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CONSTRUCTION OF OBSERVABLE AND MDP CONVOLUTIONAL CODES WITH GOOD DECODABLE PROPERTIES BY ISO REPRESENTATIONS

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This paper addresses the construction of observable and maximum distance profile convolutional codes over finite fields that exhibit good performance with some available decoding algorithms for convolutional codes. Our construction is based on the use of input/state/output representations and the invariance of certain properties of linear systems under various group actions. This framework allows us to systematically generate new convolutional codes from existing ones while preserving key decoding and distance properties.

Keywords: convolutional codes, linear systems, decoding, maximum distance profile.

1. Introduction

Since Shannon introduced entropy as a measure of information in 1948, coding theory has developed into a central area of research within information theory. Among the most important types of error-correcting codes are convolutional ones, which are designed to transmit and recover information over communication channels.

A fundamental challenge in the theory of codes is the design of codes with desirable structural and performance properties (Kuriata, 2008). In the case of convolutional codes, this includes non-catastrophicity, optimal distance profiles, and robust decoding capabilities. Over the years, researchers have approached this problem from several perspectives, leading to the development of algebraic, combinatorial, and system-theoretic frameworks.

In this article, we focus on the linear systems viewpoint that models the dynamics of the convolutional code, describing it as a free submodule $\mathcal{C} \subset \mathbb{F}[z]^n$ of rank k. This representation leads naturally to an input/state/output (ISO) model, which captures the evolution of the encoder as a reachable discrete-time linear system. ISO models are particularly useful because they enable the application of system-theoretic tools, such as reachability and observability, to analyze and construct convolutional codes.

A key advantage of ISO representations lies in their ability to characterize non-catastrophic convolutional codes (Rosenthal *et al.*, 1996). By ensuring that the system is both reachable and observable, one can guarantee that the associated encoder provides a non-catastrophic convolutional code. This idea has been successfully extended to construction of multidimensional convolutional codes (Napp *et al.*, 2010; Pinto and Simões, 2017; Climent *et al.*, 2018) and periodic time-varying ones (Napp *et al.*, 2017; 2019), showing the versatility of the ISO approach. Also, it has been applied to construct combined codes such as concatenated and product convolutional ones (Climent *et al.*, 2007; 2021; DeCastro-García and García-Planas, 2018).

Moreover, the system-theoretic structure facilitates the development of algebraic decoding algorithms, especially for erasure and noisy channels. A prominent example is the decoding strategy based on ISO representations proposed by Rosenthal (1999). This line of work has inspired further advances, including efficient algorithms tailored to the erasure channel (Tomás *et al.*, 2012; Lieb and Rosenthal, 2021), as well as adaptations for particular code structures (Muñoz Castañeda *et al.*, 2019; Martín Sánchez and Plaza-Martín, 2022).

Another essential aspect in convolutional coding

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is the distance profile of the code, which ensures an optimal recovery rate. In this context, maximum distance profile (MDP) convolutional codes have been identified as optimal in terms of column distances. Therefore, finding methods to construct such codes presents an interesting problem in coding theory. This has led to several algebraic works that provide general constructions for MDP convolutional codes (Almeida *et al.*, 2016; Lieb, 2019; Muñoz Castañeda and Plaza-Martín, 2021), including the use of ISO representations (Hutchinson *et al.*, 2005; Tomás *et al.*, 2012).

The goal of this article is to provide an algebraic method for constructing observable convolutional codes with desirable decoding and distance properties. The approach involves considering group actions on ISO representations, which allow us to generate new convolutional codes from a given one. The main advantage of these group actions is that they offer a new way to construct convolutional codes by applying well-established procedures that have proven effective in terms of distance and decoding properties.

This article is organized as follows. Section 2 presents an overview of the preliminary results. In Sections 3 and 4, we prove our main results. Finally, the conclusions and references are provided.

2. Preliminary results

In this section, we provide an overview of preliminary results related to the research. In classical convolutional coding theory, the input alphabet channel is the finite field $\mathbb{F}_q$. From now on, we will denote it by $\mathbb{F}$, assuming that $q = p^r$.

2.1. Convolutional codes. Convolutional codes were introduced by Elias (1955), and the first algebraic theoretical approach to convolutional codes was provided by Forney (1970). There are several definitions of convolutional codes, depending on the specific requirements of the message transmission process. We consider an (n, k) convolutional code $\mathcal{C}$ over a finite field $\mathbb{F}$ as a $\mathbb{F}[z]$-generated finite submodule of rank k, i.e., $\mathcal{C} \subseteq \mathbb{F}[z]^n$.

The generator matrix $G(z)$ of an (n, k) convolutional code $\mathcal{C}$ over $\mathbb{F}[z]$ is a $\mathbb{F}[z]$-linear map $G(z) : \mathbb{F}[z]^l \longrightarrow \mathbb{F}[z]^n$, $u(z) \mapsto v(z) = G(z)u(z)$, such that $\operatorname{Im} G(z) = \mathcal{C}$. An encoder $G(z)$ of $\mathcal{C}$ is a generator matrix with $l = k$ and full rank.

A given convolutional code admits several convolutional encoders. We say that two encoders are equivalent if they generate the same code. It is well known that two encoders $G(z)$ and $G'(z)$ generate the same code if and only if there exists a unimodular matrix $U(z) \in \mathbb{F}[z]^{k\times k}$ such that $G'(z) = G(z)U(z)$ (Forney, 1970, Theorem 4).

Next, we recall a natural equivalence relation in the set of convolutional codes. Two convolutional codes $\mathcal{C}, \mathcal{C}' \subset \mathbb{F}[z]^n$ of dimension k are equivalent if and only if there exists a permutation of n letters, $\sigma \in \mathcal{S}_n$, such that $\sigma(\mathcal{C}) = \mathcal{C}'$. If $G(z)$ and $G'(z)$ are generator matrices for $\mathcal{C}$ and $\mathcal{C}'$, respectively, then $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if and only if there exists a permutation matrix P and a unimodular matrix $U(z) \in \mathbb{F}[z]^{k\times k}$ such that $G'(z) = PG(z)U(z)$.

An essential property of convolutional codes is the observability, which ensures that the code is not catastrophic. Algebraically, an (n, k) convolutional code $\mathcal{C}$ over $\mathbb{F}$ is observable if and only if there exists a syndrome map $\psi : \mathbb{F}[z]^n \twoheadrightarrow \mathbb{F}[z]^{n-k}$ such that the following sequence is exact (York, 1997, Lemma 3.3.2):

$$0 \to \mathbb{F}[z]^k \xrightarrow{G(z)} \mathbb{F}[z]^n \xrightarrow{\psi} \mathbb{F}[z]^{n-k} \to 0.$$

Let $G(z) \in \mathbb{F}[z]^{n\times k}$ be an encoder of an (n, k) convolutional code and let $g_{ij}(z)$ denote the (i, j) entry of $G(z)$. We recall some important definitions:

1. The column degrees of the encoder are the k-integers $\nu_1, \nu_2, \ldots, \nu_k$, where $\nu_j = \max\{\deg(g_{ij}) \mid 1 \le i \le n \text{ and } j = 1, \ldots, k\}$ (cf. York, 1997, Definition 3.1.5). These are also called the constraint length of the j-th input of the matrix $G(z)$.

2. The sum of the column degrees, $\nu = \sum_{j=1}^{k} \nu_j$, is called the complexity of the encoder (cf. York, 1997, Definition 3.1.5), the total memory or overall constraint length, or the external degree of the encoder.

3. The highest degree of the full-size minors $k \times k$ of any encoder $G(z)$, $\delta(\mathcal{C})$, is called the complexity of the convolutional code $\mathcal{C}$ (cf. York, 1997, Definition 3.1.7).

Definition 1. (*McEliece, 1998, Definition 2.25*) $G(z)$ is a minimal encoder if it is a generator matrix for which ν is the smallest possible over all equivalent encoders.

If $G(z)$ is minimal, then $\nu = \delta(\mathcal{C})$ (cf. McEliece, 1998, Definition 2.25; York, 1997, Definition 3.18). In this case, $\{\nu_j\}_{j=1\ldots k}$ are called the Forney indices of the code, which are unique and invariant for the code. Therefore, $\nu = \delta(\mathcal{C})$ is an invariant of the code. This value is called the degree of the code, and it is usually denoted by δ. Since, when working with convolutional codes over finite fields, we can always find minimal encoders, from now on we will assume that $\mathcal{C}$ is an (n, k, δ)-convolutional code.

