is the associated sequence of S.*

(B) *If $\Phi(\tau) = S$, then $\{g^\sigma : \sigma \in J_n\}$ is a valuation basis for the restriction of v_τ to $\mathbb{R}((x))[y]$ over the x-adic valuation on $\mathbb{R}((x))$.*

(C) *If $S \in \mathcal{S}$, $m \neq \omega$, $0 < m < n$, and if $\rho \in \mathcal{M}$ has $v_\rho(g_m) > e_{m-1}q_{m-1}$, then $\Phi(\rho) \rhd \langle q_i, \theta_i \rangle_{i<m}$, g_m is irreducible over $\mathbb{R}((x))$, and on all polynomials of degree less than that of g_m, v_ρ agrees with the composition of the canonical map $\mathbb{R}((x))[y] \to \mathbb{R}((x))[y]/(g_m)$ with the unique extension of the x-adic valuation on $\mathbb{R}((x))$ to a valuation on $\mathbb{R}((x))[y]/(g_m)$.*

Note that in (B) above we are asserting that for all $a_\sigma \in \mathbb{R}((x))$ the value of v_τ on $\sum a_\sigma g^\sigma$ is the minimum of the values $v_\tau(a_\sigma g^\sigma)$. We will prove in Lemma 8.1 below, without reference to [2], a slight generalization of 3.1(B) above. The uniqueness assertion in (C) follows from the fact that $\mathbb{R}((x))$ is Henselian with its canonical valuation and $\mathbb{R}((x))[y]/(g_m)$ is an algebraic extension of $\mathbb{R}((x))$. One can show that part (C) above is valid if $m = \omega$; in this case we replace the inequality $v_\rho(g_m) > e_{m-1}q_{m-1}$ by $v_\rho(g_m) > \mathbb{Q}$.

COROLLARY 3.2. *If $\rho \in \mathcal{M}$ and $S = \Phi(\rho)$, then the value group of v_ρ is Γ_S.*

We next show how to use [2] to prove Theorem 3.1. (In Section 8 we will give an alternative path to the proof of Theorem 3.1 using [3].) Briefly, the paper [2] supplies us with a bijection from the set of equivalence classes of extensions of the canonical place on $\mathbb{R}((x))$ to a place on $\mathbb{R}((x))(y)$ to a set of equivalence classes of certain sequences of pairs (θ, q) where θ lies in a field extension of $\mathbb{R}$ and q in an ordered abelian group containing $\mathbb{Z}$. Restricting this bijection to places mapping into $\mathbb{R}$ (i.e., real places) allows great simplifications (no need for equivalence classes!) and leads easily to the bijection Φ. Now for the details.

The bijection of Theorem 3.1(A) will be constructed here as a composition of three bijections, relating in turn the set $\mathcal{M}$; the set of equivalence classes of valuations on $\mathbb{R}((x))(y)$ with residue class field $\mathbb{R}$; a set of equivalence classes of sequences (called signatures in [2] and distinct from the signatures here); and the set $\mathcal{S}$ itself.

Let τ be in $\mathcal{M}$. As usual τ determines an equivalence class $\mathcal{C}(\tau)$ of valuations on $\mathbb{R}((x))(y)$; this class uniquely determines τ since the residue class field of τ is $\mathbb{R}$, which has no automorphisms. The valuation v_τ above is a canonical representative of $\mathcal{C}(\tau)$ because of the uniqueness noted shortly before the statement of Theorem 3.1. The bijective construction of [2, Section 3 and Section 4 through Corollary 4.3] assigns to $\mathcal{C}(\tau)$ a sequence of polynomials $g := \langle g_i \rangle_{i<n}$ and an equivalence class (called a "signature" in [2] but which we will call a "class signature" here to distinguish it from the signatures introduced earlier in this section) of certain sequences; this class signature has a unique representative $S^*(\tau) := \langle \theta_i^*, q_i \rangle_{i<n}$ where the θ_i^* are in $\mathbb{R}$ and $v_\tau(g_i) = q_i \in \Gamma$ for all $i < n$. (In this construction we have chosen as a set of representatives for the value group of $\mathbb{R}((x))$ the set $A = \{x^i : i \in \mathbb{Z}\}$ and as a set of coset representatives for $\mathbb{R}((x))^\bullet/1+M$ the set $B = \{ar : a \in A, r \in \mathbb{R}^\bullet\}$ where M denotes the set of infinitesimals of $\mathbb{R}((x))$ [2, page 459].) Since any extension of the x-adic place on $\mathbb{R}((x))$ to a real place on $\mathbb{R}((x))(y)$ is totally ramified, therefore in the construction in [2], if $m + 1 < n$, then g_{m+1} takes the particularly simple form

$$g_m^{e_m} + \theta_m x^s g_0^{t_0} \cdots g_{m-1}^{t_{m-1}}.$$

Here of course $0 \neq \theta_m \in \mathbb{R}$ and we choose the exponents above with $s \in \mathbb{Z}$ and $0 \leq t_i < e_i$ for all i and

$$e_m q_m = s + \sum_{i<m} t_i q_i$$

(as earlier we let $e_i = (\Gamma_{i+1} : \Gamma_i)$ where $\Gamma_i = 0 \times \mathbb{Z} \times 0 + \sum_{j<i} \mathbb{Z} q_j$). We set $S(\tau) := \langle q_i, \theta_i \rangle_{i<n}$ if $n \leq \omega$, and if $n = \omega + 1$ then let $S(\tau)$ be the signature of length n beginning with $\langle q_i, \theta_i \rangle_{i<\omega}$ and ending with the term $((1, 0, 0), 0)$. Note

that the polynomial sequence of g_i's (for $i < \omega$ and $i < n$) is associated with $S^*(\tau)$ by the bijection of [2] and with $S(\tau)$ by the construction in this section of the sequence of polynomials associated with a signature. Let ρ denote the composition of the natural homomorphism from $\mathbb{R}((x))[y]$ onto $\mathbb{R}((x))[y]/(g_m)$ and the unique extension of the Henselian real place on $\mathbb{R}((x))$ to its algebraic extension $\mathbb{R}((x))[y]/(g_m)$. For any $m+1 < n$ we have with the above notation,

$$\begin{aligned}
\theta_m^* &= \tau(g_m^{e_m} x^{s'} g_0^{t_0'} \cdots g_{m-1}^{t_{m-1}'}) \\
&= \tau((g_{m+1} - \theta_m x^s g_0^{t_0} \cdots g_{m-1}^{t_{m-1}}) x^{s'} g_0^{t_0'} \cdots g_{m-1}^{t_{m-1}'}) \\
&= -\theta_m \tau(x^{s+s'} g_0^{t_0+t_0'} \cdots g_{m-1}^{t_{m-1}+t_{m-1}'}) \\
&= -\theta_m \rho(x^{s+s'} g_0^{t_0+t_0'} \cdots g_{m-1}^{t_{m-1}+t_{m-1}'})
\end{aligned}$$

where $s' \in \mathbb{Z}$ and the $0 \le t_i' < e_i$ are chosen so that $s' + \sum_{i<m} t_i' q_i = -e_m q_m$ (for the last equality above apply the fundamental lemma of [2, Section 3] and Lemma F of [2, Section 7]). Thus by induction the sequences $S(\tau)$ and $S^*(\tau)$ determine each other independently of τ. Hence the map taking τ to $S(\tau)$ gives a bijection from $\mathcal{M}$ to $\mathcal{S}$. Parts (B) and (C) of Theorem 3.1 now follow from the fundamental lemma and Supplement of [2, Section 3].

Remark 3.3. We now verify that the restriction map

$$\mathrm{res}\colon \mathcal{M}(\mathbb{R}((x))(y)) \to \mathcal{M}(\mathbb{R}(x,y))$$

is injective. Suppose that $\mathrm{res}(\delta) = \mathrm{res}(\tau)$. Let $S := \Phi(\delta) = \langle q_i, \theta_i \rangle_{i<n}$ have associated sequence $g = (g_i)_{i<n}$. Then $g_i \in \mathbb{R}(x,y)$ so long as $i < \omega$. By hypothesis, if $i < \omega$ and $i < n$, then $v_\tau(g_i) = v_\delta(g_i) = q_i$, which exceeds $e_{i-1}q_{i-1}$ if $i \neq 0$. Hence by the fundamental lemma (applied to all i less than n and ω), $\Phi(\tau) = \Phi(\delta)$. (Note that if $n = \omega + 1$, then $(q_\omega, \theta_\omega) = ((1,0,0),0)$ and if $n < \omega$, then $\theta_{n-1} = 0$.) And therefore, $\tau = \delta$.

4. The homeomorphism $\Phi\colon \mathcal{M} \to \mathcal{S}$

We next put a topology on $\mathcal{S}$ which will make the bijection Φ of Section 3 a homeomorphism. In outline, we present a subbasis for $\mathcal{S}$, and show in Lemma 4.3 that it makes Φ continuous and in Lemma 4.4 that it generates a Hausdorff topology on $\mathcal{S}$. Since $\mathcal{M}$ is compact we conclude the following.

THEOREM 4.1. *Φ is a homeomorphism and $\mathcal{S}$ is compact.*

We now give the subbasis for $\mathcal{S}$ and introduce some notation that will be used throughout the remainder of the paper.

NOTATION 4.2. If we denote a signature by $\widehat{S}$ then we will write $\widehat{S} = \langle \widehat{q}_i, \widehat{\theta}_i \rangle_{i<\widehat{n}}$; set $\widehat{\Gamma}_m = \mathbb{Z} + \sum_{i<m} \mathbb{Z}\widehat{q}_i$ for all $m \leq \widehat{n}$, and set $\widehat{e}_i = (\widehat{\Gamma}_{i+1} : \widehat{\Gamma}_i)$ for all $i < \widehat{n}$. We will also let $\widehat{g} = (\widehat{g}_i)_{i<\widehat{n}}$ (or $\widehat{g} = (\widehat{g}_i)_{i\leq\widehat{n}}$ if $\widehat{S}$ has finite degree) denote the sequence of polynomials associated with $\widehat{S}$ and write $\widehat{g}_{m+1} = \widehat{g}_m^{\widehat{e}_m} + \widehat{\theta}_m x^{\widehat{s}_m} \widehat{g}^{\widehat{\sigma}_m}$ (see Section 3 and equation (3.1)).

We will use similar notation if the circumflex above is replaced by another diacritical mark (or omitted completely). For example if S^* is a signature, we write $S^* = \langle q_i^*, \theta_i^* \rangle_{i<n^*}$, and if S is a signature we continue to use the notation introduced in Section 3, so $S = \langle q_i, \theta_i \rangle_{i<n}$.

Another convention will be useful; if S is a presignature and $m \leq n$, then we let $S_m = \langle q_i, \theta_i \rangle_{i<m}$ denote the initial segment of S of length m, so $S \rhd S_m$.

Suppose that S is a signature of finite length and degree. Pick $0 < \delta \in \mathbb{Q}$, $\beta \in \mathbb{Q}$, and $\alpha \in \mathbb{Q} \cup \{-\infty\}$ such that $\beta \geq \alpha$ and also such that if $n > 0$, then $\alpha > e_{n-1}q_{n-1}$. We give $\mathcal{S}$ the coarsest topology such that for all such S, δ, α and β the following three sets are open:

$$\mathcal{N}_{S,\delta} = \{\widehat{S} \in \mathcal{S} : \widehat{n} > n; \widehat{S} \rhd S_{n-1}; q_{n-1} = \widehat{q}_{n-1};$$
$$| \theta_{n-1} - \widehat{\theta}_{n-1} | < \delta; \text{ if } \theta_{n-1} = \widehat{\theta}_{n-1}, \text{ then } \widehat{q}_n \leq \delta + e_{n-1}q_{n-1}\}$$

(defined only if $n > 0$);

$$\mathcal{N}_{S,\beta,\delta} = \{\widehat{S} \in \mathcal{S} : \widehat{S} \rhd S; \widehat{q}_n \geq \beta; \text{ and } \widehat{q}_n = \beta \text{ only if } | \widehat{\theta}_n | < \delta\}$$

and

$$\mathcal{N}_{S,\alpha,\beta,\delta} = \{\widehat{S} \in \mathcal{S} : \widehat{S} \rhd S; \beta \geq \widehat{q}_n \geq \alpha; \text{ and } \widehat{q}_n = \beta \text{ only if } | \widehat{\theta}_n | > \delta\}.$$

Note that in all cases, if $\widehat{S}$ is in the given open set, then $\widehat{n} > n$. Also, although $\mathcal{N}_{S,\delta}$ is defined above only if S is nonempty, it is convenient to define $\mathcal{N}_{\emptyset,\beta}$ (for any $\beta \in \mathbb{Q}$) to be the open set

$$\mathcal{N}_{\emptyset,\beta} := \{S' \in \mathcal{S} : q_0' \leq \beta\} = \cup_{0<u\in\mathbb{Z}} \mathcal{N}_{\emptyset,-\infty,\beta,1/u}.$$

We now prove Theorem 4.1 by proving the two lemmas promised above.

