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VARIABLE PARAMETER, FRACTIONAL ORDER, DISCRETE, INERTIAL TRANSFER FUNCTION MODELS

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In the paper, new, fractional order, variable parameter discrete transfer function models of an elementary inertial plant are proposed. One uses a time variable quasi time constant while the other employs, a variable time quasi constant and variable fractional order. Both models apply the variable fractional order backward difference to approximate the fractional operator. Both variable parameters are described by discrete, bounded functions. The accuracy and stability of the models considered are discussed. Theoretical results are validated by simulations. The proposed models can be applied to the modeling of various physical phenomena, where the constant parameter transfer function is not enough.

Keywords: fractional order transfer function, Grünwald–Letnikov definition, variable order FOBD operator, stability.

1. Introduction

Variable parameter and variable fractional order models are expected to precisely describe a behaviour of physical systems or phenomena for which the classic, constant parameter model is not enough. VO FOPID[1] controllers are also required to assure better control performance as FOPID controllers with fixed orders.

The first proposal of a VO fractional derivative was developed by Ross and Samko (1993). Their definition was based on the VO RL integral and the Marchaud derivative. Later, Valerio and da Costa (2011) determined the VO dervative for the RL, C and GL definitions as well as theoretical background for VO. In addition, this paper proposes also the use of fuzzy logic to approximate VO.

Various approximation methods for variable fractional order systems have been known for years. For example, the time-continuous approximation of a VO system using the C definition is proposed by Tavares *et al.* (2016). An altenative approach to the VO operator, using the Scarpi integral and derivative, is proposed by Garrappa *et al.* (2021).

An estimation of a variable parameter fractional system with the use of the triple estimation algorithm is presented by Sierociuk and Macias (2021), while the analytical solution of VO differential equations has been proposed by Malesza *et al.* (2019). The solution of a variable fractional order state equation with distributed delays is given by Kiskinov *et al.* (2024). Al-Saidi *et al.* (2022) propose stability conditions for variable fractional dynamic systems (dedicated to chaotic systems).

Broad analysis of the state of the art in the area of VO fractional operators is presented by Patnaik *et al.* (2020). It covers problems from theoretical background to various applications of the VO operator. Applications in special issues of mechanics (e.g., viscoelastics), transport processes and anomalous diffusion, fractals, and the VO filters are recalled, along with applications of the VO FOPID controller and other control algorithms employing the VO operator.

Optimal control of VO systems was analysed, e.g., by Bahaa (2017). The use of a combination of differential evolution and a Gaussian process to parameter identification of the model of a chaotic financial system is presented by Wang *et al.* (2022).

The papers of Ostalczyk and Mozyrska (2017) as well as Mozyrska and Ostalczyk (2017) propose discrete approximation using the GL definition and FOBD approximation along with its application to construct variable fractional order inertial and oscillatory transfer

[1] The list of acronyms used in the text is included at the end of Introduction.

functions. In these papers, time variable fractional order was included in FOBD approximator factors. This approach was continued by Mozyrska *et al.* (2025), proposing the use of variable fractional order in the FOPID controller.

Variable parameter fractional systems with time-invariant order need to be mentioned, too. The latest research in this area focuses mainly on control problems. For example, Matychyn and Onyshchenko (2021) deal with time optimal control, Jolic and Konjik (2023), Sivalingam and Govindaraj (2024), Sivalingam *et al.* (2024) and Vishnukumar *et al.* (2024) discuss the controllability and observability of these systems, while Kaczorek and Ruszewski (2024) propose the extension of the Floquet–Lyapunov transformation to fractional discrete-time linear systems with periodic varying parameters.

It is characteristic that all results mentioned assume variability of fractional order or, alternatively, parameters of a model. A model using both variable order and other parameters has not been proposed yet.

This paper presents new, discrete fractional order transfer function models with time variant order and a time variable quasi time constant. Such models have not been proposed yet. In our design, the approach by Mozyrska and Ostalczyk (2017) is expanded to describe both variable order and the variable quasi time constant. The proposed models are variable parameter generalization of the elementary, fractional order inertial transfer function. The use of FOBD allows us to express time variable parameters as time invariant parameters of a high order, discrete transfer function.

The paper is organized as follows. Preliminaries draw theoretical background to present the main results. Next, both proposed transfer functions are introduced and their accuracy and stability are discussed. Finally, theoretical results are illustrated by simulations.