Another important parameter in convolutional coding theory is the distance of the code. This is a measure of its ability to protect data from errors. Several types of distances can be defined for convolutional codes:

1. The Hamming distance between $c_1, c_2 \in \mathbb{F}^n$ is defined as $d(c_1, c_2) = \text{wt}(c_1 - c_2)$, where $\text{wt}(c)$ is the Hamming weight of $c \in \mathbb{F}^n$, which is the number of nonzero components of c. Similarly, the distance between $c_1(z), c_2(z) \in \mathbb{F}[z]^n$ is defined as $d(c_1(z), c_2(z)) = \text{wt}(c_1(z) - c_2(z))$, with $\text{wt}(c(z)) = \sum_{t=0}^{\deg(c(z))} \text{wt}(c_t)$, where $wt(c(z)$ being the sum of the Hamming weights of the coefficients of the polynomial vectors of $c(z)$.

2. The free distance of a convolutional code $\mathcal{C}$ is the minimum Hamming distance between any two distinct code sequences and is given by

$$d_{\text{free}}(\mathcal{C}) := \min_{c_1(z), c_2(z) \in \mathcal{C}} \{d(c_1(z), c_2(z)) \mid c_1(z) \neq c_2(z)\}.$$

3. In addition to the free distance, convolutional codes have also column distances (Gluesing-Luerssen *et al.*, 2006). For $j \in \mathbb{N}_0$, the j-th column distance of a convolutional code is defined as

$$d_j^c(\mathcal{C}) := \min_{c(z) \in \mathcal{C}} \left\{\text{wt}(c_{[0,j]}(z)) \mid c(z) \in \mathcal{C} \text{ and } c_0 \neq 0\right\},$$

where $c_{[0,j]}(z) = c_0 + c_1 z + \ldots + c_j z^j$ represents the j-th truncation of the code vector $c(z) \in \mathcal{C}$.

There are upper bounds for the free distance and column distances. Let $\mathcal{C}$ be an (n, k, δ)-convolutional code over $\mathbb{F}$. Then, the following results hold (Gluesing-Luerssen *et al.*, 2006; Rosenthal and Smarandache, 1999):

1. $d_{\text{free}}(\mathcal{C}) \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1$ (generalized Singleton bound),

2. $d_j^c(\mathcal{C}) \leq (n-k)(j+1) + 1$ for all $j \in \mathbb{N}_0$.

An (n, k, δ)-convolutional code is called MDS (maximum distance separable) if the free distance equals the generalized Singleton bound. If the column distance equals its upper bound for $j = 0, \ldots, \lfloor \delta/k \rfloor + \lfloor \delta/(n-k) \rfloor$, the convolutional code is called MDP (Hutchinson *et al.*, 2005; Gluesing-Luerssen *et al.*, 2006).

2.2. Convolutional codes and linear systems.

Definition 2. (*York, 1997, Section 5.2; Rosenthal* et al.*, 1996, Remark 4.1*) Let $\mathcal{C} \subset \mathbb{F}[z]^n$ be an (n, k, δ)-convolutional code. An input/state/output (ISO) representation of $\mathcal{C}$ is a tuple of matrices $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta \times \delta} \times \mathbb{F}^{\delta \times k} \times \mathbb{F}^{(n-k) \times \delta} \times \mathbb{F}^{(n-k) \times k}$ such that

$$\mathcal{C} = \left\{ v(z) = \begin{pmatrix} y(z) \\ u(z) \end{pmatrix} \in \mathbb{F}[z]^n : \exists x(z) \in \mathbb{F}[z]^\delta \text{ satisfying } \begin{cases} x_{t+1} = Ax_t + Bu_t, \\ y_t = Cx_t + Du_t, \\ v_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix}, \\ x_0 = 0, \end{cases} \right\},$$

where $x_t \in \mathbb{F}^\delta$ denotes the state vector, $u_t \in \mathbb{F}^k$ represents the control/input/information vector, and $y_t \in \mathbb{F}^{n-k}$ is the output/parity vector for each time step t. The vector v_t is the code vector transmitted over the communication channel.

The existence of ISO representations for a given (n, k, δ)-convolutional code over $\mathbb{F}$ is constructive, as described by York (1997) and Rosenthal *et al.* (1996). It requires first obtaining a minimal first-order representation (Climent *et al.*, 2025).

Definition 3. (*York, 1997, Theorem 5.1.1; Rosenthal* et al.*, 1996, Theorem 3.1*) Let $\mathcal{C} \subset \mathbb{F}[z]^n$ be an (n, k, δ) convolutional code. A triple of matrices, $(\mathcal{K}, \mathcal{L}, \mathcal{M})$, is called *first-order representation* of the code $\mathcal{C}$ if the following conditions are satisfied:

(i) $$\mathcal{C} = \{v(z) \in \mathbb{F}[z]^n : \exists x(z) \in \mathbb{F}[z]^\delta$$ such that $$z\mathcal{K}x(z) + \mathcal{L}x(z) + \mathcal{M}v(z) = 0\},$$

(ii) $\mathcal{K}$ is full rank,

(iii) $(\mathcal{K}, \mathcal{M})$ is full rank.

Furthermore, if $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ also satisfies the additional property

(iv) $(z\mathcal{K} + \mathcal{L}, \mathcal{M})$ is full rank,

then the first-order representation is said to be minimal.

The triple $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is unique in the following sense: if $(\widehat{\mathcal{K}}, \widehat{\mathcal{L}}, \widehat{\mathcal{M}})$ is another first-order representation of the convolutional code $\mathcal{C}$, then there exist unique and invertible matrices T and S of the appropriate sizes such that $(\widehat{\mathcal{K}}, \widehat{\mathcal{L}}, \widehat{\mathcal{M}}) = (T\mathcal{K}S^{-1}, T\mathcal{L}S^{-1}, T\mathcal{M})$.

By properly permuting the codewords, if necessary, $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ can be transformed via elementary transformations to obtain Σ:

$$\mathcal{K}' = \begin{pmatrix} -I_\delta \\ 0 \end{pmatrix}, \quad \mathcal{L}' = \begin{pmatrix} A_{\delta \times \delta} \\ C_{(n-k) \times \delta} \end{pmatrix},$$

$$\mathcal{M}' = \begin{pmatrix} 0 & B_{\delta \times k} \\ -I_{n-k} & D_{(n-k) \times k} \end{pmatrix}.$$

$$\begin{pmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+L} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^L \end{pmatrix} x_t + \begin{pmatrix} D & 0 & 0 & \dots & 0 & 0 \\ CB & D & 0 & \dots & \vdots & \vdots \\ CAB & CB & D & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & \dots & D & 0 \\ CA^{L-1}B & CA^{L-2}B & CA^{L-3}B & \dots & CB & D \end{pmatrix} \begin{pmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+L} \end{pmatrix}, \quad (1)$$

where

$$L := \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor > 0.$$

Moreover, for any $T \in GL_\delta(\mathbb{F})$, the equality

$$\mathcal{C}(A, B, C, D) = \mathcal{C}(TAT^{-1}, TB, CT^{-1}, D)$$

holds true, as shown in Proposition 2.2.8 of Allen (1999).

Remark 1. It is implicit in the above construction that the system $\Sigma = (A, B, C, D)$ corresponds to an equivalence class (i.e., up to permutations) of codes. Therefore, if two convolutional codes are equivalent, then their ISO representations are also equivalent.

Minimality is one of the most important properties of an ISO representation of a convolutional code, as it implies greater efficiency. An ISO representation of a convolutional code $\mathcal{C}$ is considered minimal if it is a reachable linear dynamical system.

Definition 4. Let $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ be a linear system over $\mathbb{F}$ of dimension δ. Then Σ is said to be reachable if its controllability matrix

$$\Phi_\delta(A, B) = \begin{pmatrix} B & AB & \cdots & A^{\delta-2}B & A^{\delta-1}B \end{pmatrix}$$

has full rank, i.e., $\operatorname{rank}(\Phi_\delta(A, B)) = \delta$.