LEMMA 4.3. *The map* $\Phi\colon \mathcal{M} \to \mathcal{S}$ *is continuous.*

PROOF. Suppose that $a, b \in \mathbb{Z}, b > 0$, and $f \in \mathbb{R}((x))(y)$. Then for any $\tau \in \mathcal{M}$, we have $\tau(1 + (x^{-a}f^b)^2) \in (0, \infty)$ if and only if $v_\tau(f) \geq a/b$ and similarly that $\tau(1 + (x^a f^{-b})^2) \in (0, \infty)$ if and only if $v_\tau(f) \leq a/b$. Thus we will sometimes let $H(v(f) \geq a/b)$ denote the Harrison set $H(1 + (x^{-a}f^b)^2)$ and similarly for $H(v(f) \leq a/b)$, and also set $H(v(f) = a/b) = H(v(f) \geq a/b) \cap H(v(f) \leq a/b)$.

Let S, α, β, δ be as in the definitions of the subbasic open sets $\mathcal{N}_{S,\delta}, \mathcal{N}_{S,\beta,\delta}$ and $\mathcal{N}_{S,\alpha,\beta,\delta}$ above. Let $e = (\Gamma_n + \mathbb{Z}\beta : \Gamma_n)$ and pick $s \in \mathbb{Z}$ and $\sigma \in J_n$ with $e\beta = s + \sum_{j<n} \sigma(j) q_j$. We begin by showing that $\Phi(H) = \mathcal{N}_{S,\beta,\delta}$ where

$$H = H(v(g_n) \geq \beta) \cap H(\delta^2 - (g_n^e/(x^s g^\sigma)^2)).$$

Let $\tau \in \mathcal{M}$ and set $\widehat{S} = \Phi(\tau)$. We must show that $\tau \in H$ if and only if $\Phi(\tau) \in \mathcal{N}_{S,\beta,\delta}$. If $\tau \in H$, then $v_\tau(g_n) \geq \beta$ and if $n > 0$ then $\beta > e_{n-1}q_{n-1}$, so by Theorem 3.1(C) we have $\widehat{S} \rhd S$ and therefore $\widehat{g}_i = g_i$ for all $i \leq n$. (In the application of the theorem let the signature of length $n+1$ with initial segment S and last term $((0,1,0),0)$ play the role of S, n play the role of m, and τ play the role of ρ.) Hence $\widehat{q}_n = v_\tau(\widehat{g}_n) = v_\tau(g_n) \geq \beta$. If $v_\tau(\widehat{g}_n) = v_\tau(g_n) = \beta$, then $\widehat{e}_n = e$ and $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n x^s g^\sigma$, so

$$|\widehat{\theta}_n| = |\tau(\widehat{g}_{n+1}/(x^s g^\sigma) - g_n^e/(x^s g^\sigma))| = |\tau(g_n^e/(x^s g^\sigma))| < \delta$$

since $\tau \in H$. Now suppose that $\Phi(\tau) \in \mathcal{N}_{S,\beta,\delta}$. Then $\beta \leq \widehat{q}_n = v_\tau(\widehat{g}_n) = v_\tau(g_n)$. If $v_\tau(g_n) > \beta$, then $v_\tau(g_n^e/(x^s g^\sigma)) > 0$ so $\tau(\delta^2 - (g_n^e/(x^s g^\sigma))^2) \in (0, \infty)$. But if $v_\tau(g_n) = \beta$, then $\widehat{e}_n = e$ and $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n x^s g^\sigma$, so $\delta > |\widehat{\theta}_n| = |\tau(g_n^e/(x^s g^\sigma))|$ and hence in either case $\tau(\delta^2 - (g_n^e/(x^s g^\sigma)^2)) \in (0, \infty)$. Thus $\tau \in H$.

We next show that $\Phi(H) = \mathcal{N}_{S,\alpha,\beta,\delta}$ where now we set

$$H = H(\beta \geq v(g_n) \geq \alpha) \cap H(\delta^{-2} - (g_n^e/(x^s g^\sigma))^{-2}).$$

First suppose that $\tau \in H$. Arguing as in the previous paragraph, we conclude that $\widehat{S} \rhd S$, so that $g_i = \widehat{g}_i$ for all $i \leq n$. Thus $\widehat{q}_n = v_\tau(\widehat{g}_n) = v_\tau(g_n)$ is between α and β. If $\widehat{q}_n = \beta$, then $e = \widehat{e}_n$ and $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n x^s g^\sigma$, so $|\widehat{\theta}_n| = |\tau(-g_n^e/(x^s g^\sigma))| > \delta$. Hence $\widehat{S} \in \mathcal{N}_{S,\alpha,\beta,\delta}$. Now suppose that $\widehat{S} \in \mathcal{N}_{S,\alpha,\beta,\delta}$. Arguing as above we deduce that $g_i = \widehat{g}_i$ for all $i \leq n$ and $\beta \geq v_\tau(g_n) \geq \alpha$. If $v_\tau(g_n) < \beta$, then $v_\tau(x^s g^\sigma / g_n^e) > 0$, so $|\tau(g_n^e/(x^s g^\sigma))^{-1}| = 0 < \delta^{-1}$. If $v_\tau(g_n) = \beta$, then $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n x^s g^\sigma$, so $\delta < |\widehat{\theta}_n| = |\tau(g_n^e/(x^s g^\sigma))|$. Hence $\tau \in H$.

Finally we suppose that $n > 0$ and show that $\Phi(H) = \mathcal{N}_{S,\delta}$ where now (with the obvious notation) H denotes

$$H(v(g_{n-1}) = q_{n-1}, \quad v(g_n) \le \delta + e_{n-1}q_{n-1}) \cap H(\delta^2 - (g_n/(x^s g^\sigma))^2)$$

and where we now pick $s \in \mathbb{Z}$ and $\sigma \in J_{n-1}$ with $e_{n-1}q_{n-1} = s + \sum_{i<n-1} \sigma(i)q_i$. Let $\tau \in \mathcal{M}$ and first suppose that $\widehat{S} := \Phi(\tau) \in \mathcal{N}_{S,\delta}$. Since $\widehat{S} \rhd \langle q_i, \theta_i\rangle_{i<n-1}$ we have $g_i = \widehat{g}_i$ for all $i < n$ and so $v_\tau(g_i) = v_\tau(\widehat{g}_i) = \widehat{q}_i = q_i$ for all $i < n$. Hence

$$g_n = g_{n-1}^{e_{n-1}} + \theta_{n-1}x^s g^\sigma \quad \text{and} \quad \widehat{g}_n = g_{n-1}^{e_{n-1}} + \widehat{\theta}_{n-1}x^s g^\sigma.$$

Thus

$$\begin{aligned} v_\tau(g_n) - e_{n-1}q_{n-1} &= v_\tau(g_{n-1}^{e_{n-1}}/(x^s g^\sigma) + \theta_{n-1}) \\ &= v_\tau(\widehat{g}_n/(x^s g^\sigma) - \widehat{\theta}_{n-1} + \theta_{n-1}) \end{aligned}$$

which equals 0 if $\theta_{n-1} \ne \widehat{\theta}_{n-1}$ since $\widehat{q}_n > e_{n-1}q_{n-1}$ and which is at most δ if $\theta_{n-1} = \widehat{\theta}_{n-1}$. Moreover

$$|\tau(g_n/(x^s g^\sigma))| = |\tau(\widehat{g}_n/(x^s g^\sigma) - \widehat{\theta}_{n-1} + \theta_{n-1})| = |-\widehat{\theta}_{n-1} + \theta_{n-1}| < \delta,$$

so $\tau \in H$. Conversely, suppose that $\tau \in H$. Then $v_\tau(g_{n-1}) = q_{n-1}$, which if $n > 1$ is larger than $e_{n-2}q_{n-2}$, so by Theorem 3.1(C), $\widehat{S} > \langle q_i, \theta_i\rangle_{i<n-1}$ and hence $\widehat{g}_i = g_i$ for all $i \le n-1$. Hence $q_{n-1} = v_\tau(g_{n-1}) = v_\tau(\widehat{g}_{n-1}) = \widehat{q}_{n-1}$. Also

$$|\theta_{n-1} - \widehat{\theta}_{n-1}| = |\tau((g_n - \widehat{g}_n)/(x^s g^\sigma))| = |\tau(g_n/(x^s g^\sigma))| < \delta.$$

Finally, suppose that $\theta_{n-1} = \widehat{\theta}_{n-1}$, so $g_n = \widehat{g}_n$. Then

$$\widehat{q}_n - e_{n-1}q_{n-1} = v_\tau(g_n) - e_{n-1}q_{n-1} \le \delta.$$

Thus $\Phi(\tau) = \widehat{S} \in \mathcal{N}_{S,\delta}$. □

LEMMA 4.4. *$\mathcal{S}$ is Hausdorff.*

PROOF. Suppose that S^* and $\widehat{S}$ are two distinct elements of $\mathcal{S}$. Then there is a least n with $(q_n^*, \theta_n^*) \ne (\widehat{q}_n, \widehat{\theta}_n)$. Then $n < \omega$ since for any signature S of length $\omega + 1$, we have $(q_\omega, \theta_\omega) = ((1,0,0),0)$. We will find disjoint open sets $\mathcal{N}^*$ and $\widehat{\mathcal{N}}$ containing S^* and $\widehat{S}$, respectively.

First suppose that $\widehat{q}_n = q_n^*$, so that $\widehat{\theta}_n \neq \theta_n^*$. Let δ denote a positive rational less than $|\widehat{\theta}_n - \theta_n^*|$. At least one of θ_n^* and $\widehat{\theta}_n$ is nonzero, so $\widehat{q}_n = q_n^*$ is rational, so both θ^* and $\widehat{\theta}_n$ are nonzero and n^* and $\widehat{n}$ are both at least 2. Then $\widehat{S}_{n+1} = \langle \widehat{q}_i, \widehat{\theta}_i \rangle_{i \leq n}$ and $S_{n+1}^* = \langle q_i^*, \theta_i^* \rangle_{i \leq n}$ are distinct signatures of finite length and degree. If there exists $\widehat{q} \in \mathbb{Q}$ with $\widehat{q}_{n+1} > \widehat{q} > \widehat{e}_n \widehat{q}_n$, then set $\widehat{\mathcal{N}} = \mathcal{N}_{\widehat{S}_{n+1}, \widehat{q}, \delta}$; otherwise we have $\widehat{q}_{n+1} = (\widehat{e}_n \widehat{q}_n)^+$ and we set $\widehat{\mathcal{N}} = \mathcal{N}_{\widehat{S}_{n+1}, \delta/2}$. Similarly, if there exists $q^* \in \mathbb{Q}$ with $q_{n+_1}^* > q^* > e_n^* q_n^*$, then set $\mathcal{N}^* = \mathcal{N}_{S_{n+1}^*, q^*, \delta}$ and otherwise set $\mathcal{N}^* = \mathcal{N}_{S_{n+1}^*, \delta/2}$. Then clearly $\widehat{S} \in \widehat{\mathcal{N}}$ and $S^* \in \mathcal{N}^*$. Suppose $S' \in \widehat{\mathcal{N}} \cap \mathcal{N}^*$. If $\widehat{q}$ and q^* both exist then $S' \rhd S_{n+1}^*$ and $S' \rhd \widehat{S}_{n+1}$, so $\widehat{\theta}_n = \theta_n' = \theta_n^*$, a contradiction. If neither exists then $\delta < |\theta_n^* - \widehat{\theta}_n| \leq |\theta_n^* - \theta_n'| + |\theta_n' - \widehat{\theta}_n| \leq \delta$, a contradiction. Thus without loss of generality $\widehat{q}$ exists and q^* does not, so that $\theta_n' = \widehat{\theta}_n$ and $\delta/2 > |\theta_n' - \theta_n^*| = |\widehat{\theta}_n - \theta_n^*| > \delta$, a contradiction. Therefore $\mathcal{N}^*$ and $\widehat{\mathcal{N}}$ are indeed disjoint.

Hence without loss of generality we may assume that $\widehat{\theta}_n = \theta_n^*$ and $\widehat{q}_n < q_n^*$.