Acronyms. The acronyms used in the text include the following:

- FO: fractional order,
- C definition: Caputo definition,
- GL definition: Grünwald–Letnikov definition,
- FOBD: fractional order backward difference,
- FOPID controller: fractional order PID controller,
- RL definition: Riemann–Liouville definition,
- VO: variable order,
- VT constant: variable time constant,
- VOVT constant: variable order and variable time constant.

2. Preliminaries

2.1. Basics of fractional calculus. Elementary ideas from fractional calculus can be found in many books (e.g., Das, 2010; Kaczorek, 2011; Ostalczyk, 2016; Podlubny, 1999). Here, only some definitions necessary to present the main results will be recalled. First of all, the fractional-order, integro-differential operator (see, e.g., Das, 2010; Kaczorek and Rogowski, 2014; Podlubny, 1999) needs to be given. It is as follows.

Definition 1. (*Elementary fractional order operator*) The fractional-order integro-differential operator is defined as

$$ {}_{t_s}D^{\alpha}_{t_f} f(t) = \begin{cases} \dfrac{d^{\alpha} f(t)}{dt^{\alpha}}, & \alpha > 0, \\ f(t), & \alpha = 0, \\ \int\limits_{t_s}^{t_f} f(\tau)(d\tau)^{\alpha}, & \alpha < 0, \end{cases} \tag{1} $$

where t_s and t_f denote time limits for operator calculation while $\alpha \in \mathbb{R}$ is the non-integer order of the operation.

Next, recall the Gamma Euler function given, e.g., by Kaczorek and Rogowski (2014).

Definition 2. (*Gamma function*) We have

$$ \Gamma(x) = \int\limits_{0}^{\infty} t^{x-1} e^{-t} dt. \tag{2} $$

The Mittag-Leffler function is non-integer order generalization of the exponential function $e^{\lambda t}$ and plays a crucial role in solution of the FO state equation. The one parameter Mittag-Leffler function is defined as follows.

Definition 3. (*One parameter Mittag-Leffler function*) We have

$$ E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \tag{3} $$

The two parameter Mittag-Leffler function is defined as follows.

Definition 4. (*Two parameter Mittag-Leffler function*) We have

$$ E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}. \tag{4} $$

For $\beta = 1$, the two parameter function (4) turns into the one parameter function (3).

The fractional-order, integro-differential operator can be described by different definitions, given by Grünvald and Letnikov (GL defintion), Riemann and Liouville (RL definition) and Caputo (C definition). In further deliberations, only GL definitions will be used. They are recalled below.

The GL derivative along time from a function $g(t)$ is defined as follows (Caponetto *et al.*, 2010; Ostalczyk, 2012).

Definition 5. (*Grünwald–Letnikov definition*) We have

$$ {}_0^{GL}D_t^{\alpha} g(t) = \lim_{h\to 0} h^{-\alpha} \sum_{l=0}^{[\frac{t}{h}]} (-1)^l \binom{\alpha}{l} g(t-lh). \quad (5)$$

In (5), $0.0 < \alpha \leq 1.0$ is fractional order along time, h is sample time, $[\dots]$ is the nearest integer value, and $\binom{\alpha}{l}$ is the binomial coefficient,

$$\binom{\alpha}{l} = \begin{Bmatrix} 1, & l=0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{l!}, & l>0 \end{Bmatrix}. \quad (6)$$

2.2. Elementary FO transfer function. The elementary, scalar input-output differential equation using the elementary fractional operator (1) takes the following form:

$$T_{\alpha 0} D_t^{\alpha} y(t) = -y(t) + u(t), \quad (7)$$

where T_α is the quasi time constant, expressed in $[s^{\frac{1}{\alpha}}]$, $u(t)$ is the control signal and $y(t)$ is the output.

The elementary, fractional order transfer function takes the following form:

$$G_C(s) = \frac{1}{Ts^{\alpha}+1}. \quad (8)$$

For this transfer function, its impulse and step responses are as below (see, e.g., Caponetto *et al.*, 2010, p.11):

$$g_C(t) = \frac{t^{\alpha-1}}{T_\alpha} E_\alpha\left(-\frac{t^\alpha}{T_\alpha}\right), \quad (9)$$

$$y_C(t) = 1(t) - E_\alpha\left(-\frac{t^\alpha}{T_\alpha}\right). \quad (10)$$

In (9) and (10), $E_\alpha(\dots)$ is the one parameter Mittag-Leffler function (3).