Remark 2. (*York, 1997, Lemma 5.3.5*) Note that, if $(\mathcal{K}, \mathcal{L}, \mathcal{M})$ is minimal, the full rank of $(z\mathcal{K} + \mathcal{L}, \mathcal{M})$ implies that Σ is a reachable system.

Since we can obtain an encoder $G(z)$ by computing a minimal basis of the free $\mathbb{F}[z]$-module

$$\operatorname{Ker}(z\mathcal{K} + \mathcal{L}|\mathcal{M}) = \Big\{ v(z) \in \mathbb{F}[z]^n \mid \exists x(z) \in \mathbb{F}[z]^\delta : (z\mathcal{K} + \mathcal{L})x(z) + \mathcal{M}v(z) = 0 \Big\},$$

ISO representations allow us to construct observable convolutional codes based on properties of the associated system: if we start with a reachable and observable linear dynamical system, the resulting convolutional code is observable (York, 1997, Lemma 5.3.5).

Definition 5. Let $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ be a linear system over $\mathbb{F}$ of dimension δ. The system Σ is observable if $\operatorname{rank}(\Omega_\delta(A, C)) = \delta$, where $\Omega_\delta(A, C)$ is the observability matrix defined by

$$\Omega_\delta(A, C) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\delta-1} \end{pmatrix}.$$

2.3. Decoding process using ISO representations. The ISO representations of convolutional codes provide an algebraic relation between the input sequence u_t and the output sequence y_t. For any code sequence v_t, which must satisfy the dynamics of the system, if we have an (n, k, δ)-convolutional code $\mathcal{C} \subseteq \mathbb{F}[z]^n$ described by the system Σ, Eqn. (1) must hold (Rosenthal, 1999, Proposition 2.6). The state evolution is modeled by the following equation:

$$x_\lambda = A^{\lambda-t}x_t + \begin{pmatrix} A^{\lambda-t-1}B & \cdots & AB & B \end{pmatrix} \begin{pmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+L} \end{pmatrix},$$

where $\lambda = t+1, t+2, \dots, t+L$. This system can also be written as Eqn. (2), which we denote by

$$\Omega_{L+1}(C, A) \cdot \mathbf{x_t} + (-I_{L+1}|F_L) \cdot \mathbf{v_t} = 0, \quad (3)$$

where F_L is a block Toeplitz matrix.

Let $u(z)$ be an information word and $G(z)$ an encoder for an (n, k)-convolutional code. The codeword $v(z) = G(z) \cdot u(z)$ is transmitted through the channel, potentially with errors. If no errors occur, v_t is a valid trajectory, and the error value is zero. Otherwise, if there is an error, the received sequence is not a codeword and does not belong to the code family. The goal of

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^L \end{pmatrix} x_t + \left(\begin{array}{c|cccccc} & D & 0 & 0 & \dots & 0 & 0 \\ & CB & D & 0 & \dots & \vdots & \vdots \\ -I_{L+1} & CAB & CB & D & \dots & 0 & 0 \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & \dots & D & 0 \\ & CA^{L-1}B & CA^{L-2}B & CA^{L-3}B & \dots & CB & D \end{array} \right) \begin{pmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+L} \\ u_t \\ u_{t+1} \\ \vdots \\ u_{t+L} \end{pmatrix} = 0. \qquad (2)$$

the decoding process is to recover v_t from the erroneous codeword $\widehat{v_t}$. The most well-known decoding algorithm for convolutional codes is Viterbi. However, the existence of minimal ISO representations allows the development of new decoding algorithms. For a noisy channel, with substitution errors, the first decoding algorithms using ISO representations were introduced by Rosenthal (1999). In this context, the original information sequence u_t, which has been corrupted by error, is recovered more efficiently than with the Viterbi algorithm, particularly for convolutional codes based on Reed–Solomon, BCH (Rosenthal and York, 1999), or strongly MDS codes (Gluesing-Luerssen *et al*., 2006). We recall the main details of the decoding algorithm: to apply it on a convolutional code of complexity δ described by an ISO representation $\Sigma = (A, B, C, D)$, we have to assume that there are natural numbers $T > \Theta$ such that the following holds:

1. A is invertible and the reachability matrix

$$\Phi_T(A, B) := \begin{pmatrix} B & AB & A^{T-1}B \end{pmatrix}$$

is full row rank (surjective). Its rows form a parity check matrix of a block code. Note that, if Σ is reachable, then $\Phi_\delta(A, B)$ is full row rank and $\Phi_T(A, B)$ is full row rank $\forall\, T > \Theta$.

2. The observability matrix

$$\Psi_\Theta(A, C)^t := \begin{pmatrix} C^t & (CA)^t & (CA^{\Theta-1})^t \end{pmatrix}$$

is full column rank (injective). Its columns form an encoder of another block code. Note that, if Σ observable, then $\Psi_\delta(A, B)$ is full column rank and $\Psi_T(A, B)$ is full column rank $\forall\, T > \Theta$.

The theoretical performance of Rosenthal's decoding algorithm depends on the theoretical distances of the block codes determined by $\Phi_T(A, B)$ and $\Psi_\Theta(A, C)$. On the other hand, the efficiency of that algorithm depends on the existence of efficient decoding algorithms for the block codes determined by $\Phi_T(A, B)$ and $\Psi_\Theta(A, C)$. The decoding algorithm leaves the selection of decoding algorithms for the above block code open-ended.

This algorithm is still being studied and analyzed (Martín Sánchez and Plaza-Martín, 2022; Muñoz Castañeda *et al*., 2019). For example, García-Planas *et al*. (2014) describe another algorithm following the same idea than Rosenthal's algorithm but using generalized inverse matrices such as the Moore–Penrose pseudoinverse matrix in a perturbation scenario to find the nearest possible $\widehat{u_t}$ to u_t. This method is quite effective for error detection.

For erasure channels, the problem is not that the received message $\widehat{v_t}$ is incorrect, but that v_t is incomplete due to lost symbols. For this case, an algorithm based on ISO representations was proposed by Tomás *et al.* (2012; 2010) and improved by Lieb and Rosenthal (2021). The advantage of this algorithm is that it reduces the decoding delay and computational effort in the erasure recovery process. We consider the case where both the state and the symbols in v_t that are erased can be recovered simultaneously. Then, the decoding algorithm requires certain assumptions on the matrix $(\Omega_{L+1}(A, C) \mid F_L)$. Let $\alpha = (L + 1)(n - k) - \delta$ and $\mathcal{X} \leq \binom{(L+1)n}{\alpha}$. Assume that $\{j_1, \dots, j_\alpha\}$ represents the set of indices of the columns of $(-I \mid F_L)$ corresponding to the erased symbols in v_t, where $1 \leq j_1 < j_2 < \ldots < j_\alpha \leq \mathcal{X}$. Also, suppose that $\{j_1, \dots, j_\alpha\} \subset \{i_1, i_2, \dots, i_{(L+1)(n-k)}\}$ with $i_{s(n-k)} \leq sn$ for $s = 1, 2, \dots, (L + 1)(n - k)$.

Let $U_{j_1,\dots,j_\alpha}$ be the subspace spanned by the columns indexed by $\{j_1, \dots, j_\alpha\}$, and let $\langle \Omega_{L+1}(A, C) \rangle$ be the space generated by the columns of $\Omega_{L+1}(A, C)$.

Theorem 1. (Tomás, 2009, Theorem 4.3.) *If, for all such subspaces $U_{j_1,\dots,j_\alpha}$ the condition*

$$U_{j_1,\dots,j_\alpha} \oplus \langle \Omega_{L+1}(A, C) \rangle = \mathbb{F}^{(L+1)(n-k)}$$

holds, then it is possible to recalculate the state of the system (Eqn. (3)) when we observe a window of length $(L + 1)n$, provided that no more than $(L + 1)(n - k) - \delta$ erasures occur.

Remark 3. The above hypothesis theorem implies that the columns of $\Omega_{L+1}(A, C)$ and any α columns of $(-I_{L+1} \mid F_L)$ must be linearly independent. Let F_α

be the matrix formed by the columns of F_L indexed by $\{j_1, \ldots, j_\alpha\}$. Note that, if Σ is observable, $\Omega_{L+1}(A, C)$ has full rank and $(\Omega_{L+1}(A, C) \mid F_L)$ is full rank, then $(\Omega_{L+1}(A, C) \mid F_\alpha)$ is full rank.