Let $S = \langle \widehat{q}_i, \widehat{\theta}_i \rangle_{i<n} = \langle q_i^*, \theta_i^* \rangle_{i<n}$. Suppose that there exist distinct rationals α and β with $\widehat{q}_n < \alpha < \beta < q_n^*$. If there exists a rational δ with $\widehat{q}_n > \delta$ (and $\delta > e_{n-1} q_{n-1}$ if $n > 0$), then we can take $\widehat{\mathcal{N}} = \mathcal{N}_{S,\delta,\alpha,1}$ and $\mathcal{N}^* = \mathcal{N}_{S,\beta,1}$. (If $\langle q_i, \theta_i \rangle_{i<t} \in \widehat{\mathcal{N}} \cap \mathcal{N}^*$, then $\beta \leq q_n \leq \alpha < \beta$, a contradiction.) If no such δ exists, then either $n = 0$ and $\widehat{q}_0 = (-1, 0, 0)$, in which case we can take $\mathcal{N}^* = \mathcal{N}_{\emptyset,\beta,1}$ and $\widehat{\mathcal{N}} = \mathcal{N}_{\emptyset,-\infty,\alpha,1}$ or else $n > 0$ and $\widehat{q}_n = (e_{n-1} q_{n-1})^+ < \alpha$, which we now assume to be the case. Then $S^* \in \mathcal{N}^* := \mathcal{N}_{S,\beta,1}$ and $\widehat{S} \in \widehat{\mathcal{N}} := \mathcal{N}_{S,\beta-\alpha}$. If $\langle q_i, \theta_i \rangle_{i<t} \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$, then $q_n > \beta$ and since $\theta_{n-1} = \theta_{n-1}^* = \widehat{\theta}_{n-1}$, we have $q_n \leq \beta - \alpha + e_{n-1} q_{n-1}$, so

$$\beta - \alpha \geq q_n - e_{n-1} q_{n-1} > \beta - e_{n-1} q_{n-1} > \beta - \widehat{q}_n > \beta - \alpha,$$

a contradiction. Hence we may suppose that there do not exist distinct rationals between q_n^* and $\widehat{q}_n$. Hence there must exist a rational q such that either:

(1) $q_n^* = q^+$ and $\widehat{q}_n = q^-$, or
(2) $q_n^* = q^+$ and $\widehat{q}_n = q$, or
(3) $q_n^* = q$ and $\widehat{q}_n = q^-$.

In all three cases there is a rational β with $e_{n-1} q_{n-1} < \beta$ if $n \neq 0$ and $\beta < \widehat{q}_n$. In the first case above we take $\widehat{\mathcal{N}} = \mathcal{N}_{S,\beta,q,1}$ and $\mathcal{N}^* = \mathcal{N}_{S,q,1}$. If $\langle q_i, \theta_i \rangle_{i<t} \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$, then $q \leq q_n \leq q$, so $1 < |\theta_n| < 1$, a contradiction. Thus $\mathcal{N}^*$ and $\widehat{\mathcal{N}}$ are disjoint. We next consider the second case above where we now take $\mathcal{N}^* = \mathcal{N}_{S,q,|\widehat{\theta}_n|/2}$ and $\widehat{\mathcal{N}} = \mathcal{N}_{S,q,q,|\widehat{\theta}_n|/2}$. (Note that $\widehat{q}_n = q \in \mathbb{Q}$ implies that $\widehat{\theta}_n \neq 0$.) If $\langle q_i, \theta_i \rangle_{i<t} \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$, then $q_n = q$ so $|\widehat{\theta}_n|/2 < |\theta_n| < |\widehat{\theta}_n|/2$, a contradiction. Finally, in the third case above we set $\mathcal{N}^* = \mathcal{N}_{S,q,2|\theta_n^*|}$ (note

that $\theta_n^* \neq 0$ since $q_n^* \in \mathbb{Q}$) and $\widehat{\mathcal{N}} = \mathcal{N}_{S,\beta,q,2|\theta_n^*|}$. Then if $\langle q_i, \theta_i\rangle_{i<t} \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$ we have that $q \leq q_n \leq q$, so $2|\theta_n^*| < |\theta_n| < 2|\theta_n^*|$, so $\widehat{\mathcal{N}}$ and $\mathcal{N}^*$ are disjoint. □

5. Path-connectedness

We now prove Theorem 1.1.

Throughout this section (except in the proof of Theorem 1.1) $S = \langle q_i, \theta_i\rangle_{i<n}$ will denote a signature of finite length and degree. We use the notation of Section 4. If $i = 0$, set $e_{i-1}q_{i-1} = -\infty$ and $(-\infty)^+ = (e_{i-1}q_{i-1})^+ = (-1,0,0)$. We give $\Gamma \times \mathbb{R}$ the lexicographic order and consider its ordered subset

$$A_S = \{(q,\theta) : q \geq (e_{n-1}q_{n-1})^+, \theta \leq 0, \text{ and } \theta = 0 \text{ iff } q \notin \mathbb{Q}\}.$$

Thus A_S has maximum $((1,0,0),0)$ and minimum $((e_{n-1}q_{n-1})^+,0)$. For any $(q,\theta) \in \Gamma \times \mathbb{R}$ we set $S(q,\theta) = S \oplus (q,\theta)$ if $q \notin \mathbb{Q}$ and $S(q,\theta) = S \oplus (q,\theta) \oplus ((eq)^+,0)$ if $q \in \mathbb{Q}$ where $e = (\Gamma_S + \mathbb{Z}q : \Gamma_S)$. (For sequences A and B we let $A \oplus B$ denote the concatenation of A and B; if $B = \{b\}$ is a singleton, we will often write $A \oplus b$ for $A \oplus B$.) Note that if $(q, -|\theta|) \in A_S$, then $S(q,\theta) \in \mathcal{S}$. We will show that A_S is order isomorphic to the interval $[0,1]$ and that the function $(q,\theta) \mapsto S(q,\theta)$ maps A_S (given the order topology) continuously into $\mathcal{S}$.

One checks that the set $\{(q,\theta) \in A_S : q \in \mathbb{Q} \text{ and } \theta \in \mathbb{Q}\}$ is dense in A_S (the first $\mathbb{Q}$ above of course abbreviates $0 \times \mathbb{Q} \times 0$); hence

LEMMA 5.1. *A_S has a countable dense subset.*

We next prove

LEMMA 5.2. *A_S has the least upper bound property.*

PROOF. Suppose that $\emptyset \neq B \subset A_S$, so that $((1,0,0),0)$ is an upper bound for B. Without loss of generality $((1,0,0),0)$ is not a least upper bound for B. Therefore B has an upper bound of the form $((0,b,c,),d)$. Without loss of generality $((-1,0,0),0) \notin B$. Then b is an upper bound for the image B' of B under the composition of projections

$$\Gamma \times \mathbb{R} \to \Gamma \to \mathbb{R}.$$

(The second map above is the restriction of the natural projection $\mathbb{Z}\times\mathbb{R}\times\mathbb{Z} \to \mathbb{R}$.) Then B' has a least upper bound, say q. If $q \in \mathbb{R} \setminus \mathbb{Q}$, then $((0,q,0),0)$ is

a least upper bound for B, so suppose that $q \in \mathbb{Q}$. Without loss of generality, $(q^+, 0) \notin B$ (otherwise $(q^+, 0)$ would be a least upper bound for B) and either $V := \{\theta \in (-\infty, 0] : (q, \theta) \in B\}$ is empty or it has a least upper bound $\theta' < 0$ (otherwise $(q^+, 0)$ would be a least upper bound for B). If $V = \emptyset$, then $(q^-, 0)$ is a least upper bound for B, and if not then (q, θ') is a least upper bound for B. □

COROLLARY 5.3. *A_S is order isomorphic (and hence homeomorphic) to* $[0, 1]$.

The Corollary follows from the previous two lemmas and [16, Theorem 2.30] (either apply the theorem to the interior of A_S or as suggested in the paragraph just before [16, Exercise 2.37, page 40], apply the proof of Theorem 2.30 of [16]).

REMARK 5.4. One obtains an explicit order isomorphism $\delta\colon A_S \to [0, 1]$ by first picking an enumeration $r_1, r_2, \cdots$ of the set $\mathbb{Q} \cap (e_{n-1}q_{n-1}, \infty)$ and then setting $\delta(q, \theta)$ equal to $\sum_{r_i<q} \frac{1}{2^i}$ if $q \notin \mathbb{Q}$ and equal to $\frac{1}{(1-\theta)2^j} + \sum_{r_i<q} \frac{1}{2^i}$ if $q = q_j \in \mathbb{Q}$ (recall that we have $q > e_{n-1}q_{n-1}$; as usual, $e_{n-1}q_{n-1} = -\infty$ if $n = 0$). This construction was inspired by an argument that the second author learned while an undergraduate at University of Maryland from Professor William Kirwan.

Let $\mathcal{N}_S = \{S' \in \mathcal{S} : S' \rhd S\}$.

LEMMA 5.5. *If $\mathcal{N}$ is a subbasic open set, so that $\mathcal{N}$ has one of the forms $\mathcal{N}_{M,\delta}$, $\mathcal{N}_{M,\beta,\delta}$ or $\mathcal{N}_{M,\alpha,\beta,\delta}$ for some signature M, and if $\mathcal{N} \cap \mathcal{N}_S \notin \{\emptyset, \mathcal{N}_S\}$, then $M \rhd S$.*

PROOF. Write $M = \langle q_i^M, \theta_i^M \rangle_{i<m}$. Just suppose that $M \rhd S$ is false. By hypothesis there exists an element $\widehat{S} \in \mathcal{N}_S \cap \mathcal{N}$. Suppose $S^* \in \mathcal{N}_S$. We show that $S^* \in \mathcal{N}$, so $\mathcal{N}_S \cap \mathcal{N} = \mathcal{N}_S$, a contradiction.

Case 1: $\mathcal{N} = \mathcal{N}_{M,\beta,\delta}$. Then $\widehat{S} \rhd M$ and $\widehat{S} \rhd S$ so $S \rhd M$ and $S \neq M$ (recall that $M \rhd S$ is assumed to be false). Hence $m < n$. Thus $q_m^* = q_m = \widehat{q}_m \geq \beta$ with equality if and only if $|\theta_m^*| = |\theta_m| = |\widehat{\theta}_m| < \delta$. Thus $S^* \in \mathcal{N}$, as claimed.

Case 2: $\mathcal{N} = \mathcal{N}_{M,\alpha,\beta,\delta}$. Argue as in the previous case.

Case 3: $\mathcal{N} = \mathcal{N}_{M,\delta}$. Then $\widehat{S} \rhd M_{m-1}$ and $\widehat{S} \rhd S$, so $S \rhd M_{m-1}$ and $n \geq m$. Hence $q_{m-1}^* = q_{m-1} = \widehat{q}_{m-1} = q_{m-1}^M$ and $\theta_{m-1}^* = \widehat{\theta}_{m-1}$, and hence $|\theta_{m-1}^* - \theta_{m-1}^M| = |\widehat{\theta}_{m-1} - \theta_{m-1}^M| < \delta$. If $\theta_{m-1}^* \neq \theta_{m-1}^M$, then by definition $S^* \in \mathcal{N}$, and we are finished. So suppose $\theta_{m-1}^* = \theta_{m-1}^M$. Then $S \rhd M$ and $S \neq M$, so $n > m$. Thus $q_m^* = q_m = \widehat{q}_m \leq \delta + e_{m-1}^M q_{m-1}^M$. So again we have $S^* \in \mathcal{N}$. □

LEMMA 5.6. *The map $E = E_S\colon A_S \to \mathcal{N}_S$ taking each (q,θ) to $S(q,\theta)$ is continuous.*

PROOF. Consider a subbasic open set $\mathcal{N}$ of $\mathcal{S}$, say either

$$\mathcal{N} = \mathcal{N}_{\widehat{S},\beta,\delta} \text{ or } \mathcal{N}_{\widehat{S},\beta,\gamma,\delta} \text{ or } \mathcal{N}_{\widehat{S},\delta}$$

and suppose that $E^{-1}(\mathcal{N}) \neq \emptyset$; we show that $E^{-1}(\mathcal{N})$ is an open interval of A_S.

First suppose that $n > \widehat{n}$. By Lemma 5.5

$$\mathcal{N} \cap E(A_S) = \mathcal{N} \cap \mathcal{N}_S \cap E(A_S) = \mathcal{N}_S \cap E(A_S) = E(A_S),$$

so $E^{-1}(\mathcal{N}) = A_S$. Thus we may suppose that $n \leq \widehat{n}$. For some $(q,\theta) \in A_S$ we have $S(q,\theta) \in \mathcal{N}$, so in all cases we have

$$n+2 \geq \text{length of } S(q,\theta) > \widehat{n} \geq n,$$

so $\widehat{n} = n$ or $\widehat{n} = n+1$.

Next assume that $\widehat{n} = n+1$. Just suppose that $\mathcal{N}$ equals either $\mathcal{N}_{\widehat{S},\beta,\delta}$ or $\mathcal{N}_{\widehat{S},\beta,\gamma,\delta}$. If $S(q,\theta) \in \mathcal{N}$, then $S(q,\theta) \rhd \widehat{S}$, so $q \in \mathbb{Q}$; then $eq < \beta \leq (eq)^+$, a contradiction since $\beta \in \mathbb{Q}$. (Here as above, $e = (\Gamma_S + \mathbb{Z}q : \Gamma_S)$.) Hence $\mathcal{N} = \mathcal{N}_{\widehat{S},\delta}$. Let $(q,\theta) \in A_S$. If $S^* := S(q,\theta) \in \mathcal{N}$, then since $S^* \rhd \langle \widehat{q}_i, \widehat{\theta}_i \rangle_{i<\widehat{n}-1}$ and $S^* \rhd S$, therefore $S = \widehat{S}_n$. Also $q = q_n^* = q_{\widehat{n}-1}^* = \widehat{q}_{\widehat{n}-1} = \widehat{q}_n \in \mathbb{Q}$ and $|\theta - \widehat{\theta}_n| < \delta$, so (q,θ) is in the open interval

$$((\widehat{q}_n, \widehat{\theta}_n - \delta), (\widehat{q}_n, \min(\widehat{\theta}_n + \delta, \theta/2))) \tag{5.1}$$

of A_S. Conversely, if (q,θ) is in this interval, then $q_n^* = q = \widehat{q}_n$ and $|\theta - \widehat{\theta}_n| < \delta$ and $q_{n+1}^* = (eq)^+ \leq eq + \delta$. That is, $S(q,\theta) \in \mathcal{N}$. Thus $E^{-1}(\mathcal{N})$ is the interval of display (5.1).