2.3. FOBD approximator and its variable order generalization. The GL definition is the limit case for $h \to 0$ of FOBD, commonly employed in discrete FO calculations (see, e.g., Ostalczyk, 2016, p. 68).

Definition 6. (*Fractional order backward difference along time*) We have

$$\Delta^{\alpha} g(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} g(t-lh). \quad (11)$$

In (11), h is sample time and L denotes a memory length necessary for correct approximation of a non-integer order operator. Unfortunately, good accuracy of approximation requires the use of long memory L, which can make implementation difficult.

Denote coefficients $(-1)^l\binom{\alpha}{l}$ by d_l:

$$d_l = (-1)^l \binom{\alpha}{l}. \quad (12)$$

The coefficients (12) can be also computed using the following, equivalent, recursive formula given, e.g., by Caponetto *et al.* (2010, p. 12), which is useful in numerical calculations:

$$\begin{aligned} d_0 &= 1, \\ d_l &= \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1,\dots,L. \end{aligned} \quad (13)$$

Busłowicz and Kaczorek (2009) proved that

$$\sum_{l=1}^{\infty} d_l = 1-\alpha, \quad (14)$$

$$\sum_{l=0}^{\infty} d_l = 0. \quad (15)$$

Using (12), the operator (11) can be expressed in shorter form:

$$\Delta^{\alpha} g(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} d_l g(t-lh), \quad (16)$$

and, consequently, its discrete transfer function $G(z^{-1})$ takes the following form:

$$G(z^{-1}) = \frac{1}{h^\alpha} \sum_{l=0}^{L} d_l z^{-l}. \quad (17)$$

Next, assume that order α can take various values at discrete time instants:

$$\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_L\}. \quad (18)$$

Variable order generalization of the operator (11) using the various orders (18) was proposed by Ostalczyk (2010), followed by Ostalczyk and Mozyrska (2017). It is employed also by Mozyrska *et al.* (2025) in the form analogical to (16):

$$\Delta^{\alpha} g(t) = \sum_{l=0}^{L} a_l g(t-lh), \quad (19)$$

where

$$\begin{aligned} a_0 &= \frac{1}{h^{\alpha_0}}, \\ a_l &= \frac{d_{\alpha_l}}{h^{\alpha_l}}, \quad l = 1,\dots,L. \end{aligned} \quad (20)$$

Coefficients d_{α_l} are computed using the recursive formula (13) and variable order α:

$$\begin{aligned} d_{\alpha_0} &= 1, \\ d_{\alpha_l} &= \left(1 - \frac{1+\alpha_l}{l}\right) d_{\alpha_{l-1}}, \quad l = 1,\dots,L. \end{aligned} \quad (21)$$

The transfer function (8) using (19) takes the following form, analogical to that given by Ostalczyk and Mozyrska (2017):

$$G_{VO}(z^{-1}) = \frac{1}{T\sum_{l=0}^{L} a_l z^{-l} + 1}. \tag{22}$$

Elementary properties of this transfer function are discussed by Ostalczyk and Mozyrska (2017). Some new results as well as generalization to VOVT form are proposed in the next section.

2.4. Discrete systems: Selected results. Next, some theorems from the theory of discrete time dynamic systems, necessary to present the main results, should be recalled. These are the inital value theorem (IVT), the final value theorem (FVT) and the necessary condition for the asymptotic stability of a system described by a discrete transfer function $G^+(z)$. They are given below.

Theorem 1. (Initial value theorem for discrete time) *Let $g(k)$ be a discrete function of time, defined in k time instants, and $G(z)$ be its z-transform. If the limit* (23) *exists, then the initial value $g(0)$ takes the following form:*

$$g(0) = \lim_{k\to 0} g(k) = \lim_{z\to\infty} G(z). \tag{23}$$

Theorem 2. (Final value theorem for discrete time) *Let $g(k)$ be a discrete function of time, defined in k time instants, and $G(z)$ be its z-transform. Assume that $G^+(z)$ has*

1. *no poles outside the unit circle,*
2. *maximally one pole on the unit circle: $z = 1$.*