3. Construction of observable and good decodable property convolutional codes by ISO representations

Definition 6. (*Good decodable property*) An (n, k, δ) convolutional code $\mathcal{C}$ over $\mathbb{F}$ is said to have a good decodable property (GDP) if it is constructed using a reachable and observable linear system $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ as an ISO representation, and it verifies that

(i) A is invertible in the case of noisy channels;

(ii) $(\Omega_{L+1}(A, C) \mid F_\alpha)$ is full rank for $\alpha = (L + 1)(n - k) - \delta$ in the case of erasure channels. Here is sufficient that $(\Omega_{L+1}(A, C) \mid F_L)$ is full rank.

This article explores alternative methods for obtaining ISO representations that satisfy the above conditions via group actions.

We now consider the following group transformations applied to $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$:

1. group actions in the state vector: $\Sigma_1 = (A_1, B_1, C_1, D_1) = (S^{-1}AS, S^{-1}B, CS, D)$, where S is an invertible matrix in $\mathbb{F}^{\delta\times\delta}$;
2. group actions in the parity vector: $\Sigma_2 = (A_2, B_2, C_2, D_2) = (A, BQ, C, DQ)$, where Q is an invertible matrix in $\mathbb{F}^{k\times k}$;
3. group actions in the information vector: $\Sigma_3 = (A_3, B_3, C_3, D_3) = (A, B, H^{-1}C, H^{-1}D)$, where H is an invertible matrix in $\mathbb{F}^{(n-k)\times(n-k)}$.

We denote by $\mathcal{C}_i$ the associated convolutional code obtained by taking Σ_i as the ISO representation, for $i = 1, 2, 3$, and $\mathcal{C}$ the associated convolutional code to Σ.

Proposition 1. *Let $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ be a linear system. The reachability of Σ is invariant under the following group actions:*

(i) group actions in the state vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_1 = (S^{-1}AS, S^{-1}B, CS, D)$ for some invertible matrix $S \in \mathbb{F}^{\delta\times\delta}$;

(ii) group actions in the parity vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_2 = (A, BQ, C, DQ)$ for some invertible matrix $Q \in \mathbb{F}^{k\times k}$;

(iii) group actions in the information vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_3 = (A, B, H^{-1}C, H^{-1}D)$ for some invertible matrix $H \in \mathbb{F}^{(n-k)\times(n-k)}$.

Proof. We assume that $\Sigma = (A, B, C, D)$ is reachable, i.e., $\operatorname{rank}(\Phi_\delta(A, B)) = \delta$.

(i) This is a known result in classical systems theory (see, e.g., Hazewinkel and Kalman, 1976, Eqn. 3.1.3).

The proof in the remaining cases is analogous.

(ii) Since $\Phi_\delta(A_2, B_2) = \Phi_\delta(A, B) \cdot \operatorname{diag}(Q, Q, \ldots, Q)$, it follows that $\operatorname{rank}(\Phi_\delta(A_2, B_2)) = \operatorname{rank}(\Phi_\delta(A, B)) = \delta$.

(iii) Since $\Phi_\delta(A_3, B_3) = \Phi_\delta(A, B)$, it follows that $\operatorname{rank}(\Phi_\delta(A_3, B_3)) = \operatorname{rank}(\Phi_\delta(A, B)) = \delta$. ∎

Corollary 1. *If Σ is an ISO representation of a convolutional code $\mathcal{C}$ over $\mathbb{F}$, then Σ_i is an ISO representation of the convolutional code $\mathcal{C}_i$ for $i = 1, 2, 3$ over $\mathbb{F}$.*

Proof. It follows from the fact that, if Σ is an ISO representation of a convolutional code $\mathcal{C}$ over $\mathbb{F}$, then Σ is reachable. By Proposition 1, Σ_i are reachable and, then, they can be considered ISO representations for convolutional codes $\mathcal{C}_i$ for $i = 1, 2, 3$. ∎

Example 1. Let Σ be the following system over $\mathbb{Z}_3$ with $\delta = 3$, $k = 2$, and $n = 3$:

$$\Sigma = A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The system Σ is reachable because

$$\Phi_3(A, B) = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 & 2 \end{pmatrix}$$

has full rank. Thus, we can take Σ as a minimal ISO representation of a convolutional code. An associated encoder for the convolutional code $\mathcal{C}$ constructed by Σ is

$$G(z) = \begin{pmatrix} 1 + z + 2z^2 & 2 + z \\ 1 + 2z & z \\ 2 + z + 2z^2 & 0 \end{pmatrix}.$$

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ be an invertible matrix over $\mathbb{Z}_3$. We apply the group transformation $\Sigma_2 = (A_2, B_2, C_2, D_2) = (A, BQ, C, DQ)$ to Σ. Then,

$$\Sigma_2 = A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 2 & 0 \end{pmatrix}.$$

The transformed system Σ_2 is also reachable because

$$\Phi_3(A_2,B_2)=\begin{pmatrix}B_2 & A_2B_2 & A_2^2B_2\end{pmatrix}=\begin{pmatrix}0&0&2&1&2&1\\2&1&2&1&0&0\\1&1&2&1&0&0\end{pmatrix}$$

has full rank. Thus, we can take Σ_2 as a minimal ISO representation to compute an encoder for the convolutional code $\mathcal{C}_2$,

$$G_2(z)=\begin{pmatrix}1+z+2z^2 & 2+z\\ z^2 & 2z\\ 1+2z+2z^2 & 2z\end{pmatrix}.$$

Let $H=\begin{pmatrix}2\end{pmatrix}$ be an invertible matrix in $\mathbb{Z}_3^{1\times 1}$. We apply the group transformation $\Sigma_3=(A_3,B_3,C_3,D_3)=(A,B,H^{-1}C,H^{-1}D)$ to Σ. Then,

$$\Sigma_3=A_3=\begin{pmatrix}0&1&0\\2&1&0\\2&1&0\end{pmatrix},\qquad B_3=\begin{pmatrix}0&0\\0&2\\1&0\end{pmatrix},$$
$$C_3=\begin{pmatrix}2&2&1\end{pmatrix},\qquad D_3=\begin{pmatrix}2&2\end{pmatrix}.$$

The transformed system Σ_3 is also reachable because $\Phi_3(A_3,B_3)=\Phi_3(A,B)$ has full rank. Thus, we can take Σ_3 as a minimal ISO representation to compute an encoder for the convolutional code $\mathcal{C}_3$,

$$G_3(z)=\begin{pmatrix}2+2z+z^2 & 1+2z\\ 1+2z & z\\ 2+z+2z^2 & 0\end{pmatrix}.$$

♦

Proposition 2. *Let $\Sigma=(A,B,C,D)\in\mathbb{F}^{\delta\times\delta}\times\mathbb{F}^{\delta\times k}\times\mathbb{F}^{(n-k)\times\delta}\times\mathbb{F}^{(n-k)\times k}$ be a linear dynamical system. The observability of Σ is invariant under the following group actions:*

(i) group actions in the state vector: $\Sigma=(A,B,C,D)\mapsto\Sigma_1=(S^{-1}AS,S^{-1}B,CS,D)$ for some invertible matrix $S\in\mathbb{F}^{\delta\times\delta}$;

(ii) group actions in the parity vector: $\Sigma=(A,B,C,D)\mapsto\Sigma_2=(A,BQ,C,DQ)$ for some invertible matrix $Q\in\mathbb{F}^{k\times k}$;

(iii) group actions in the information vector: $\Sigma=(A,B,C,D)\mapsto\Sigma_3=(A,B,H^{-1}C,H^{-1}D)$ for some invertible matrix $H\in\mathbb{F}^{(n-k)\times(n-k)}$.

Proof. Assume that $\Sigma=(A,B,C,D)$ is observable, i.e., $\operatorname{rank}\Omega_\delta(A,C)=\delta$.

(i) This is a known result in classical systems theory (see, e.g., Falb, 1999, Proposition 11.13) for the unidimensional case. The multidimensional one is analogous.

The remaining proofs follow the same idea as in the above case.