Finally assume that $\widehat{n} = n$. If $\mathcal{N}$ is $\mathcal{N}_{\widehat{S},\beta,\delta}$ or $\mathcal{N}_{\widehat{S},\beta,\gamma,\delta}$, then since $E^{-1}(\mathcal{N}) \neq \emptyset$, we have $S = \widehat{S}$. Let $(q,\theta) \in A_S$. Then $E(q,\theta) = S(q,\theta) \in \mathcal{N}_{\widehat{S},\beta,\delta}$ if and only if $q \geq \beta$ and if $q = \beta$, then $|\theta| < \delta$, i.e., if and only if $(q,\theta) \in ((\beta,-\delta),((1,0,0),0)]$. Similarly, $E(q,\theta) \in \mathcal{N}_{\widehat{S},\beta,\gamma,\delta}$ if and only if $(q,\theta) \in ((\beta^-,0),(\gamma,-\delta))$. Now assume that $\mathcal{N} = \mathcal{N}_{\widehat{S},\delta}$. Since $E^{-1}(\mathcal{N}) \neq \emptyset$, we may assume that $q_i = \widehat{q}_i$ for all $i < n$, $\theta_i = \widehat{\theta}_i$ for all $i < n-1$, and $|\theta_{n-1} - \widehat{\theta}_{n-1}| < \delta$. Let $(q,\theta) \in A_S$, so $q > e_{n-1}q_{n-1}$. Then $S^* := S(q,\theta) \in \mathcal{N}$ if and only if $q \leq e_{n-1}q_{n-1} + \delta$ if $\theta_{n-1} = \widehat{\theta}_{n-1}$. If $\theta_{n-1} \neq \widehat{\theta}_{n-1}$, then $E^{-1}(\mathcal{N}) = A_S$ (an interval) and if $\theta_{n-1} = \widehat{\theta}_{n-1}$, then $S(q,\theta) \in \mathcal{N}$ if and only if $(q,\theta) \in [((e_{n-1}q_{n-1})^+,0),((e_{n-1}q_{n-1}+\delta)^+,0))$. □

We need another class of homeomorphisms. Let $\epsilon = \langle \epsilon_i \rangle_{0 \le i \le \omega}$ be a sequence with each $\epsilon_i \in \{\pm 1\}$. Define a map $\Psi_\epsilon \colon \mathcal{S} \to \mathcal{S}$ by the formula

$$\Psi_\epsilon(S') = \langle q'_i, \epsilon_i \theta'_i \rangle_{i<n'} \tag{5.2}$$

for all $S' \in \mathcal{S}$.

LEMMA 5.7. *Ψ_ϵ is a homeomorphism.*

PROOF. Ψ_ϵ is continuous since for all appropriate T, α, β, δ we have

$$\Psi_\epsilon(\mathcal{N}_{T,\delta}) = \mathcal{N}_{\Psi_\epsilon(T),\delta},$$

$$\Psi_\epsilon(\mathcal{N}_{T,\beta,\delta}) = \mathcal{N}_{\Psi_\epsilon(T),\beta,\delta},$$

and

$$\Psi_\epsilon(\mathcal{N}_{T,\alpha,\beta,\delta}) = \mathcal{N}_{\Psi_\epsilon(T),\alpha,\beta,\delta}.$$

(In the above formulas we use equation (5.2) to define $\Psi_\epsilon(T)$ even though T is not in $\mathcal{S}$.) But $\mathcal{S}$ is compact and Ψ_ϵ is its own inverse, so Ψ_ϵ is a homeomorphism. □

We now give the proof of Theorem 1.1.

PROOF. Suppose $S \in \mathcal{S}$. It suffices to show there is a path connecting $\langle((-1,0,0),0)\rangle$ (a signature of length one) to S, i.e., a continuous map $\pi \colon [0,1] \to \mathcal{S}$ with $\pi(0) = \langle((-1,0,0),0)\rangle$ and $\pi(1) = S$. Now let $\epsilon_i = -1$ if $i < n$ and $\theta_i > 0$ and let $\epsilon_i = 1$ otherwise. If π_0 were a path connecting $\langle((-1,0,0),0)\rangle$ to $\Psi_\epsilon(S)$, then $\Psi_\epsilon \circ \pi_0$ would be a path connecting $\langle((-1,0,0),0)\rangle$ to $\Psi_\epsilon\Psi_\epsilon(S) = S$. Thus we may assume without loss of generality that $\theta_i \le 0$ for all $i < n$.

Set $S_0^+ = \langle((-1,0,0),0)\rangle \in \mathcal{S}$, and if $0 < m \le n$ and $m < \omega$ set $S_m^+ = S_{m-1}(q_{m-1}, \theta_{m-1})$ (so if $n < \infty$ then $S_n^+ = S$). By the Corollary 5.3 above and Lemma 5.6 for each $m < n$ we have a path from S_m^+ to $S_m((1,0,0),0)$ passing through $S_m(q_m, \theta_m) = S_{m+1}^+$ and hence a path from S_m^+ to S_{m+1}^+ whose image is in $\mathcal{N}_{S_m}$. If n is finite we simply combine these paths to get the required path from $\langle((1,0,0),0)\rangle$ to S. Now suppose n is infinite. We have continuous maps $\pi_i \colon [1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}] \to \mathcal{N}_{S_i}$ with $\pi_i(1 - \frac{1}{2^i}) = S_i^+$ and $\pi_i(1 - \frac{1}{2^{i+1}}) = S_{i+1}^+$. Combining these maps gives a continuous map $\pi \colon [0,1) \to \mathcal{S}$. It remains to verify that if we extend this map π to the interval $[0,1]$ by mapping 1 to S, then the resulting map, which we also call π, is continuous at 1. Let $\mathcal{N}_1, \cdots, \mathcal{N}_k$ be a finite collection of sets containing S with each $\mathcal{N}_i$ having one of the forms

$\mathcal{N}_{T(i),\delta}, \mathcal{N}_{T(i),\beta,\delta}, \mathcal{N}_{T(i),\alpha,\beta,\delta}$ for some signatures $T(i)$ of length, say, $n_i < \infty$. Let $m > n_i$ for all i. If $\rho \in (1-1/2^m, 1)$, then $\rho \in [1-1/2^u, 1-1/2^{u+1}]$ for some $u \geq m$. By Lemma 5.5 and the definition of π, $\pi(\rho) \in \mathcal{N}_{S_u} \subset \mathcal{N}_{S_m} \subset \mathcal{N} := \cap_{i \leq k} \mathcal{N}_i$. So $\pi((1-1/2^m, 1]) \subset \mathcal{N}$. Thus π is indeed continuous at 1. □

REMARK 5.8. It is not obvious to us how to account directly for the vast collection of involutions of $\mathcal{M}$ suggested by Lemma 5.7 without using the homeomorphism of $\mathcal{M}$ and $\mathcal{S}$ from Section 4.

6. Dense subsets

In this section we list all the possible value groups of real places on $\mathbb{R}(x,y)$ (i.e., of the valuations associated with these real places) and show that for each of these, the set of real places on $\mathbb{R}(x,y)$ whose value groups are order isomorphic to it is a dense subset of $\mathcal{M}(\mathbb{R}(x,y))$. For example, the set of real places in $\mathcal{M}(\mathbb{R}(x,y))$ with value group isomorphic to $\mathbb{Z}$ is dense in $\mathcal{M}(\mathbb{R}(x,y))$, as is the set of those with value group isomorphic to $\mathbb{Q}$. We begin by considering the corresponding facts for $\mathbb{R}((x))(y)$.

For any $\sigma \in \mathcal{M}$ the value group of the associated valuation v_σ is Γ_S where $S = \Phi(\sigma)$, so the collection of such groups is the set of groups of the form $\mathbb{Z}\frac{1}{M} + \mathbb{Z}\gamma$ where γ is a nonrational member of Γ, together with the nondiscrete subgroups of $\mathbb{Q}$ (see Corollary 3.2).

PROPOSITION 6.1. *For any of the groups of the paragraph above, the set of real places $\sigma \in \mathcal{M}$ with value group order isomorphic to that group and with the length of $\Phi(\sigma)$ at most ω is dense in $\mathcal{M}$.*

We begin with a key lemma.

LEMMA 6.2. *If U is a nonempty open subset of $\mathcal{S}$, then there is a signature S^* of finite degree and length such that if $S^* < S' \in \mathcal{S}$, then $S' \in U$.*

PROOF. There exists a finite set $\mathcal{N}_1, \cdots, \mathcal{N}_d$ of subbasic open sets and a signature $S \in \mathcal{S}$ with $S \in \cap_{i \leq d} \mathcal{N}_i \subseteq U$. For each $i \leq d$ there exists a signature T_i of finite length and degree such that $\mathcal{N}_i$ has one of the forms $\mathcal{N}_{T_i,\delta}$, $\mathcal{N}_{T_i,\beta,\delta}$ or $\mathcal{N}_{T_i,\alpha,\beta,\delta}$ (see Section 4). Let M_i denote the length of T_i and $M = \max_{i \leq d} M_i$, so $n \geq M+1$. For any $i \leq d$ and signature $S' \in \mathcal{S}$, if S' is in $\mathcal{N}_i$, then $n' > M_i$, and if $n' > M_i$, then whether or not S' is in $\mathcal{N}_i$ depends only on the subsequence S'_{M_i+1}. Thus if $n > M+1$, then we can simply set $S^* = S_{M+1}$. Hence without loss of generality we may assume $n = M+1$, so in

particular $q_M \notin \mathbb{Q}$. We will show that there exists $q \in \mathbb{Q}$ with $q > e_{M-1}q_{M-1}$ such that we can take $S^* = S_M \oplus (q, 1)$.

Case: $q_M = r^-$ *where* $r \in \mathbb{Q}$. Suppose that $q \in (r - \epsilon, r) \cap \mathbb{Q}$ where $0 < \epsilon \in \mathbb{Q}$. Let $i \leq d$. It suffices in this case to show that if $S_M \oplus (q, 1) \lhd S' \in \mathcal{S}$ and ϵ is sufficiently small, then $S' \in \mathcal{N}_i$. This is obvious if $M > M_i$, so suppose that $M = M_i$. First suppose that $\mathcal{N}_i = \mathcal{N}_{T_i,\beta,\delta}$. Then $S' \rhd S_M = T_i$ and $e_{M-1}q_{M-1} < r^- < r \leq \beta$, so $e_{M-1}q_{M-1} < q < \beta$ if ϵ is sufficiently small; hence for such ϵ we have $S' \in \mathcal{N}_i$. The case when $\mathcal{N}_i = \mathcal{N}_{T_i,\alpha,\beta,\delta}$ is similar: we have $\alpha \leq r^- < r \leq \beta$, so picking ϵ small enough will guarantee that $\alpha < q < \beta$. Similarly if $\mathcal{N}_i = \mathcal{N}_{T_i,\delta}$ it suffices to pick ϵ small enough that $q > e_{M-1}q_{M-1}$.

Case: $q_M = r^+$ *where* $r \in \mathbb{Q}$. We will argue as in the previous case except that in this case we show that it will suffice to pick $q \in (r, r + \epsilon) \cap \mathbb{Q}$ for a sufficiently small positive rational number ϵ. For any such ϵ and q we will of course have that $S_M \oplus (q, 1)$ is a signature. Suppose that $i \leq d$ and $M_i = M$. If $\mathcal{N}_i = \mathcal{N}_{T_i,\beta,\delta}$, then $q_M = r^+ \geq \beta$, so $r \geq \beta$ and hence for positive rational ϵ and $r < q < r + \epsilon$ we have $q > \beta$. If $\mathcal{N}_i = \mathcal{N}_{T_i,\alpha,\beta,\delta}$, then the inequalities $\alpha \leq r^+ \leq \beta$ imply that $\alpha < q < \beta$ for $q \in (r, r + \epsilon)$ for sufficiently small ϵ. Finally if $\mathcal{N}_i = \mathcal{N}_{T_i,\delta}$ and $q_M = r^+ \leq e_{M-1}q_{M-1} + \delta$, then for sufficiently small ϵ and rational $q \in (r, r + \epsilon)$, we have $q \leq e_{M-1}q_{M-1} + \delta$. Thus we can again take $S^* = S_M \oplus (q, 1)$.