Then,

$$\lim_{k\to\infty} g(k) = \lim_{z\to 1}(z-1)G(z). \tag{24}$$

Theorem 3. (Necessary condition for the asymptotic stability of the discrete polynomial $w(z)$) *Consider the characteristic polynomial of a discrete system: $w(z) = a_N z^N + \cdots + a_1 z + a_0$. The necessary condition for its asymptotic stability is as follows:*

$$\begin{aligned} &w(1) > 0,\\ &w(-1) > 0,\ N - \textit{even},\\ &w(-1) < 0,\ N - \textit{odd},\\ &|a_0| < a_N. \end{aligned} \tag{25}$$

Next, two sufficient conditions for instability are given, e.g., by Byrski (2007).

Theorem 4. (Sufficient condition for the unstability of the discrete polynomial $w(z)$) *Consider the characteristic polynomial of a discrete system: $w(z) = z^N + z^{N-1}b_{N-1} + \cdots + b_1 z + b_0$. The polynomial is unstable if*

$$\begin{aligned} &|b_{N-1}| > N,\\ &\vee\\ &|b_0| > 1. \end{aligned} \tag{26}$$

3. Main results

3.1. Initial and final values of the step response of the $G_{VO}(z^{-1})$ transfer function. First, the formulae of the inial and final values of the step response for the $G_{VO}(z^{-1})$ transfer function (22) can be given. They allow us to estimate the accuracy of approximation, because for the transfer function (8) the initial value of its step response equals zero and the final value of its step response tends to one.

Remark 1. (*Initial value of the step response*) Consider the $G_{VO}(z^{-1})$ transfer function (17). The initial value y_0 of its step response is

$$y_0 = \frac{h^{\alpha_0}}{T + h^{\alpha_0}}, \tag{27}$$

where T and h are a time constant and sample time, respectively, and α_0 is the initial value of the order.

This condition follows directly from the initial value theorem (23). It can be observed immediately that

$$\lim_{h\to 0} y_0 = 0. \tag{28}$$

Remark 2. (*Steady-state value of the step response*) Consider the $G_{VO}(z^{-1})$ transfer function (17). The steady-state value of its step response y_{ss} is

$$y_{ss} = \frac{1}{T\sum_{l=0}^{L} a_l + 1}, \tag{29}$$

where a_l are coefficients described by (20).

The condition (29) follows directly from the FVT (24).

3.2. Transfer function $G_{VT}(z^{-1})$. Next, time-variant generalization of the transfer function (17) can be proposed. Assume that the order α of the transfer fuction is constant, but the quasi time constant T can take various values from bounded interval $[\underline{T}; \overline{T}]$ for particular time instants $l = 0, \ldots, L$. This can be written as follows:

$$G_{VT}(z^{-1}) = \frac{1}{h^{-\alpha}\sum_{l=0}^{L} T_l d_l z^{-l} + 1}, \tag{30}$$

where d_l are described by (13) and T_l are bounded values of T for the time instants $l = 0, \dots, L$:

$$T = \{T_0, T_1, \dots, T_L\} \in [\underline{T}; \overline{T}] \subset \mathbb{R}^+,\ \underline{T} \le \overline{T},\ \overline{T} < \infty. \quad (31)$$

Note that the approximated transfer function (30) containing the time-variant T_l really is the typical transfer function of order L with various time constants $T_0 - T_L$.

The step response of the transfer function (30) is computed analogically as for the transfer function with an invariant time constant:

$$y_{VT}(k) = \mathbb{Z}^{-1}\{\frac{G_{VT}(z^{-1})}{1 - z^{-1}}\}. \quad (32)$$

The initial and final values of the step response (32) are described by the following remarks, directly derived from (27) and (29).

Remark 3. (*Initial value of the step response of the $G_{VT}(z^{-1})$ transfer function*) Consider the transfer function (30). The initial value y_{0VT} of its step response is

$$y_{0VT} = \frac{h^{\alpha}}{T_0 + h^{\alpha}}, \quad (33)$$

where α is fractional order, T_0 is the initial value of the time constant and h is sample time.

The property (28) is kept for the proposed transfer function, too.

Remark 4. (*Steady-state value of the step response of the $G_{VT}(z^{-1})$ transfer function*) Consider the transfer function (30). The steady-state value of its step response y_{ssVT} is

$$y_{ssVT} = \frac{1}{\sum_{l=0}^{L} T_l d_l + 1}, \quad (34)$$

where d_l are coefficients described by (13) and T_l are values of the time constant for the time moments $l = 0, \dots, L$.