(ii) Since $\Omega_\delta(A_2,C_2)=\Omega_\delta(A,C)$, it follows that

$$\operatorname{rank}(\Omega_\delta(A_2,C_2))=\operatorname{rank}(\Omega_\delta(A,C))=\delta.$$

(iii) Since

$$\Omega_\delta(A_3,C_3)=\operatorname{diag}(H^{-1},\dots,H^{-1})\cdot\Omega_\delta(A,C),$$

we conclude that

$$\operatorname{rank}(\Omega_\delta(A_3,C_3))=\operatorname{rank}(\Omega_\delta(A,C))=\delta.$$

■

Corollary 2. *If Σ is an observable ISO representation of a convolutional code $\mathcal{C}$ over $\mathbb{F}$, Σ_i for $i=1,2,3$ can be considered observable ISO representations for observable convolutional codes $\mathcal{C}_i$ over $\mathbb{F}$.*

Proof. If Σ is an observable ISO representation of a convolutional code $\mathcal{C}$ over $\mathbb{F}$, Σ is a reachable and observable linear system. Then, by Propositions 1 and 2, Σ_i are reachable and observable ISO representations of the convolutional code $\mathcal{C}_i$ for $i=1,2,3$ over $\mathbb{F}$. By Lemma 5.3.5 of York (1997), $\mathcal{C}_i$ are observable convolutional codes. ■

Example 2. Let Σ and Σ_2 be the systems defined over $\mathbb{Z}_3$ as given in Example 1. The system Σ is observable because

$$\Omega_3(A,C)=\begin{pmatrix}C\\CA\\CA^2\end{pmatrix}=\begin{pmatrix}1&1&2\\0&1&0\\2&1&0\end{pmatrix}$$

has full rank. Therefore, $\mathcal{C}$ is an observable convolutional code. Similarly, the system Σ_2 is also observable because

$$\Omega_3(A_2,C_2)=\begin{pmatrix}C_2\\C_2A_2\\C_2A_2^2\end{pmatrix}=\Omega_3(A,C).$$

Thus, $\mathcal{C}_2$ is an observable convolutional code.

On the other hand, let Σ and Σ_3 be the systems defined over $\mathbb{Z}_3$ as given in Example 1. The system Σ_3 is observable because the observability matrix

$$\Omega_3(A_3,C_3)=\begin{pmatrix}C_3\\C_3A_3\\C_3A_3^2\end{pmatrix}=\begin{pmatrix}2&2&1\\0&2&0\\1&2&0\end{pmatrix}$$

has full rank. Thus, $\mathcal{C}_3$ is an observable convolutional code. ♦

The first question that arises is whether the convolutional codes $\mathcal{C}_i$ are equivalent to $\mathcal{C}$. In fact, this is not our primary concern, as we aim to obtain the largest possible number of distinct convolutional codes with good properties. Trivially, we know that, when the first transformation is applied to the representation, the resulting ISO systems are equivalent by similarity. Consequently, the codes they generate are equal. However, this statement does not hold for transformations obtained through Σ_2 and Σ_3. Let us examine some examples below.

Example 3. Let Σ and Σ_2 be the ISO representations given in Example 1. Let

$$G(z) = \begin{pmatrix} 1+z+2z^2 & 2+z \\ 1+2z & z \\ 2+z+2z^2 & 0 \end{pmatrix},$$
$$G_2(z) = \begin{pmatrix} 1+z+2z^2 & 2+z \\ z^2 & 2z \\ 1+2z+2z^2 & 2z \end{pmatrix}$$

be the encoders for $\mathcal{C}$ and $\mathcal{C}_2$, respectively, obtained from Σ and Σ_2. Then, $\mathcal{C}$ is not equivalent to $\mathcal{C}_2$. Let us consider this: assume $v(z) \in \mathbb{Z}_3[z]^3$ to be the codeword defined as

$$v(z) = \begin{pmatrix} 1+z+z^2 \\ z^2 \\ 1+2z+2z^2 \end{pmatrix}.$$

Clearly, $v(z) \in \mathcal{C}_2$, but $v(z) \notin \mathcal{C}$. If $v(z) \in \mathcal{C}$, then there would exist

$$x(z) = \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} \in \mathbb{Z}_3[z]^2$$

such that

$$\begin{pmatrix} 1+z+2z^2 & 2+z \\ 1+2z & z \\ 2+z+2z^2 & 0 \end{pmatrix} \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = \begin{pmatrix} 1+z+z^2 \\ z^2 \\ 1+2z+2z^2 \end{pmatrix}.$$

Solving this equation, we find that $(2+z+2z^2)\cdot a(z) = 1+2z+2z^2$, implying $a(z) = \lambda \in \mathbb{Z}_3$. However, $(2+z+2z^2)\cdot\lambda = 1+2z+2z^2$ has no solution, so $v(z) \notin \mathcal{C}$. This can also be seen for words $\tilde{v(z)}$ obtained from $v(z)$ by permuting is components. Thus, the codes are not equivalent. ♦

Example 4. Let Σ and Σ_3 be the ISO representations given in Example 1. Let

$$G(z) = \begin{pmatrix} 1+z+2z^2 & 2+z \\ 1+2z & z \\ 2+z+2z^2 & 0 \end{pmatrix},$$
$$G_3(z) = \begin{pmatrix} 2+2z+z^2 & 1+2z \\ 1+2z & z \\ 2+z+2z^2 & 0 \end{pmatrix}$$

be the encoders for $\mathcal{C}$ and $\mathcal{C}_3$, respectively, obtained from Σ and Σ_3. Then, $\mathcal{C}$ is not equivalent to $\mathcal{C}_3$. Let us consider this: assume $v(z) \in \mathbb{Z}_3[z]^3$ to be the codeword defined as

$$v(z) = \begin{pmatrix} 2+2z+z^2 \\ 1+2z \\ 2+z+2z^2 \end{pmatrix}.$$

Clearly, $v(z) \in \mathcal{C}_3$, but $v(z) \notin \mathcal{C}$. If $v(z) \in \mathcal{C}$, then there would exist

$$x(z) = \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} \in \mathbb{Z}_3[z]^2$$

such that

$$\begin{pmatrix} 1+z+2z^2 & 2+z \\ 1+2z & z \\ 2+z+2z^2 & 0 \end{pmatrix} \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = \begin{pmatrix} 2+2z+z^2 \\ 1+2z \\ 2+z+2z^2 \end{pmatrix}.$$

Solving the above equation, we find that $(2+z+2z^2)\cdot a(z) = 2+z+2z^2$, implying $a(z) = 1 \in \mathbb{Z}_3$. Then, substituting into the equation $(1+2z)\cdot a(z) + z\cdot b(z) = 1+2z$, we obtain $b(z) = 0$. But, with these values, $(1+z+2z^2)\cdot a(z) + (2+z)\cdot b(z) \neq 2+2z+2z^2$. So, $v(z) \notin \mathcal{C}$. This can also be seen for words $\tilde{v(z)}$ obtained from $v(z)$ by permuting is components. Thus, the codes are not equivalent. ♦

Due to the results obtained above, the possible equivalence between the convolutional codes will depend on the transformation matrices Q and H, but there will exist some properties on them that let us obtain no equivalent convolutional codes that keep the good properties. For this reason, we continue only with the group transformations on the parity and information vectors, as these are the transformations that can lead to different convolutional codes compared to the one we initially had.

Recall that we want to explore possible methods to construct observable and GDP convolutional codes. From the systems theory perspective, the solvability of Eqn. (1) can be studied by analyzing the characterization of the ISO representation Σ. We are going to prove that the GDP is invariant under the defined group actions, both noisy and erasure channels.

$$T_L = \begin{pmatrix} C & D & 0 & 0 & \dots & 0 & 0 \\ CA & CB & D & 0 & \dots & \vdots & \vdots \\ CA^2 & CAB & CB & D & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ CA^{L-1} & CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & \dots & D & 0 \\ CA^L & CA^{L-1}B & CA^{L-2}B & CA^{L-3}B & \dots & CB & D \end{pmatrix}. \tag{4}$$

Proposition 3. *Let $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ be a reachable and observable system over $\mathbb{F}$ with A an invertible matrix. We consider the following group actions over Σ:*

(i) group actions in the parity vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_2 = (A, BQ, C, DQ)$ for some invertible matrix $Q \in \mathbb{F}^{k\times k}$;

(ii) group actions in the information vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_3 = (A, B, H^{-1}C, H^{-1}D)$ for some invertible matrix $H \in \mathbb{F}^{(n-k)\times(n-k)}$.