Case: $q_M \in \mathbb{R}$ *and* $q_M \notin \mathbb{Q}$. In this case we show that it suffices to pick $q \in (q_M - \epsilon, q_M + \epsilon) \cap \mathbb{Q}$ for a sufficiently small positive rational number ϵ. Suppose that $i \leq d$ and $M_i = M$. If $\mathcal{N}_i = \mathcal{N}_{T_i,\beta,\delta}$, then $\beta < q_M$, so for small enough ϵ we are guaranteed that $q > \beta$. If $\mathcal{N}_i = \mathcal{N}_{T_i,\alpha,\beta,\delta}$, then $\alpha < q_M < \beta$, so these inequalities are true with q_M replaced by q if ϵ is small enough. Finally if $\mathcal{N}_i = \mathcal{N}_{T_i,\delta}$ and $q_M \leq e_{M-1}q_{M-1} + \delta$, then for sufficiently small ϵ we have $q \leq e_{M-1}q_{M-1} + \delta$ (keep in mind that $q_M \neq e_{M-1}q_{M-1} + \delta$). Thus we can again in all cases take $S^* = S_M \oplus (q, 1)$.

Case: $q_M = (1, 0, 0)$. In this case it suffices to choose q to be sufficiently large. For example if $\mathcal{N}_i = \mathcal{N}_{T_i,\beta,\delta}$, then we simply need to take $q > \beta$. The case $\mathcal{N}_i = \mathcal{N}_{T_i,\alpha,\beta,\delta}$ cannot occur, and in the case that $\mathcal{N}_i = \mathcal{N}_{T_i,\delta}$ we just need to pick $q > e_{M-1}q_{M-1}$ (notice that we will never have θ_{M-1} equal to the second coordinate of the last term of T_i).

Case: $q_M = (-1, 0, 0)$. Then $n = 0$. Each $\mathcal{N}_i$ must have the form $\mathcal{N}_{\emptyset,-\infty,\beta,\delta}$, in which case it suffices to take q to be less than the least such β. □

We can now prove Proposition 6.1.

Proof of Proposition 6.1. Let Δ be one of the ordered groups in question. By Corollary 3.2 and Theorem 4.1 it suffices to show that the set of signatures $S \in \mathcal{S}$ of length at most ω with Γ_S order isomorphic to Δ is dense

in $\mathcal{S}$. Let U be a nonempty open subset of $\mathcal{S}$; let S be a signature of finite degree and length as in the lemma above. Write $M = [\Gamma_S : \mathbb{Z}]$, so that $\Gamma_S = \mathbb{Z}\frac{1}{M}$. First suppose that Δ has rational rank one. We may without loss of generality assume that Δ contains $\mathbb{Z}$ (it is isomorphic to a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$). Let $\delta_1, \delta_2 \cdots$ be an enumeration of the countable set Δ. There exist integers $L_1, L_2, \cdots$ such that

$$\widehat{S} = S \oplus (L_1 + \frac{\delta_1}{M}, 1) \oplus (L_2 + \frac{\delta_2}{M}, 1) \oplus \cdots$$

is a signature in $\mathcal{S}$ of length ω with value group

$$\Gamma_{\widehat{S}} = \mathbb{Z}\frac{1}{M} + \sum_i \mathbb{Z}\frac{\delta_i}{M}.$$

Multiplication by M is an order isomorphism of $\Gamma_{\widehat{S}}$ onto $Z + \Delta = \Delta$. So by the above lemma, U has a member $\widehat{S}$ of length at most ω with $\Gamma_{\widehat{S}}$ order isomorphic to Δ.

Next suppose that Δ is not of rational rank one. Then it has the form $\mathbb{Z}\frac{1}{L} + \mathbb{Z}\gamma$ where $0 < L \in \mathbb{Z}$ and $\gamma \in \Gamma$ is not rational; we may also assume without loss of generality that $\gamma > 0$. We can write $\Gamma_S = \mathbb{Z}\frac{1}{M}$ for a positive integer M. Let $\gamma' = \gamma$ if $\gamma \in \mathbb{R}$, and $\gamma' = \frac{M}{L}\gamma$ if $\gamma = (1,0,0)$, and let $\gamma' = (0, q, \pm\frac{M}{L})$ if $\gamma = q^{\pm}$ for some $q \in \mathbb{Q}$. We can pick a nonnegative $T \in \mathbb{Z}$ with $T + \gamma' > \frac{M}{L} e_{n-1} q_{n-1}$. Then

$$\Delta \cong \mathbb{Z}\frac{1}{L} + \mathbb{Z}(T + \gamma') \cong \mathbb{Z}\frac{1}{M} + \mathbb{Z}(\frac{L}{M}(T + \gamma')) = \Gamma_{\widehat{S}}$$

where $\widehat{S} = S \oplus (\frac{L}{M}(T + \gamma'), 0)$. (Note that $\frac{L}{M}\gamma' \in \Gamma$ is not rational and $\widehat{S} \in \mathcal{S}$. The isomorphism above is given by multiplication by $\frac{L}{M}$.) Thus by the above lemma again U has a member $\widehat{S}$ of length at most ω with $\Gamma_{\widehat{S}}$ order isomorphic to Δ. □

The above proposition will be used to prove the main theorem for this section.

THEOREM 6.3. *The value groups of real places on $\mathbb{R}(x,y)$ are, up to order isomorphism, the group $\mathbb{Z}$ together with the value groups of elements in $\mathcal{M}$. For any such ordered group Δ, the set of real places on $\mathbb{R}(x,y)$ with value group order isomorphic to Δ is dense in $\mathcal{M}(\mathbb{R}(x,y))$.*

PROOF. Suppose that Δ is the value group of some $\sigma \in \mathcal{M}(\mathbb{R}(x,y))$ which is not discrete rank one. We show that it is a value group of an element of $\mathcal{M}$; it suffices to show that it is isomorphic to Γ_S for some $S \in \mathcal{S}$. For some $t \in \mathbb{R}(x)$ with $\mathbb{R}(x) = \mathbb{R}(t)$ we have $\sigma(t) = 0$. Without loss of generality we may assume that $x = t$. By Lemma 2.2 σ extends to a real place σ' on $\mathbb{R}((x))(y)$. Suppose that $S = \Phi(\sigma')$. If $n \leq \omega$, then the associated sequence $g = (g_i)_{i<n}$ is in $\mathbb{R}(x,y)$ (see Section 3). Hence the value group Γ_S of σ' satisfies

$$\Delta \subseteq \Gamma_S = \mathbb{Z} + \sum \mathbb{Z} q_i = \mathbb{Z} + \sum \mathbb{Z} v_\sigma(g_i) \subseteq \Delta$$

so $\Delta = \Gamma_S$. Next suppose that $n = \omega + 1$, so $\Gamma_S = \Gamma_\omega + \mathbb{Z}(1,0,0)$. Arguing as above we have that Δ contains Γ_ω (itself isomorphic to $\mathbb{Z}$) and is contained in the free abelian group $\Gamma_\omega + \mathbb{Z}(1,0,0)$. Hence it is itself a free abelian group of dimension 2 with Γ_ω a direct summand, so we can write Δ in the form $\Gamma_\omega + \mathbb{Z}\gamma$ where without loss of generality $0 < \gamma \in \Gamma_S$ and $\gamma \notin \Gamma_\omega$ so $\gamma > \Gamma_\omega$ and hence $\Delta \cong \Gamma_\omega + \mathbb{Z}(1,0,0) = \Gamma_S$ as required.

Now suppose that Δ is the value group of some real place in $\mathcal{M}$ and that U is a nonempty open set in $\mathcal{M}(\mathbb{R}(x,y))$, say with $\sigma \in U$. Since $\sigma(t) = 0$ for some $t \in \mathbb{R}(x)$ with $\mathbb{R}(t) = \mathbb{R}(x)$, we may assume without loss of generality that $\sigma(x) = 0$. Thus by Lemma 2.2 there is an extension of σ to a real place $\sigma' \in \mathcal{M}$. Then σ' is in the open set $\mathrm{res}^{-1}(U)$, so by the proposition above there exists a real place $\tau \in \mathrm{res}^{-1}(U)$ with length at most ω and value group order isomorphic to Δ. If we set $S = \Phi(\tau)$, then the associated sequence $(g_i)_{i<n}$ of S has all its entries in $\mathbb{R}(x,y)$. Thus all $q_i = v_\tau(g_i)$ are in the value group Δ_0 of $\mathrm{res}(\tau)$ and hence

$$\Delta_0 \subseteq \Gamma_S = \mathbb{Z} + \sum_i \mathbb{Z} q_i \subseteq \Delta_0.$$

Hence the set of real places with value group isomorphic to Δ is dense in $\mathcal{M}(\mathbb{R}(x,y))$.

It remains to show that the set of real places on $\mathcal{M}(\mathbb{R}(x,y))$ with discrete rank one value group is dense in $\mathcal{M}(\mathbb{R}(x,y))$. So let U denote a nonempty open set in $\mathcal{M}(\mathbb{R}(x,y))$. As in the previous paragraph we may assume that $\mathrm{res}^{-1}(U)$ is a nonempty open set in $\mathcal{M}$ and hence that $\Phi(\mathrm{res}^{-1}(U))$ is a nonempty open set in $\mathcal{S}$. By Lemma 6.2 there is a signature S of finite length and degree such that if $\tau \in \mathcal{M}$ has $\Phi(\tau) \rhd S$, then $\tau \in \mathrm{res}^{-1}(U)$, so $\mathrm{res}(\tau) \in U$. Pick $L \in \mathbb{Z}$ with $L > e_{n-1} q_{n-1}$, so that

$$S' := S \oplus \langle L + i, \frac{1}{i!} \rangle_{0 \leq i < \omega} \oplus ((1,0,0),0) \in \mathcal{S},$$

and set $\tau = \Phi^{-1}(S')$. If $(g_i)_{i\le\omega}$ is the associated sequence of S', then we have

$$g_\omega = g_n + x^L(\sum_{i=0}^{\infty} \frac{x^i}{i!}) = g_n + x^L e^x$$

(making the obvious abbreviation). Let $w = v_\tau$; it has value group $\Gamma_S + \mathbb{Z}(1,0,0)$, which itself has the isolated subgroup Γ_S. Thus w induces a valuation, say w', on $\mathbb{R}((x))(y)$ which has value group $\mathbb{Z}(1,0,0)$ and maps to 0 all nonzero polynomials in $\mathbb{R}((x))[y]$ of degree less than $\deg S = \deg g_n$. In particular the place, call it μ, corresponding to w' is injective on $\mathbb{R}((x))$. Let w_0 denote the restriction of w to $\mathbb{R}(x,y)$. Since for all $i < n$ we have $g_i \in \mathbb{R}(x,y)$, therefore the value group of w_0 contains all the q_i, $i < n$, i.e., it contains Γ_S. If this containment is actually equality, then we are done: $\mathrm{res}(\tau) \in U$ and its valuation w_0 is discrete rank one. So just suppose the containment is proper. Then there exists a monic $f(y) \in \mathbb{R}(x)[y]$ of least degree with $w(f(y)) \notin \Gamma_S$. We can write

$$f(y) = \sum A_i g_n^i = \sum A_i (g_\omega - x^L e^x)^i$$

where each $A_i \in R(x)[y]$ has degree less than $\deg g_n$ and hence is 0 or is a unit with respect to w'. Now $w(f(y)) \ge \min_i w(A_i g_n^i) \in \Gamma_S$, so $w'(f(y)) > 0$. Therefore applying the place μ we obtain

$$0 = \sum \mu(A_i)\mu(-x^L e^x)^i = \sum \mu(A_i(-1)^i x^{iL})\mu(e^x)^i$$

which says that $\mu(e^x) = \sum \frac{\mu(x)^i}{i!}$ is algebraic over the residue class field of the restriction of μ to $\mathbb{R}(x,y)$. But this restriction is not the trivial place (since $w'(f(y)) \ne 0$) and hence its residue class field is isomorphic to an algebraic extension of $\mathbb{R}(x)$. But this says that $\sum \frac{\mu(x)^i}{i!}$ is algebraic over the rational function field $\mathbb{R}(\mu(x))$, a contradiction (see [1, exercise 3.1(a) on page 173]). Hence w_0 is indeed discrete rank one. □

7. Self-similarity

The space of real places on $\mathbb{R}(x)$ is a simple closed curve, and hence it has large classes of subsets (e.g., the class of closed arcs) such that any two elements of one of those classes are homeomorphic. We now consider similar behavior for $\mathcal{M}$. We discussed the possibility of such "self-similarity" with K. and F.-V. Kuhlmann for $\mathbb{R}(x,y)$ in 2011. Of course for any field F the

image of any subset of $\mathcal{M}(F)$ under a group of automorphisms of F would give such a class; these are not the type of examples considered here. (However, we do note that $\mathbb{R}(x,y)$ has a large class of automorphisms indeed.)

Let $(g_i)_{i\le n}$ be the sequence of polynomials associated with a signature S of finite length and degree and suppose that $e_{n-1}q_{n-1} < q = \frac{a}{b} \in \mathbb{Q}$ where $a,\ b \in \mathbb{Z}$ with $b > 0$; as usual we set $e_{-1}q_{-1} = -\infty$.