For constant $T = T_c\ \forall l = 0, \dots, L$, assuming $L \to \infty$ and using the condition (15), we obtain

$$\lim_{L\to\infty} y_{ssVT} = 1. \quad (35)$$

3.3. Transfer function $G_{VOVT}(z^{-1})$.

Finally, consider the order-variant and time-variant generalization of the transfer function (17). It is described as follows:

$$G_{VOVT}(z^{-1}) = \frac{1}{\sum_{l=0}^{L} T_l a_l z^{-l} + 1}, \quad (36)$$

where a_l and T_l are given by (20) and (31), respectively.

The transfer function (36) is also the discrete transfer function with constant parameters of order L.

The step response of the transfer function (36) is computed analogically as for the transfer function with an invariant time constant:

$$y(k) = \mathbb{Z}^{-1}\{\frac{G_{VT}(z^{-1})}{1 - z^{-1}}\}. \quad (37)$$

The initial and final values of the step response (37) are described by the following remarks, analogical to (27) and (29).

Remark 5. (*Initial value of the step response of the $G_{VT}(z^{-1})$ transfer function*) Consider the transfer function (36). The initial value y_0 of its step response is

$$y_0 = \frac{h^{\alpha_0}}{T_0 + h^{\alpha_0}}, \quad (38)$$

where α_0 and T_0 are the initial values of the order and the time constant, respectively, and h is sample time.

The property (28) is kept for the proposed transfer function, too.

Remark 6. (*Steady-state value of the step response of the $G_{VT}(z^{-1})$ transfer function*) Consider the transfer function (36). The steady-state value of its step response y_{ss} is

$$y_{ss} = \frac{1}{\sum_{l=0}^{L} T_l a_l + 1}, \quad (39)$$

where a_l are coefficients described by (20) and T_l are values of the time constant for the time moments $l = 0, \dots, L$.

For constant α and T, the property (35) is kept, too.

3.4. Stability.

The stability analysis for the proposed VT and VOVT transfer functions requires adopting the conditions (25)–(26). Such adaptations are proposed below. They use the characteristic polynomial of the transfer function (30) or (36).

First, we deal with the VT transfer function. Its characteristic polynomial takes the following form:

$$w_{VT}(z^{-1}) = (h^{\alpha} + T_0) + \sum_{l=1}^{L} T_l d_l z^{-l}, \quad (40)$$

where d_l and T_l are expressed by (13) and (31), respectively.

The conditions for the stability of the polynomial (40) are given below.

Remark 7. (*Necessary condition for the asymptotic stability of the transfer function $G_{VT}(z^{-1})$*) Consider the characteristic polynomial (40). The necessary condition for its stability requires meeting all of the following conditions together:

$$(h^\alpha + T_0) > -\sum_{l=1}^{L} T_l d_l, \tag{41}$$

$$\begin{aligned}(h^\alpha + T_0)(-1)^L + \sum_{l=1}^{L}(-1)^{L-l} T_l d_l > 0, \quad L - \text{even},\\ (h^\alpha + T_0)(-1)^L + \sum_{l=1}^{L}(-1)^{L-l} T_l d_l < 0, \quad L - \text{odd},\end{aligned} \tag{42}$$

$$|T_L d_L| < (h^\alpha + T_0). \tag{43}$$

In turn, the sufficient unstability conditions (26) can be formulated as follows.

Remark 8. (*Sufficient condition for unstability*) Consider the characteristic polynomial (40). It will be unstable if

$$\left|\frac{T_1 d_1}{h^\alpha + T_0}\right| > L, \tag{44}$$

$$\vee$$

$$\left|\frac{T_L d_L}{h^\alpha + T_0}\right| > 1, \tag{45}$$

where d_1 and d_L can be computed with the use of (13).

The necessary condition for stability or, equivalently, the sufficient condition for unstability for the range of the quasi time constant T is described by the following proposition.

Proposition 1. (Sufficient condition of unstability for the range of the quasi time constant T) *Consider the characteristic polynomial* (40) *with the variable quasi time constant T described by* (31). *Assume that the time constant is monotonically increasing: $\underline{T} = T_0 \le T_1 \le \cdots \le T_L = \overline{T}$. The polynomial* (47) *will be unstable if*

$$\overline{T} > \frac{h^\alpha + \underline{T}}{|d_L|}. \tag{46}$$

where d_L is described by (13).