Then, Σ_i $(i = 2, 3)$ are reachable and observable linear systems over $\mathbb{F}$, and A_2 and A_3 are invertible matrices.

Proof. If Σ is a reachable and observable linear system over $\mathbb{F}$, by Propositions 1 and 2, Σ_2 and Σ_3 are reachable and observable linear systems over $\mathbb{F}$. Also, since the group actions that provide us with Σ_2 and Σ_3 do not affect the matrix A, then A_2 and A_3 will be invertible. ■

Corollary 3. *Let $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ be a reachable and observable system over $\mathbb{F}$ with A an invertible matrix. Let $\mathcal{C}$ be the observable and GDP convolutional code over a noisy channel associated to Σ. The GDP of $\mathcal{C}$ is invariant under the defined group actions over Σ.*

Proof. In the case of noisy channels, the necessary conditions for applying the Rosenthal decoding algorithm (Rosenthal, 1999) is that Σ be a reachable and observable linear system and that A be invertible. Then, if $\mathcal{C}$ is the associated convolutional code of Σ, $\mathcal{C}$ is observable and GDP for noisy channels. By Proposition 3, the systems Σ_2 and Σ_3 are reachable and observable linear systems, and A_2 and A_3 are invertible matrices. Then, Σ_2 and Σ_3 are ISO representations for observable and GDP convolutional codes over noisy channels. ■

For erasure channels, the solvability of Eqn. (1) can be studied by analyzing the characterization of the ISO representation Σ of the code as an *output observable system* (García-Planas *et al.*, 2013; 2014; García-Planas and Domíguez-García, 2013, Tomăs, 2010; Lieb and Rosenthal, 2021). For convenience, we denote the matrix $(\Omega_{L+1}(A, C) \mid F_L)$ by T_L.

Definition 7. A system Σ is *output observable* if the sequence of states $x(0), \dots, x(L)$ is uniquely determined by the knowledge of the output sequence $y(0), \dots, y(L)$ for a finite number of steps $L \in \mathbb{N}$. If the matrix T_L is defined as Eqn. (4), then Σ is *output observable* if and only if $\mathrm{rank}(T_L)$ is maximum for all $L \in \mathbb{N}$.

Proposition 4. *Let Σ be a linear system over $\mathbb{F}$. The output observability of the system Σ over $\mathbb{F}$ is invariant under the following group actions:*

(i) group actions on the parity vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_2 = (A, BQ, C, DQ)$ for some invertible matrix $Q \in \mathbb{F}^{k\times k}$;

(ii) group actions on the information vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_3 = (A, B, H^{-1}C, H^{-1}D)$ for some invertible matrix $H \in \mathbb{F}^{(n-k)\times(n-k)}$.

Proof. We denote by $T_L^{(i)}$ the matrix $(\Omega_{L+1}(A_i, C_i) \mid F_L^i)$ obtained from Σ_i. It suffices to prove that $rank(T_L] = rank(T_L^{(i)})$, and then, if Σ is output observable, we conclude that Σ_i is also output observable.

(i) Since Eqn. (5) holds and since $\mathcal{Q}$ is invertible, we conclude that $\mathrm{rank}(T_L) = \mathrm{rank}(T_L^{(2)})$.

(ii) Since Eqn. (6) holds and since $\mathcal{H}$ is invertible, we conclude that $\mathrm{rank}(T_L) = \mathrm{rank}(T_L^{(3)})$. ■

Corollary 4. *If Σ is an observable ISO representation of a GDP convolutional code $\mathcal{C}$ over $\mathbb{F}$ in an erasure channel, Σ_i for $i = 1, 2, 3$ are ISO representations for GDP convolutional codes $\mathcal{C}_i$ over $\mathbb{F}$.*

Proof. It follows from Definition 6, Remark 3 and Proposition 4. ■

Example 5. Let Σ and Σ_2 be the systems obtained in Example 1. The system Σ is output observable because

$$T_L = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 1 \end{pmatrix},$$

which has full rank. Also, Σ_2 is output observable because

$$T_L^{(2)} = \begin{pmatrix} 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 0 \end{pmatrix},$$

$$
\begin{aligned}
T_L^{(2)} &= \begin{pmatrix}
C & DQ & 0 & 0 & \dots & 0 & 0 \\
CA & CBQ & DQ & 0 & \dots & \vdots & \vdots \\
CA^2 & CABQ & CBQ & DQ & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\
CA^{L-1} & CA^{L-2}BQ & CA^{L-3}BQ & CA^{L-4}BQ & \dots & DQ & 0 \\
CA^{L} & CA^{L-1}BQ & CA^{L-2}BQ & CA^{L-3}BQ & \dots & CBQ & DQ
\end{pmatrix} \\
&= \begin{pmatrix}
C & D & 0 & 0 & \dots & 0 & 0 \\
CA & CB & D & 0 & \dots & \vdots & \vdots \\
CA^2 & CAB & CB & D & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\
CA^{L-1} & CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & \dots & D & 0 \\
CA^{L} & CA^{L-1}B & CA^{L-2}B & CA^{L-3}B & \dots & CB & D
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & Q & 0 & \vdots & \vdots \\
0 & 0 & Q & 0 & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & Q
\end{pmatrix} \\
&= T_L \mathcal{Q},
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
T_L^{(3)} &= \begin{pmatrix}
H^{-1}C & H^{-1}D & 0 & 0 & \dots & 0 & 0 \\
H^{-1}CA & H^{-1}CB & H^{-1}D & 0 & \dots & \vdots & \vdots \\
H^{-1}CA^2 & H^{-1}CAB & H^{-1}CB & H^{-1}D & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H^{-1}CA^{L-1} & H^{-1}CA^{L-2}B & H^{-1}CA^{L-3}B & H^{-1}CA^{L-4}B & \dots & H^{-1}D & 0 \\
H^{-1}CA^{L} & H^{-1}CA^{L-1}B & H^{-1}CA^{L-2}B & H^{-1}CA^{L-3}B & \dots & H^{-1}CB & H^{-1}D
\end{pmatrix} \\
&= \begin{pmatrix}
H^{-1} & 0 & 0 & 0 & 0 \\
0 & H^{-1} & 0 & 0 & 0 \\
0 & 0 & H^{-1} & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & H^{-1}
\end{pmatrix}
\begin{pmatrix}
C & D & 0 & 0 & \dots & 0 & 0 \\
CA & CB & D & 0 & \dots & \vdots & \vdots \\
CA^2 & CAB & CB & D & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\
CA^{L-1} & CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & \dots & D & 0 \\
CA^{L} & CA^{L-1}B & CA^{L-2}B & CA^{L-3}B & \dots & CB & D
\end{pmatrix} \\
&= \mathcal{H} T_L.
\end{aligned}
\tag{6}
$$

which also has full rank. Therefore, since $\mathcal{C}$ is GDP, the convolutional code $\mathcal{C}_2$ is also GDP.

Additionally, let Σ_3 be the system obtained in Example 1. Then, Σ_3 is output observable because

$$
T_L^{(3)} = \begin{pmatrix}
2 & 2 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2
\end{pmatrix},
$$

which also has full rank. Therefore, since $\mathcal{C}$ is GDP over an erasure channel, the convolutional code $\mathcal{C}_3$ is also GDP. ♦

Since reachability, observability, and output observability are invariant under the described group actions on the parity and codeword vectors, if we start with a reachable, observable, and output observable ISO representation Σ over $\mathbb{F}$, applying the group transformations described above will yield a reachable, observable, and output observable ISO representation Σ_i, which allows us to construct observable and GDP convolutional codes $\mathcal{C}_i$ over erasure channels. Also, if A is invertible, then $\mathcal{C}$ would be GDP for noisy channels. The proposal developed by Tomás *et al.* (2012), Tomás (2010), Lieb and Rosenthal (2021) as well as Lieb (2019) involves obtaining Σ by imposing that $(\Omega_{L+1}(A, C) \mid F_L)$ is a superregular matrix. Since these matrices have the property that all non-trivial minors are non-zero, the matrix $(\Omega_{L+1}(A, C) \mid F_L)$ has full rank. Thus, if B is full rank, Σ is reachable, and if $\Omega_L(A, C)$ has full rank, then Σ is observable. Consequently, we can apply the group actions described for obtaining ISO

$$F_L^{(2)} = \begin{pmatrix} D & 0 & 0 & 0 & 0 \\ CB & D & 0 & 0 & 0 \\ CAB & CB & D & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{L-1}B & CA^{L-2}B & \dots & CB & D \end{pmatrix} \begin{pmatrix} Q & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & Q \end{pmatrix}, \tag{7}$$

$$F_L^{(3)} = \begin{pmatrix} H^{-1} & 0 & 0 & 0 & 0 \\ 0 & H^{-1} & 0 & 0 & 0 \\ 0 & 0 & H^{-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & H^{-1} \end{pmatrix} \begin{pmatrix} D & 0 & 0 & 0 & 0 \\ CB & D & 0 & 0 & 0 \\ CAB & CB & D & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{L-1}B & CA^{L-2}B & \dots & CB & D \end{pmatrix}. \tag{8}$$

representations for observable and GDP convolutional codes.