THEOREM 7.1. *The set*

$$H_{q,g_n} := H((x^{-a}g_n^b)^2 + 1) = H(v(g_n) \ge q)$$

is up to homeomorphism independent of the choice of S and q; indeed it is homeomorphic to $H(v(y) \ge 0) = H(1 + y^2)$.

We have used above the notation introduced in the proof of Lemma 4.3; the displayed set is of course independent of the choice of a and b. The theorem says, for example, that $H(1 + y^2)$ is also homeomorphic to $H(1 + (x^{-3}y^4 + 2x^{-2}y^2 + x^{-1}y + x^{-1})^2)$ (taking $q = 3$ and g_n to be the polynomial associated with the signature $\langle(1/2, 1), (5/4, 1)\rangle$).

We will remark later on other classes of subsets of $\mathcal{M}$ with the same property.

REMARK 7.2. The monic irreducible polynomials over $\mathbb{R}((x))$ are bijective with the signatures of finite degree [2, Theorem 5.1]; the bijection maps each such signature S' to g'_n where $(g'_i)_{i\le n'}$ is the sequence associated with S'. The restriction in Theorem 7.1 to polynomials associated with signatures of finite length is somewhat artificial and arises in part from our explicit choice of $\{x^i : i \in \mathbb{Z}\}$ as a set of representatives in $\mathbb{R}((x))$ for its value group $\mathbb{Z}$.

Let $\mathcal{S}_{q,S} = \{S' \in \mathcal{S} : S' \rhd S \text{ and } q'_n \ge q\}$.

LEMMA 7.3. *The homeomorphism $\Phi\colon \mathcal{M} \to \mathcal{S}$ carries H_{q,g_n} onto $\mathcal{S}_{q,S}$.*

PROOF. Since $q > e_{n-1}q_{n-1}$, Theorem 3.1 implies that for any $\sigma \in \mathcal{M}$, say with $S' = \Phi(\sigma)$, we have $v_\sigma(g_n) \ge q$ if and only if $S' \rhd S$ and $q'_n \ge q$. □

Therefore in order to prove Theorem 7.1 it suffices to prove

PROPOSITION 7.4. *$\mathcal{S}_{q,S}$ is homeomorphic to $\mathcal{S}_{0,\emptyset}$.*

In order to prove Proposition 7.4 it is convenient to consider the natural bijection $\Upsilon_{q,S} : \mathcal{S}_{q,S} \to S_q$, where S_q is the set of signatures S' with $S' \rhd S$ and $q'_i \ne (1, 0, 0)$ for all $i < n'$ and, finally, if $n' > n$ then $q'_n \ge q$; the bijection

$\Upsilon_{q,S}$ just deletes any term of the form $((1,0,0),0)$ from any element of $\mathcal{S}_{q,S}$ and $\Upsilon_{q,S}^{-1}$ maps each element of $S_q \cap \mathcal{S}$ to itself and any other element S' of S_q to $S' \oplus ((1,0,0),0)$. (The notation "S_q" introduced above will only be used in this section; the notation "S_m" introduced in Notation 4.2 will never occur in this section.)

Let $\mathcal{P}$ denote the set of all presignatures $\widehat{S}$ of length at most ω having $(1,0,0) > \widehat{q}_i \geq 0$ whenever $0 \leq i < \widehat{n}$ with equality only if $\widehat{n} > i = 0$. For any $S' \in S_q$, and $n \leq j < n'$ we will let $\gamma_j^{S',q,S}$ denote q if $j = n$ and denote $e'_{j-1}q'_{j-1}$ if $n < j < n'$. Notice that if $S' \rhd S^* \in S_q$ and $n \leq j < n^*$, then $S' \in S_q$ and

$$\gamma_j^{S',q,S} = \gamma_j^{S^*,q,S}. \tag{7.1}$$

We can now define $\Psi_{q,S}\colon S_q \to \mathcal{P}$ by the formula

$$\Psi_{q,S}(S') = \langle q'_{i+n} - \gamma_{i+n}^{S',q,S}, \theta'_{i+n}\rangle_{0\leq i<n'-n}$$

(where we equate $\omega - n$ with ω) and define $\Omega_{q,S}\colon \mathcal{P} \to S_q$ by the formula

$$\widehat{S} := \Omega_{q,S}(S') = S \oplus \langle q'_i + \gamma_{i+n}^{\widehat{S},q,S}, \theta'_i\rangle_{0\leq i<n'}.$$

Note that this definition of $\widehat{S}$ is inductive (not circular!). For example, $\widehat{q}_n = q'_0 + q$, $\widehat{q}_{n+1} = q'_1 + \widehat{e}_n\widehat{q}_n$, and so on.

Lemma 7.5. *$\Omega_{q,S}$ is the inverse of $\Psi_{q,S}$.*

Proof. Let $S' \in \mathcal{P}$ and set

$$S'' := \Omega_{q,S}(S') = S \oplus \langle q'_i - \gamma_{i+n}^{S'',q,S}, \theta'_i\rangle_{0\leq i<n'}$$

so that

$$\widehat{S} := \Psi_{q,S}\Omega_{q,S}(S') = \langle q'_i + \gamma_{i+n}^{S'',q,S} - \gamma_{i+n}^{S'',q,S}, \theta'_i\rangle_{0\leq i<n'} = S'$$

and on the other hand, if $S' \in S_q$, then

$$\widehat{S} := \Omega_{q,S}\Psi_{q,S}(S') = S \oplus \langle q'_{i+n} - \gamma_{i+n}^{S',q,S} + \gamma_{i+n}^{\widehat{S},q,S}, \theta'_{i+n}\rangle_{0\leq i<n'-n} = S'$$

since by induction on i (where $0 \leq i < n - n'$) we have $\gamma_{i+n}^{S',q,S} = \gamma_{i+n}^{\widehat{S},q,S}$. After all, for $i = 0$ both sides of the equation equal q and if the equality holds for all $0 \leq j < i < n' - n$, then $\widehat{q}_k = q'_k = q_k$ for $k < n$ and for $0 \leq j < i$

we have $\widehat{q}_{n+j} = q'_{n+j} - \gamma_{n+j}^{S',q,S} + \gamma_{n+j}^{\widehat{S},q,S} = q'_{n+j}$, so $\widehat{e}_{n+j} = e'_{n+j}$, and hence $\gamma_{i+n}^{S',q,S} = \gamma_{i+n}^{\widehat{S},q,S}$. □

Suppose that $T = \langle q_i^T, \theta_i^T \rangle_{i<t}$ is a signature of finite length and degree and that $e_{t-1}q_{t-1} < r \in \mathbb{Q}$. Define $\mathcal{S}_{r,T}$ and T_r analogously to $\mathcal{S}_{q,S}$ and S_q. Let $\Delta := \Delta_{q,S,r,T}$ be the composition

$$\mathcal{S}_{q,S} \xrightarrow{\Upsilon_{q,S}} S_q \xrightarrow{\Psi_{q,S}} \mathcal{P} \xrightarrow{\Omega_{r,T}} T_r \xrightarrow{\Upsilon_{r,T}^{-1}} \mathcal{S}_{r,T}.$$

Using Lemma 7.5 one checks that $\Delta_{r,T,q,S}$ is the inverse of $\Delta := \Delta_{q,S,r,T}$. Thus in order to prove Proposition 7.4–and hence also Theorem 7.1–it suffices to prove the

PROPOSITION 7.6. Δ *is continuous.*

The remainder of this section is devoted to the proof of the above proposition and to a remark that Theorem 7.1 can be extended to some other classes of subsets of $\mathcal{M}$, one of which includes $\mathcal{M}$ itself.

Another notational scheme for presignatures will be useful. For any presignature M we will write $M = \langle q_i^M, \theta_i^M \rangle_{i<|M|}$ (so $|M|$ denotes the length of M). (We used much of this notation for the fixed signature T above.)

It will be convenient to set $\Delta_0 = \Omega_{r,T}\Psi_{q,S}$, so $\Delta = \Upsilon_{r,T}^{-1}\Delta_0\Upsilon_{q,S}$. The map Δ_0 is a bijection from S_q onto T_r which preserves the relation $\rhd$. For any $S' \in S_q$ we have

$$\Delta_0(S') = T \oplus \langle q'_{i+n} - \gamma_{i+n}^{S',q,S} + \gamma_{i+t}^{\Delta_0(S'),r,T}, \theta'_{i+n} \rangle_{0 \le i < n'-n}$$

so if $t \le j < t + n' - n$, then

$$q_j^{\Delta_0(S')} = q'_{j+n-t} - \gamma_{j+n-t}^{S',q,S} + \gamma_j^{\Delta_0(S'),r,T}.$$

If $S' \rhd M \in S_q$ and M has finite length and degree, then

$$|\Delta_0(M)| = |M| - n + t,$$

so if $t \le j < |M| - n + t$, then by equation (7.1) we also have

$$q_j^{\Delta_0(S')} = q'_{j+n-t} - \gamma_{j+n-t}^{M,q,S} + \gamma_j^{\Delta_0(M),r,T}. \tag{7.2}$$

Now suppose that $\mathcal{N}$ is a subbasic open set in $\mathcal{S}$, so it has one of the forms $\mathcal{N}_{M,\delta}$, $\mathcal{N}_{M,\beta,\delta}$ or $\mathcal{N}_{M,\alpha,\beta,\delta}$ (see Section 4). It suffices to show that $\Delta(\mathcal{N} \cap \mathcal{S}_{q,S})$

is open in $\mathcal{S}_{r,T}$ (since $\Delta_{r,T,q,S}$ and $\Delta_{q,S,r,T}$ are inverses); indeed we shall show that $\Delta(\mathcal{N}\cap\mathcal{S}_{q,S})$ is essentially one of our subbasic open subsets of $\mathcal{S}_{r,T}$.

Without loss of generality we may assume that $\mathcal{N}\cap\mathcal{S}_{q,S}\notin\{\emptyset,\mathcal{S}_{q,S}\}$; hence by Lemma 5.5 we may assume $M\rhd S$ in all cases. (After all, if the assertion "$M\rhd S$" is false, then $\mathcal{N}\cap\mathcal{N}_S\in\{\emptyset,\mathcal{N}_S\}$, so $\mathcal{N}\cap\mathcal{S}_{q,S}=\mathcal{N}\cap\mathcal{S}_{q,S}\cap\mathcal{N}_S\in\{\emptyset,\mathcal{S}_{q,S}\}$.) We consider four cases.

First suppose that $\mathcal{N}=\mathcal{N}_{M,\beta,\delta}$. Let $S'\in S_q$ (so that $\Upsilon_{q,S}^{-1}(S')$ is an arbitrary element of $\mathcal{S}_{q,S}$). Then $\Upsilon_{q,S}^{-1}(S')\in\mathcal{N}$ if and only if $S'\rhd M$ and if $S'\neq M$ then $q'_{|M|}\geq\beta$ with equality only if $|\theta'_{|M|}|<\delta$, i.e., if and only if (applying equation (7.2)) $\Delta_0(S')\rhd\Delta_0(M)$ and if $\Delta_0(S')\neq\Delta_0(M)$, then

$$q_{|\Delta_0(M)|}^{\Delta_0(S')}=q'_{|M|}+\gamma\geq\beta+\gamma$$

where $\gamma=-\gamma_{|M|}^{M,q,S}+\gamma_{|\Delta_0(M)|}^{\Delta_0(M),r,T}$, with equality holding only if

$$\delta>|\theta'_{|M|}|=|\theta_{|\Delta_0(M)|}^{\Delta_0(S')}|,$$

that is, if and only if $\Upsilon_{r,T}(\Delta_0(S'))\in\mathcal{N}_{\Delta_0(M),\beta+\gamma,\delta}$. Thus $\Delta(\mathcal{N}\cap\mathcal{S}_{q,S})=\mathcal{N}_{\Delta_0(M),\beta+\gamma,\delta}\cap\mathcal{S}_{r,T}$.