This proposition follows directly from (43) or, equivalently, from (44).

Next, the VTVO transfer function is considered. Its characteristic polynomial is

$$w_{VTVO}(z^{-1}) = (1 + T_0 a_0) + \sum_{l=1}^{L} T_l a_l z^{-l}, \tag{47}$$

where a_l and T_l are expressed by (20) and (31), respectively.

Suitable stability conditions are as follows.

Remark 9. (*Necessary condition for the asymptotic stability of the transfer function $G_{VTVO}(z^{-1})$*) Consider the characteristic polynomial (47). The necessary condition for its stability requires to meet all of the following conditions:

$$(1 + T_0 a_0) > -\sum_{l=1}^{L} T_l a_l, \tag{48}$$

$$\begin{aligned}(1 + T_0 a_0)(-1)^L + \sum_{l=1}^{L}(-1)^{L-l} T_l a_l > 0, \quad L - \text{even},\\ (1 + T_0 a_0)(-1)^L + \sum_{l=1}^{L}(-1)^{L-l} T_l a_l < 0, \quad L - \text{odd},\end{aligned} \tag{49}$$

$$|T_L a_L| < (1 + T_0 a_0). \tag{50}$$

In turn, the sufficient unstability conditions (26) can be formulated as follows.

Remark 10. (*Sufficient condition for unstability*) Consider the characteristic polynomial (47). It will be unstable if

$$\left|\frac{T_1 a_1}{1 + T_0 a_0}\right| > L, \tag{51}$$

$$\vee$$

$$\left|\frac{T_L a_L}{1 + T_0 a_0}\right| > 1. \tag{52}$$

The necessary condition for stability or, equivalently, the sufficient condition for unstability for the range of the time constant T is described by the following proposition.

Proposition 2. (Sufficient condition of unstability for the range for the time constant T) *Consider the characteristic polynomial* (47) *with the variable time constant T described by* (31). *Assume that the time constant is monotonically increasing: $\underline{T} = T_0 \le T_1 \le \cdots \le T_L = \overline{T}$. The polynomial* (47) *will be unstable if*

$$\overline{T} > \frac{1 + \underline{T} a_0}{|a_L|}, \tag{53}$$

where a_0 and a_L are described by (20).

This proposition follows directly from (50) or, equivalently, from (52).

4. Numerical tests and simulations

The simulations were performed for all transfer functions considered and various functions describing the order α and time constant T. For all experiments, the memory length was equal $L = 100$. This value is a compromise between accuracy and numerical complexity of FOBD approximation.

Table 1. Steady state values y_{ss} of the step response for all scenarios of changes in α and $T\ [s^{\frac{1}{\alpha}}]$.

	$T_{const} = 2$	T_{in}	T_{de}	$T_{\min}$	$T_{\max}$
α_{const}	0.7363	1.0842	0.4196	0.4060	1.4666
α_{in}	0.6016	1.3530	0.2753	0.2685	2.1653
α_{de}	0.8617	0.9835	0.6501	0.6376	1.0797
$\alpha_{\max}$	0.6348	1.1511	0.2879	0.2664	35.7277
$\alpha_{\min}$	0.7008	1.0364	0.4064	0.4018	1.1364

The variable order α was described by the following, piecevise constant functions:

$$\alpha_{const} = 0.50, \quad l = 0, \ldots, 100, \tag{54}$$

$$\alpha_{in} = \begin{cases} 0.25, & l = 0, \ldots, 30, \\ 0.50, & l = 31, \ldots, 70, \\ 0.75, & l = 71, \ldots, 100, \end{cases} \tag{55}$$

$$\alpha_{de} = \begin{cases} 0.75, & l = 0, \ldots, 30, \\ 0.50, & l = 31, \ldots, 70, \\ 0.25, & l = 71, \ldots, 100, \end{cases} \tag{56}$$

$$\alpha_{max} = \begin{cases} 0.25, & l = 0, \ldots, 30, \\ 0.75, & l = 31, \ldots, 70, \\ 0.25, & l = 71, \ldots, 100, \end{cases} \tag{57}$$

$$\alpha_{min} = \begin{cases} 0.50, & l = 0, \ldots, 30, \\ 0.25, & l = 31, \ldots, 70, \\ 0.50, & l = 71, \ldots, 100. \end{cases} \tag{58}$$