4. Some conditions about the construction of MDP convolutional codes

Regarding the distance properties, Σ is an ISO representation of an MDP convolutional code if and only if each minor of F_L, which is not trivially zero, is non-zero (cf. Hutchinson *et al.*, 2005, Theorem 2.4.). Here, it is clear that $D \neq O$ so that Σ can be the ISO of an MDP convolutional code. Thus, if F_L is superregular, the associated convolutional code is an MDP one (Almeida *et al.*, 2016; Lieb, 2019; Lieb and Rosenthal, 2021; Tomás, 2010; Tomás *et al.*, 2012). A natural question arises as to whether this property is invariant under group actions. The invariance of superregularity under group transformations might require additional conditions on the matrices used to perform the action. For this reason, we aim to propose some conjectures, starting with the necessary condition for the MDP property of a convolutional code to remain invariant under the transformations. Specifically, F_L must be of full rank.

Proposition 5. *Let $\Sigma = (A, B, C, D)$ be a minimal ISO representation of an MDP convolutional code defined over $\mathbb{F}$ with $D \neq 0$. We consider the following group actions:*

(i) group actions on the parity vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_2 = (A, BQ, C, DQ)$ for some invertible matrix $Q \in \mathbb{F}^{k\times k}$;

(ii) group actions on the information vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_3 = (A, B, H^{-1}C, H^{-1}D)$ for some invertible matrix $H \in \mathbb{F}^{(n-k)\times(n-k)}$.

Let $F_L^{(i)}, i = 2, 3$ be the corresponding matrices, defined as

$$F_L^{(i)} = \begin{pmatrix} D_i & 0 & 0 & 0 & 0 \\ C_iB_i & D_i & 0 & 0 & 0 \\ C_iA_iB_i & C_iB_i & D_i & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_iA_i^{L-1}B_i & C_iA_i^{L-2}B_i & \dots & C_iB_i & D_i \end{pmatrix},$$

Then, $F_L^{(i)}$ is full rank.

Proof. We start by stating that, since $\mathcal{C}$ is MDP, then F_L is full rank.

(i) We have Eqn. (7) above, which implies that $\operatorname{rank}(F_L^{(2)}) = \operatorname{rank}(F_L)$. Therefore, if $\mathcal{C}$ is MDP, the necessary condition for $\mathcal{C}_2$ to be MDP is satisfied.

(ii) Similarly, since Eqn. (8) holds, we obtain $\operatorname{rank}(F_L^{(3)}) = \operatorname{rank}(F_L)$. Therefore, if $\mathcal{C}$ is MDP, the necessary condition for $\mathcal{C}_3$ to be MDP is satisfied. ■

Example 6. Let Σ and Σ_2 be the systems given in Example 1. We can verify that, since F_L has full rank, then $F_L^{(2)}$ is also full rank, where

$$F_L = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 1 & 1 \end{pmatrix},$$

$$F_L^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 & 0 & 2 & 0 \end{pmatrix}.$$

Furthermore, let Σ_3 be the system given in Example 1. We can verify that $F_L^{(3)}$ has full rank, where

$$F_L^{(3)} = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

♦

A sufficient condition for the proposed transformations to preserve the MDP property would require that the condition of every trivially non-zero determinant of F_L being non-zero remain invariant under the group actions. Although we have shown that the maximum rank of F_L is preserved, proving that this result holds for all non-trivially non-zero minors in F_L is not straightforward. The first condition that is necessary is already determined by the singularity of the matrices Q and H. However, multiplying an invertible matrix by a superregular one does not generally preserve the superregularity of the product. Only if the invertible matrix is diagonal are we sure that it preserves superregularity by matrix multiplication.

Proposition 6. *Let $\Sigma = (A, B, C, D)$ over $\mathbb{F}$ be an ISO representation of an MDP convolutional code. The MDP property is invariant under*

(i) group actions on the parity vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_2 = (A, BQ, C, DQ)$ for some diagonal invertible matrix $Q \in \mathbb{F}^{k\times k}$;

(ii) group actions on the information vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_3 = (A, B, H^{-1}C, H^{-1}D)$ for some diagonal invertible matrix $H \in \mathbb{F}^{(n-k)\times(n-k)}$.

Proof.
(i) Let N be a $k \times n$ matrix and Q an invertible diagonal matrix of size n with diagonal elements $d_1, \ldots, d_n$. Suppose $N(I, J)$ is the $l\times l$ submatrix of N corresponding to rows with indices in I and columns with indices in J. Then, the $l \times l$ submatrix $(N \cdot Q)(I, J)$ of $N \cdot Q$ corresponding to the same rows and columns is computed by multiplying the rows of N with indices in I with the columns of Q with indices in J. Thus, the j-th column of $(N \cdot Q)(I, J)$ is equal to the j-th column of $N(I, J)$ multiplied by the j-th element of the diagonal of Q. Therefore,

$$\det((N \cdot q)(I, J)) = \prod_{j\in J} d_i M(I, J).$$

So,

$$\det((N \cdot Q)(I, J)) \neq 0$$

if and only if $\det(N(I, J)) \neq 0$. The result follows from this fact.

(ii) It follows from an argument similar to the above but with left multiplication. ■

Corollary 5. *Let Σ be an ISO representation of an MDP convolutional code $\mathcal{C}$ over $\mathbb{F}$. Then, Σ_i for $i = 2, 3$ are ISO representations for MDP convolutional codes $\mathcal{C}_i$ over $\mathbb{F}$.*

Proof. It follows from Proposition 6. ■

Remark 4. It would be interesting to know if the largest subgroup of the general linear group (of the appropriate size) that preserves the MDP property under the group actions defined in Proposition 6 is the subgroup of invertible diagonal matrices.

Finally, we give the main result.

Theorem 2. *Let $\Sigma = (A, B, C, D) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{(n-k)\times\delta} \times \mathbb{F}^{(n-k)\times k}$ be a reachable and observable linear system over $\mathbb{F}$ with F_L a superregular matrix. We consider the following group transformations over Σ:*

(i) group actions on the parity vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_2 = (A, BQ, C, DQ)$ for some diagonal invertible matrix $Q \in \mathbb{F}^{k\times k}$;

(ii) group actions on the information vector: $\Sigma = (A, B, C, D) \mapsto \Sigma_3 = (A, B, H^{-1}C, H^{-1}D)$ for some diagonal invertible matrix $H \in \mathbb{F}^{(n-k)\times(n-k)}$.

Then,

(1) if Σ is output-observable, Σ_i with $i = 2, 3$ are observable ISO representations for observable, GDP and MDP convolutional codes over erasure channels;

(2) if A is invertible, Σ_i with $i = 2, 3$ are observable ISO representations for observable, GDP and MDP convolutional codes over noisy channels.

Proof. It follows from Corollaries 1, 2, 4, 3 and 5. ■

5. Conclusions

Convolutional codes are deeply connected to several areas of mathematics, including algebraic systems theory, module theory, and symbolic dynamics. These connections not only enable a deeper understanding of encoder dynamics, but also open the door to systematic methods for constructing families of convolutional codes with guaranteed structural properties. Such methods are crucial in the context of modern communication systems, where decoding performance must be balanced with algebraic robustness and implementation efficiency.