Next suppose that $\mathcal{N}=\mathcal{N}_{M,\alpha,\beta,\delta}$. Arguing as in the previous paragraph we see that

$$\Delta(\mathcal{N}\cap\mathcal{S}_{q,S})=\mathcal{N}_{\Delta_0(M),\alpha+\gamma,\beta+\gamma,\delta}\cap\mathcal{S}_{r,T}.$$

Now assume that $\mathcal{N}=\mathcal{N}_{M,\delta}$ and $S=M$. Let $S'\in S_q$. Then $\Upsilon_{q,S}^{-1}(S')\in\mathcal{N}$ if and only if $q'_{|M|}\leq e_{|M|-1}q_{|M|-1}+\delta$ (since $S'\rhd M$) and hence if and only if

$$\begin{aligned}q_{|\Delta_0(M)|}^{\Delta_0(S')}&=q_t^{\Delta_0(S')}=q'_{|M|}-\gamma_n^{S',q,S}+\gamma_t^{\Delta_0(S'),r,T}\\&=q'_{|M|}-q+r\leq e_{|M|-1}q_{|M|-1}+\delta-q+r.\end{aligned}$$

Note that $e_{n-1}q_{n-1}+\delta-q\geq 0$ if $\Upsilon_{q,S}^{-1}(S')\in\mathcal{N}$. Since $\Delta_0(S')\rhd\Delta_0(S)=\Delta_0(M)$, it follows that $\Upsilon_{q,S}^{-1}(S')\in\mathcal{N}$ if and only if

$$\Delta(\Upsilon_{q,S}^{-1}(S'))\in\mathcal{N}_{\Delta_0(M),e_{n-1}q_{n-1}+\delta-q+r}$$

if $T=\emptyset$ (in this case $\Delta_0(M)=\emptyset$); and

$$\Delta(\Upsilon_{q,S}^{-1}(S'))\in\mathcal{N}_{\Delta_0(M),e_{n-1}q_{n-1}+\delta-q+r-e_{t-1}^Tq_{t-1}^T}$$

if $T \neq \emptyset$ (note that $q_j^{\Delta_0(S')} = q_j^T$ and so $e_j^{\Delta_0(S')} = e_j^T$ for all $j < t$). Thus in either case $\Delta(\mathcal{N} \cap \mathcal{S}_{q,S})$ is an open set in $\mathcal{S}_{r,T}$.

Finally, consider the case that $\mathcal{N} = \mathcal{N}_{M,\delta}$ and $M \neq S$ (so $|M| > n$). It suffices to show that $\Delta(\mathcal{N} \cap \mathcal{S}_{q,S}) = \mathcal{N}_{\Delta_0(M),\delta} \cap \mathcal{S}_{r,T}$. Note that $\Delta_0(M)_{|\Delta_0(M)|-1} = \Delta_0(M_{|M|-1})$ since both are subsequences of $\Delta_0(M)$ with length $|M|-n+t-1$. Let $S' \in S_q$. Then $\Upsilon_{q,S}^{-1}(S') \in \mathcal{N}$ if and only if $S' \rhd M_{|M|-1}$, $q'_{|M|-1} = q^M_{|M|-1}$, $|\theta'_{|M|-1} - \theta^M_{|M|-1}| < \delta$, and if $\theta'_{|M|-1} = \theta^M_{|M|-1}$, then $q'_{|M|} \leq e'_{|M|-1} q'_{|M|-1} + \delta$. Thus $\Upsilon_{q,S}^{-1}(S') \in \mathcal{N}$ if and only if $\Delta_0(S') \rhd \Delta_0(M)_{|\Delta_0(M)|-1}$, and

$$\begin{aligned} q^{\Delta(S')}_{|\Delta_0(M)|-1} &= q'_{|M|-1} - \gamma^{M,q,S}_{|M|-1} + \gamma^{\Delta_0(M),r,T}_{|\Delta_0(M)|-1} \\ &= q^M_{|M|-1} - \gamma^{M,q,S}_{|M|-1} + \gamma^{\Delta_0(M),r,T}_{|\Delta_0(M)|-1} = q^{\Delta_0(M)}_{|\Delta_0(M)|-1} \end{aligned}$$

and

$$|\theta^{\Delta_0(S')}_{|\Delta_0(M)|-1} - \theta^{\Delta_0(M)}_{|\Delta_0(M)|-1}| = |\theta'_{|M|-1} - \theta^M_{|M|-1}| < \delta$$

and, finally, if $\theta^{\Delta_0(S')}_{|\Delta_0(M)|-1} = \theta^{\Delta_0(M)}_{|\Delta_0(M)|-1}$, then

$$\begin{aligned} q^{\Delta_0(S')}_{\Delta_0(M)} &= q'_{|M|} - \gamma^{M,q,S}_{|M|} + \gamma^{\Delta_0(M),r,T}_{|\Delta_0(M)|-1} \\ &\leq \delta + e'_{m-1} q'_{m-1} - \gamma^{M,q,S}_{|M|} + \gamma^{\Delta_0(M),r,T}_{|\Delta_0(M)|} = \delta + e^{\Delta_0(M)}_{|\Delta_0(M)|-1} q^{\Delta_0(M)}_{|\Delta_0(M)|-1} \end{aligned}$$

since $|\Delta_0(M)| = |M| - n + t > t$. Thus as required, $\Upsilon_{q,S}^{-1}(S') \in \mathcal{N}$ if and only if $\Delta(\Upsilon_{q,S}^{-1}(S')) = \Upsilon_{r,T}^{-1}(\Delta_0(S')) \in \mathcal{N}_{\Delta_0(M),\delta}$. This completes the proof of Proposition 7.6.

REMARK 7.7. Let $q \in \mathbb{Q}$ and S be as above. We define subsets

$$\mathcal{S}^{(1)}_{q,S} \supseteq \mathcal{S}^{(2)}_{q,S} \supseteq \mathcal{S}^{(3)}_{q,S} \supseteq \mathcal{S}^{(4)}_{q,S}$$

of $\mathcal{S}$ by setting

$$\mathcal{S}^{(i)}_{q,S} = \{S' \in \mathcal{S} : S' \rhd S \text{ and } q'_n \succ_i q\}$$

where the relations $\succ_i$ on Γ are defined by

$$a \succ_1 b \text{ if } a \geq b^- \text{ and } b \in \mathbb{Q};$$
$$a \succ_2 b \text{ if } a \geq b \text{ and } b \in \mathbb{Q};$$
$$a \succ_3 b \text{ if } a \geq b^+ \text{ and } b \in \mathbb{Q};$$

$$a \succ_4 b \text{ if } a > b^+ \text{ and } b \in \mathbb{Q}.$$

Then for each $i \leq 4$ the sets $\mathcal{S}^{(i)}_{q,S}$ are up to homeomorphism independent of the choice of q and S. The space $\mathcal{S}$ itself, and indeed each of the spaces $\mathcal{N}_S$, is homeomorphic to each of the spaces $\mathcal{S}^{(3)}_{q,S}$. Also $\mathcal{S}^{(1)}_{q,S} = \mathcal{S}^{(2)}_{q,S} \cup \{S \oplus (q^-, 0)\}$ is closed (and hence compact) with interior $\mathcal{S}^{(2)}_{q,S} = \mathcal{S}_{q,S}$, while $\mathcal{S}^{(3)}_{q,S} = \mathcal{S}^{(4)}_{q,S} \cup \{S \oplus (q^+, 0)\}$ is closed with interior $\mathcal{S}^{(4)}_{q,S}$. The equalities asserted here can be proved using Theorem 3.1; the assertions about closures and interiors follow easily from the properties of $\mathcal{M}$ and the bijection Φ. The independence claims above can now be proven from the independence established earlier for the sets $\mathcal{S}_{q,S} = \mathcal{S}^{(2)}_{q,S}$ (note that the homeomorphism $\Delta_{q,S,r,T}$ maps $\mathcal{S}^{(i)}_{q,S}$ to $\mathcal{S}^{(i)}_{r,T}$ for $i = 3$ and $i = 4$ and that if S is nontrivial, then $\mathcal{N}_S$ is the one-point compactification of a union of sets of the form $\mathcal{S}_{q,S}$). The argument that the set $\mathcal{S}$ is homeomorphic to, say, $\mathcal{S}^{(3)}_{0,\emptyset}$ is similar to the arguments used to prove Proposition 7.4 but also uses an order isomorphism $f\colon \Gamma \to [0^+, (1,0,0)]$ which maps $\mathbb{Q}$ onto $\mathbb{Q} \cap (0,\infty)$ to define a homeomorphism from $\mathcal{S}$ to $\mathcal{S}^{(3)}_{0,\emptyset}$.

8. Strict systems and the bijection $\Phi\colon \mathcal{M} \to \mathcal{S}$

In this section we relate the signatures of this paper to the "strict systems of polynomial extensions" of [3, Definition 2]. This connection is then exploited to give a proof independent of the results of [2] of Theorem 3.1, which established the bijection $\Phi\colon \mathcal{M} \to \mathcal{S}$ and its basic properties. The arguments of [2] are made in the context of the extensions of the valuation on an *arbitrary* maximal field F (e.g., $\mathbb{R}((x))$) to its rational function field $F(y)$. Operating at this level of generality requires conceptual machinery which can be avoided in the special situation of this paper. We continue to use the notation introduced in Notation 4.2; v will denote the canonical x-adic valuation on $\mathbb{R}((x))$. We begin with a counting argument.

LEMMA 8.1. *Let S be a signature and w be an extension of v to a valuation on $\mathbb{R}((x))[y]$ such that $w(g_i) = q_i$ for all $i < n$. Then $\{g^\sigma : \sigma \in J_n\}$ is a basis for the $\mathbb{R}((x))$-linear space "R_S" of all polynomials in $\mathbb{R}((x))[y]$ of degree less than $\deg S$. Moreover, for any $a_\sigma \in \mathbb{R}((x))$ (all but a finite number of which are zero) we have*

$$w\Big(\sum_{\sigma \in J_n} a_\sigma g^\sigma\Big) = \min_{\sigma \in J_n} \big(v(a_\sigma) + \sum_{i<n} \sigma(i) q_i\big). \tag{8.1}$$

If $\deg S < \infty$, *then*

$$w(R_S) = \{\infty\} \cup (\mathbb{Z} + \sum_{i<n} \mathbb{Z}q_i) = \{\infty\} \cup \Gamma_n. \tag{8.2}$$

Finally, if $\deg S = \infty$, *then there is a unique element of* $\mathcal{M}$ *inducing* w.

(Valuations on commutative rings are defined just as in the field case, except that nonzero elements can have value ∞ [1, Definition 1, p. 101].)

Proof. The first assertion follows from the fact that for each $m < \deg S$ there exists a unique $\sigma \in J_n$ with $\deg g^\sigma = m$ (after all, $\deg g_{i+1} = e_i \deg g_i$ if $i + 1 < n$). Similarly, suppose that $e_i < \infty$ for all $i < n$; then for each $\rho \in \mathbb{Z} + \sum_{i<n} \mathbb{Z}q_i$ there exists a unique $s \in \mathbb{Z}$ and $\sigma \in J_n$ with $\rho = s + \sum \sigma(i)q_i$ (recall that if $i < n$ then $e_i = (\Gamma_{i+1} : \Gamma_i)$). Hence the polynomials $(g^\sigma : \sigma \in J_n)$ have distinct values modulo $\mathbb{Z} = v(\mathbb{R}((x))^\bullet)$; formula (8.1) above is an immediate consequence. The situation when $e_i = \infty$ for some i is similar; in this case $i = n - 1$ and the polynomials g^σ still have distinct values modulo $\mathbb{Z}$.

Equation (8.2) follows directly from equation (8.1). Another consequence of equation (8.1) is the fact that any unit of w is congruent to an element of $\mathbb{R}((x))$, and hence the residue class ring of w is $\mathbb{R}$. It follows that w is induced by an unique element of $\mathcal{M}$. □

The next lemma shows that each signature of finite degree determines a "strict system of polynomial extensions" in the sense of [3, Definition 2]. In the same lemma we will establish a fundamental property of associated polynomials (cf. [2, the fundamental lemma]). Proving both parts of Lemma 8.2 together facilitates its inductive proof.

Lemma 8.2. *Let* $S \in \mathcal{S}$. *If* $0 \le m < n$, *then:*

(I) *Suppose that* w *is an extension of* v *to* $\mathbb{R}((x))[y]$ *with* $w(g_m) > \mathbb{Q}$ *if* $m = \omega$ *and* $w(g_m) > e_{m-1}q_{m-1}$ *if* $\omega > m > 0$. *Then* $w(g_i) = q_i$ *for all* $i < m$.

(II) *Let* $u_0 < u_1 < \cdots < u_t$ *be the distinct elements in the set* $\{i \le m : i = m \text{ or } e_i \ne 1\}$. *Pick extensions* w_i *of* v *to a valuation on* $\mathbb{R}((x))[y]$ *with* $w_i(g_{u_i}) = \infty$. *Set* $\gamma_0 = -\infty$, *and for* $i > 0$ *set* $\gamma_i = e_{u_{i-1}}q_{u_{i-1}}$. *Then* $(g_{u_i}, w_i, \gamma_i)_{i \le t}$ *is a strict system of polynomial extensions over* $(\mathbb{R}((x)), v)$.

Proof. Both conditions are valid if $m = 0$. We suppose inductively that $m > 0$ and that they are both valid for all smaller values. We begin by proving (II) by verifying each of the six conditions (A) through (F) of [3, Definition 2, p. 2170]. In terms of the notation of this paper we have:

(A) Note that $\deg g_{u_0} = \prod_{i<u_0} e_i = 1$, so $g_{u_0} = y - a$ for some $a \in \mathbb{R}((x))$; also if $f(y) = \sum b_i(y-a)^i$, then $w_0(f(y)) = w_0(b_0) = v(f(a))$.

Now suppose that $0 \le i < t$. Then

(B) $\deg g_{u_{i+1}} = e_{u_i} \deg g_{u_i} > \deg g_{u_i}$, and

(D) $w_{i+1}(g_{u_{i+1}}) = \infty$.

We can write

$$g_{u_{i+1}} = g_{u_i}^{e_{u_i}} + \sum_{u_i \le j < u_{i+1}} \theta_j x^{s_j} g^{\sigma_j}$$

or more properly (at least in the case that $u_{i+1} = \omega$; see the definition of g_ω in Section 3)

$$= g_{u_i}^{e_{u_i}} + \sum_r (\sum_\tau (\sum_k \theta_k x^{s_k})(g^\tau / g_{u_i}^r)) g_{u_i}^r$$

(where in the above sums $0 \le r < e_{u_i}$, $\tau \in J_{u_{i+1}}$ with $\tau(u_i) = r$, and $u_i \le k < u_{i+1}$ with $\sigma_k = \tau$), which we will as usual write in the form

$$= g_{u_i}^{e_{u_i}} + \sum_{0 \le r < e_{u_i}} A_r g_{u_i}^r.$$

For any term $\theta_k x^{s_k} g^\tau / g_{u_i}^r$ of A_r, we have

$$w_i(\theta_k x^{s_k} g^\tau / g_{u_i}^r) = s_k + (\sum_{0 \le j \le u_i} \tau(j) q_j) - r q_{u_i} = e_k q_k - r q_{u_i}$$

(using our induction hypotheses; since $w_i(g_{u_i}) = \infty$, therefore $w_i(g_j) = q_j$ for all $j < u_i$). Now if $k = u_i$, then $\sigma_k = \sigma_{u_i}$ and $r = 0$, so

$$e_k q_k - r q_{u_i} = e_{u_i} q_{u_i}$$

and otherwise $k > u_i$ and hence

$$e_k q_k - r q_{u_i} = q_k - r q_{u_i} > (e_{u_i} - r) q_{u_i}.$$

Thus

$$w_i(g_{u_{i+1}}) = w_i(A_0) = e_{u_i} q_{u_i} = \gamma_{i+1},$$

proving (C) of [3, Definition 2]. Also, for all r,

$$w_i(A_r)/(e_{u_i} - r) \ge (e_{u_i} - r) q_{u_i} / (e_{u_i} - r) = q_{u_i} = w_i(A_0)/e_{u_i} > \gamma_i,$$

proving (E) of [3, Definition 2] (the last inequality above follows from the fact that $S \in \mathcal{S}$). Finally if $e > 0$ is minimal with $ew_i(A_0) \in e_{u_i} w_i(\mathbb{R}((x))[y])$, i.e., with

$$ee_{u_i} q_{u_i} \in e_{u_i}(\mathbb{Z} + \sum_{j<u_i} \mathbb{Z} q_j)$$

(see equation (8.2) in Lemma 8.1), then $e = e_{u_i}$ by the choice of e and e_{u_i}, so the polynomial of equation (1) of [3, Definition 2] is linear and hence irreducible. This proves part (F) of [3, Definition 2] and hence proves that $(g_{u_i}, w_i, \gamma_i)_{i \le t}$ is indeed a strict system as claimed.

We next prove part (I) of our lemma. First suppose that $\deg g_m = 1$, so $g_m = y + \sum_{0 \le j < m} \theta_j x^{s_j}$, where for each $j < m$ we have $e_j = 1$ and $s_j = q_j$, so $q_0 < q_1 < \cdots < q_m$. For each $0 \le j < m$ we have

$$\begin{aligned} w(g_j) &= w(g_m - \sum_{j \le k < m} \theta_k x^{s_k}) \\ &= \min(w(g_m), w(\sum_{j \le k < m} \theta_k x^{s_k})) = q_j \end{aligned}$$

since by hypothesis either $w(g_m) > e_{m-1} q_{m-1} \ge q_j$ or $w(g_m) > \mathbb{Q}$. Next suppose that $\deg g_m > 1$, so $t \ge 1$. Now $w(g_{u_t}) > \mathbb{Q}$ if $m = \omega$ and otherwise

$$w(g_{u_t}) = w(g_m) > e_{m-1} q_{m-1} \ge e_{u_{t-1}} q_{u_{t-1}} = \gamma_t$$

so by [3, Proposition 5], for $i = t - 1$ we have

$$w(g_{u_i}) = \gamma_{i+1}/e_{u_i} = e_{u_i} q_{u_i}/e_{u_i} = q_{u_i} > e_{u_i - 1} q_{u_i - 1} \ge e_{u_{i-1}} q_{u_{i-1}} = \gamma_i.$$

Thus by our induction hypothesis we also have $w(g_j) = q_j$ for all $j \le u_{t-1}$. If $u_t = u_{t-1} + 1$, we are finished, so suppose that $u_t > u_{t-1} + 1$. Suppose that $u_t > j > u_{t-1}$; we must show that $w(g_j) = q_j$. Write

$$g_m = g_j + \sum_{j \le k < m} \theta_k x^{s_k} g^{\sigma_k}$$

where if $j \le i < m$ we have

$$q_i = e_i q_i = s_i + \sum_{0 \le k < i} \sigma_i(k) q_k = w(\theta_i x^{s_i} g^{\sigma_i})$$

(recall that $0 \le \sigma_i(k) < e_k = 1$ if $u_{t-1} < k < i$). Since $w(g_m) > \mathbb{Q}$ or $m < \omega$ and $q_j < q_{j+1} \le \cdots \le q_{m-1} = e_{m-1}q_{m-1} < w(g_m)$, therefore

$$w(g_j) = w(g_m - \sum_{j \le i < m} \theta_i x^{s_i} g^{\sigma_i}) = q_j$$

as required. □

Let w be an extension of v to a valuation on $\mathbb{R}((x))[y]$. We say a signature S is *associated with* w if, first, $w(g_i) = q_i$ for all $i < n$ and, second, if S has finite nonzero length and finite degree, then $w(g_n) > e_{n-1}q_{n-1}$ and if S has length ω and finite degree, then $w(g_\omega) > \mathbb{Q}$. We also say that S is *associated with* $\sigma \in \mathcal{M}$ if it is associated with v_σ (or, more precisely, with the restriction of v_σ to $\mathbb{R}((x))[y]$) in the above sense.

We now use Lemmas 8.1 and 8.2 to reprove Theorem 3.1 without reference to [2]*. In particular we show that there is a unique element of $\mathcal{S}$ associated with each element of $\mathcal{M}$ and that each element of $\mathcal{S}$ is associated with a unique element of $\mathcal{M}$, so that the map Φ of Theorem 3.1 is well-defined and bijective.*

Proof. Let $\sigma \in \mathcal{M}$. By a (barely transfinite) induction we see that there is a signature S maximal with respect to $\rhd$ (possibly of length 0) associated with σ. Just suppose $S \notin \mathcal{S}$. First assume that $n = \omega$. Then g_n is the "limit" of the g_k $(k < \omega)$, so that for sufficiently large finite k we have

$$v_\sigma(g_n) = v_\sigma(g_k + g_n - g_k) = v_\sigma(g_k + \sum_{r \ge k} \theta_r x^{s_r} g^{\sigma_r}) \ge q_k.$$

Thus $v_\sigma(g_n) = (1,0,0)$ (see Section 3; note that g_n is monic of minimal degree with $v_\sigma(g_n) \notin \mathbb{Q}$) and $S \oplus ((1,0,0),0)$ is associated with σ, contradicting the maximality of S. So assume $n \ne \omega$, so $n < \omega$. If $v_\sigma(g_n) \notin \mathbb{Q}$, then $S \oplus (v_\sigma(g_n),0)$ is associated with σ, again a contradiction. Thus $v_\sigma(g_n) \in \mathbb{Q}$, say with $e > 0$ least with $ev_\sigma(g_n) = s + \sum_{i<n} \tau(i)q_i$ for some $s \in \mathbb{Z}$ and $\tau \in J_n$. Then for some $\theta \in \mathbb{R}\backslash\{0\}$ we have $\sigma(g_n^e/x^s g^\tau) = -\theta$, so

$$v_\sigma(g_n^e + \theta x^s g^\tau) > v_\sigma(g_n^e) = ev_\sigma(g_n).$$

Hence $S \oplus (v_\sigma(g_n),\theta)$ is associated with σ, again a contradiction. (Note that $g_n^e + \theta x^s g^\tau$ is a polynomial associated with the signature $S \oplus (v_\sigma(g_n),\theta)$.) Hence $S \in \mathcal{S}$.

Next, suppose that a second signature $S^* \in \mathcal{S}$ is associated with σ. Suppose inductively that $q_i = q_i^*$ and $\theta_i = \theta_i^*$ for all $i < m$ for some $m < \min(n, n^*)$. If $m = \omega$, then $q_\omega = (1,0,0) = q_\omega^*$ and $\theta_\omega = 0 = \theta_\omega^*$ so $S = S^*$. So assume $m < \omega$. Then $g_i = g_i^*$ for all $i \le m$, so $q_m = v_\sigma(g_m) = q_m^*$ and

$\theta_m = \sigma(-g_m^{e_m}/x^{s_m}g^{\sigma_m}) = \theta_m^*$. We conclude that $S = S^*$, so the map Φ of Theorem 3.1 is well-defined. Φ is also injective by Lemma 8.1 (formula (8.1) shows that S determines v_σ, which is induced by a unique element of $\mathcal{M}$). We now prove that Φ is surjective.

Let $S \in \mathcal{S}$. We first suppose that $n \neq \omega$. In this case there exists an m with $n = m + 1$. By Theorem 2.2.1 of [7, page 31] there is a valuation $w\colon \mathbb{R}((x))(g_m) \to \Gamma_S \cup \{\infty\}$ with $w(\sum a_i g_m^i) = \min_i(v(a_i) + iq_m)$ for all $a_i \in \mathbb{R}((x))$. The valuation w extends to a valuation w_S on $\mathbb{R}((x))(y)$. Since $w_S(g_m) = q_m > e_{m-1}q_{m-1}$ if $m < \omega$ and $w_S(g_m) = (1, 0, 0) > \mathbb{Q}$ if $m = \omega$, therefore by (I) of Lemma 8.2 the signature S is associated with w_S. By the last part of Lemma 8.1, w_S is the valuation associated with an element of $\mathcal{M}$.

We next suppose that $n = \omega$. For each $i < n$ let w_i be the extension of v to $\mathbb{R}((x))[y]$ with $w_i(g_i) = \infty$. This extension is unique since $\mathbb{R}((x))$ is Henselian and g_i is irreducible in $\mathbb{R}((x))[y]$ [3, Proposition 5(c)]. Note that if $s < t$ then w_s and w_t agree on all polynomials of degree less than $\deg g_s$ (use Lemma 8.2 to apply Lemma 8.1). Since $\deg g_i \to \infty$ as $i \to \infty$, we can therefore define $w^*\colon \mathbb{R}((x))[y] \to \mathbb{Q} \cup \{\infty\}$ by letting $w^*(h)$ be the common value of the $w_i(h)$ for all i with $\deg g_i > \deg h$ for any $h \in \mathbb{R}((x))[y]$. Then w^* is easily checked to be a valuation on $\mathbb{R}((x))[y]$ extending v. For example, if $a, b \in \mathbb{R}((x))[y]$ and $r > \deg a + \deg b$, then

$$w^*(ab) = w_r(ab) = w_r(a) + w_r(b) = w^*(a) + w^*(b).$$

Since 0 is the only element of $\mathbb{R}((x))[y]$ which w^* maps to ∞, therefore w^* extends in the obvious way to a valuation $w\colon \mathbb{R}((x))(y) \to \mathbb{Q} \cup \{\infty\}$. Now $w(g_i) = q_i$ for all $i < \omega$, so S is associated with w and hence with the unique real place inducing w (see Lemma 8.1). This completes the proof of surjectivity and hence the proof of part (A) of Theorem 3.1. Part (B) is just a special case of Lemma 8.1. Now suppose that S, m, and ρ are as in the statement of part (C). Write $S' = \Phi(\rho)$. By Lemma 8.2(I) we have $v_\rho(g_i) = q_i$ for all $i < m$. We first show that $S' \rhd S_m$. We may assume by induction that $S' \rhd S_{m-1}$. Then $g_i = g_i'$ for all $i < m$. Now $v_\rho(g_m) > e_{m-1}q_{m-1} = v_\rho(g_{m-1}^{e_{m-1}})$; therefore $\theta_{m-1} = \rho(-g_{m-1}^{e_{m-1}}/x^{s_{m-1}}g^{\sigma_{m-1}})$. We get the analogous expression for θ'_{m-1} by replacing S by S', so $\theta_{m-1} = \theta'_{m-1}$ and hence $S' \rhd S_m$ as required. Now let w be an extension of v to $\mathbb{R}((x))[y]$ with $w(g_m) = \infty$. If g_m properly factored $g_m = kh$, then by Lemma 8.1 both h and k and hence g_m would have finite value under w, a contradiction. So g_m is irreducible. To prove the last assertion of part (C) apply part (I) of Lemma 8.2 to both v_ρ and the valuation w above and then use formula (8.1). □

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