In this paper, we focused on using group actions and the ISO representation of convolutional codes to construct new classes of observable and good decodable convolutional codes. This approach builds on system-theoretic principles and algebraic transformations to preserve essential properties such as reachability, observability, and output observability under group actions. By proving the invariance of these properties under well-defined transformations, we provided a solid theoretical framework that supports the construction of new convolutional encoders without compromising decoding performance (GDP convolutional code).

Moreover, we addressed the construction of MDP convolutional codes, which are of particular interest due to their optimality in distance properties. Our results give sufficient conditions, based on matrix superregularity and structured diagonal transformations, that ensure the preservation of the MDP property during the construction process. These contributions enrich the existing theory with constructive tools from an algebraic perspective. This approach is directly applicable in the design of efficient error-correcting schemes for both erasure and noisy channels.

As part of future work, if we also want the constructed convolutional codes to be non-equivalent; it is likely that we will need to impose additional conditions on the matrices Q and H that hold the desired properties. On the other hand, once the ISO representations of convolutional codes over modular integer rings are fully developed, it will be natural to explore the application of the proposed construction to the decoding problem for convolutional codes over these rings.

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References

Allen, B. (1999). *Linear Systems Analysis and Decoding of Convolutional Codes*, Thesis dissertation, University of Notre Dame, Notre Dame `http://user.math.uzh.ch/rosenthal/Paper/ThesisBrian.pdf`.

Almeida, P.J., Napp, D. and Pinto, R. (2016). Superregular matrices and applications to convolutional codes, *Linear Algebra and Its Applications* **499**: 1–25.

Climent, J.-J., Herranz, V. and Perea, C. (2007). A first approximation of concatenated convolutional codes from linear systems theory viewpoint, *Linear Algebra and Its Applications* **425**(2): 673–699.

Climent, J.-J., Napp, D., Pinto, R. and Requena, V. (2021). Minimal state-space representation of convolutional product codes, *Mathematics* **9**(12): 1410.

Climent, J.-J., Napp, D., Pinto, R. and Simóes, R. (2018). Series concatenation of 2D convolutional codes by means of input-state-output representations, *International Journal of Control* **91**(12): 2682–2691, DOI: 10.1080/00207179.2017.1410573.

Climent, J.-J., Napp, D. and Requena, V. (2025). An algorithm to compute a minimal input-state-output representation of a convolutional code, *Linear Algebra and Its Applications* **721**: 715–735.

DeCastro-García, N. and García-Planas, M.I. (2018). Concatenated linear systems over rings and their application to construction of concatenated families of convolutional codes, *Linear Algebra and Its Applications* **542**: 624–647.

Elias, P. (1955). Coding for two noisy channels, *IRE WESCON Convention Record* **4**: 37–46.

Falb, P. (1999). *Methods of Algebraic Geometry in Control Theory. Part II: Multivariable Linear Systems and Projective Algebraic Geometry*, Systems & Control: Foundations & Applications, Birkhäuser, Boston.

Forney, G. (1970). Convolutional codes I: Algebraic structure, *IEEE Transactions on Information Theory* **16**(6): 720–738.

García-Planas, M. and Domínguez-García, J. (2013). Alternative tests for functional and pointwise output-controllability of linear time-invariant systems, *Systems & Control Letters* **62**(5): 382–387.

García-Planas, M., Souidi, E.M. and Um, L. (2013). Convolutional codes under control theory point of view: Analysis of output-observability, *12th WSEAS International Conference on Recent Advances in Circuits, Communications and Signal Processing, Cambridge, UK*, pp. 131–137.

García-Planas, M., Souidi, E. and Um, L. (2014). Decoding algorithm for convolutional codes under linear systems point of view, *8th WSEAS International Conference on Recent Advances in Circuits, Systems, Signal Processing and Communications, Tenerife, Spain*, pp. 17–24.

Gluesing-Luerssen, H., Rosenthal, J. and Smarandache, R. (2006). Strongly-MDS convolutional codes, *IEEE Transactions on Information Theory* **52**(2): 584–598.

Hazewinkel, M. and Kalman, R.E. (1976). On invariants, canonical forms and moduli for linear, constant, finite dimensional, dynamical systems, *in* G. Marchesini and S.K. Mitter (Eds), *Mathematical Systems Theory*, Springer, Berlin/Heidelberg, pp. 48–60.

Hutchinson, R., Rosenthal, J. and Smarandache, R. (2005). Convolutional codes with maximum distance profile, *Systems & Control Letters* **54**(1): 53–63.

Kuriata, E. (2008). Creation of unequal error protection codes for two groups of symbols, *International Journal of Applied Mathematics and Computer Science* **18**(2): 251–257, DOI: 10.2478/v10006-008-0023-x.

Lieb, J. (2019). Complete MDP convolutional codes, *Journal of Algebra and Its Applications* **18**(6): 1950105.

Lieb, J. and Rosenthal, J. (2021). Erasure decoding of convolutional codes using first-order representations, *Mathematics of Control, Signals, and Systems* **33**: 499–513.

Martín Sănchez, S. and Plaza-Martín, F.J. (2022). A decoding algorithm for convolutional codes, *Mathematics* **10**(9): 1573.

McEliece, R.L. (1998). The algebraic theory of convolutional codes, *in* V.S. Pless and W.C. Huffman (Eds), *Handbook of Coding Theory I*, Amsterdam, Elsevier, pp. 1065–1138.

Muñoz Castañeda, A.L., Muñoz-Porras, J.M. and Plaza-Martín, F.J. (2019). Rosenthal's decoding algorithm for certain 1-dimensional convolutional codes, *IEEE Transactions on Information Theory* **65**(12): 7736–7741.

Muñoz Castañeda, A.L. and Plaza-Martín, F.J. (2021). On the existence and construction of maximum distance profile convolutional codes, *Finite Fields and Their Applications* **75**: 101877.

Napp, D., Perea, C. and Pinto, R. (2010). Input-state-output representations and constructions of finite support 2D convolutional codes, *Advances in Mathematics of Communications* **4**(4): 533–545.

Napp, D., Pereira, R., Pinto, R. and Rocha, P. (2019). Realization of 2D (2,2)-periodic encoders by means of 2D periodic separable Roesser models, *International Journal of Applied Mathematics and Computer Science* **29**(3): 527–539, DOI: 10.2478/amcs-2019-0039.

Napp, D., Pereira, R. and Rocha, P. (2017). A state space approach to periodic convolutional codes, *in* Á. Barbero *et al.* (Eds), *Coding Theory and Applications*, Lecture Notes in Computer Science, Vol. 10495, Springer, Cham, pp. 238–247, DOI: 10.1007/978-3-319-66278-7_20.

Pinto, R. and Simões, R. (2017). On minimality of ISO representation of basic 2D convolutional codes, *in* Á. Barbero *et al.* (Eds), *Coding Theory and Applications*, Lecture Notes in Computer Science, Vol. 10495, Springer, Cham, pp. 357-271, DOI: 10.1007/978-3-319-66278-7_22.

Rosenthal, J. (1999). An algebraic decoding algorithm for convolutional codes, *in* G. Picci and D.S. Gilliam (Eds), *Dynamical Systems, Control, Coding, Computer Vision*, Birkhäuser, Basel, pp. 343–360, DOI: 10.1007/978-3-0348-8970-4_16.

Rosenthal, J., Schumacher, J. and York, E. (1996). On behaviors and convolutional codes, *IEEE Transactions on Information Theory* **42**(6): 1881–1891.

Rosenthal, J. and Smarandache, R. (1999). Maximum distance separable convolutional codes, *Applicable Algebra in Engineering, Communication and Computing* **10**(1): 15–32.

Rosenthal, J. and York, F. (1999). BCH convolutional codes, *IEEE Transactions on Information Theory* **45**(6): 1833–1844.

Tomás, V. (2010). *Complete-MDP Convolutional Codes over the Erasure Channel*, Thesis dissertation, University of Alicante, Alicante, `https://rua.ua.es/dspace/bitstream/10045/18325/1/tesis_Tomas.pdf`.

Tomás, V., Rosenthal, J. and Smarandache, R. (2012). Decoding of convolutional codes over the erasure channel, *IEEE Transactions on Information Theory* **58**(1): 90–108.

York, E. (1997). *Algebraic Description and Construction of Error Correcting Codes: A Systems Theory Point of View*, Thesis dissertation, University of Notre Dame, Notre Dame.

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