The quasi time constants considered were examined according to

$$T_{const} = 1, 2, 5\ [s^{\frac{1}{\alpha}}], \quad l = 0, \ldots, 100, \tag{59}$$

$$T_{in} = \begin{cases} 1[s^{\frac{1}{\alpha}}], & l = 0, \ldots, 30, \\ 2[s^{\frac{1}{\alpha}}], & l = 31, \ldots, 70, \\ 5[s^{\frac{1}{\alpha}}], & l = 71, \ldots, 100, \end{cases} \tag{60}$$

$$T_{de} = \begin{cases} 5[s^{\frac{1}{\alpha}}], & l = 0, \ldots, 30, \\ 2[s^{\frac{1}{\alpha}}], & l = 31, \ldots, 70, \\ 1[s^{\frac{1}{\alpha}}], & l = 71, \ldots, 100, \end{cases} \tag{61}$$

$$T_{max} = \begin{cases} 1[s^{\frac{1}{\alpha}}], & l = 0, \ldots, 30, \\ 5[s^{\frac{1}{\alpha}}], & l = 31, \ldots, 70, \\ 2[s^{\frac{1}{\alpha}}], & l = 71, \ldots, 100, \end{cases} \tag{62}$$

$$T_{min} = \begin{cases} 5[s^{\frac{1}{\alpha}}], & l = 0, \ldots, 30, \\ 1[s^{\frac{1}{\alpha}}], & l = 31, \ldots, 70, \\ 2[s^{\frac{1}{\alpha}}], & l = 71, \ldots, 100. \end{cases} \tag{63}$$

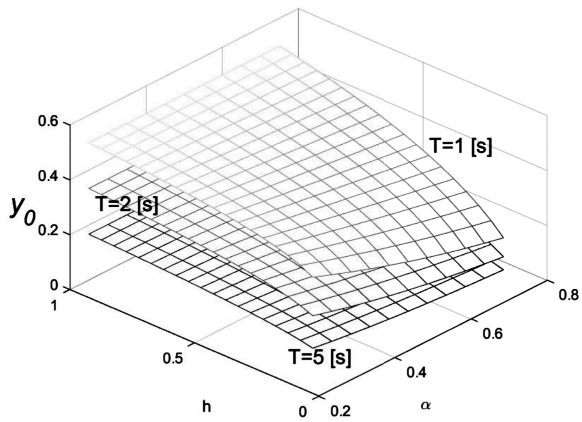

Fig. 1. Initial value of the step response y_0 as a function of order α and sample time h for various quasi time constants T_{const}.

4.1. Initial and final accuracy.

First, the initial and final accuracy of the approximated transfer function will be examined. This will be done with the use of the initial and final values of the step responses (33) and (34).

In general, the initial accuracy, expressed by (27), (33) or (38), is determined by the initial values of α or T and by sample time h. This does not depend on the model.

Figure 1 presents the 3D diagrams of y_0 for $h \in [0.05; 1.0][s], \alpha \in [0.25; 0.75]$ and all quasi time constants T_{const} (59). It shows that the initial accuracy descreases for descreasing α and an increasing quasi time constant T; it increases for increasing sample time h.

The values of steady-state responses y_{ss} expressed by (34)–(38) and the parameters (54)–(63) are included in Table 1.

The table shows that the steady-state value of the step response is strongly determined by both variable parameters of the proposed model. The most accurate in the sense of its steady-state response is the model with increasing T and decreasing α.

4.2. Stability.

Stability was examined for both transfer functions $G_{VT}(\ldots)$, $G_{VTVO}(\ldots)$ and numerical data (54)–(63). During the test, first the conditions (41)–(43)

Table 2. Meeting of the necessary stability conditions (41)–(43) and (48)–(50) for all scenarios of changes in α and T $[s^{\frac{1}{\alpha}}]$.

	$T_{const} = 2$	T_{in}	T_{de}	$T_{\min}$	$T_{\max}$
α_{const}	Y	Y	Y	Y	Y
α_{in}	Y	Y	Y	Y	Y
α_{de}	Y	Y	Y	Y	Y
$\alpha_{\max}$	Y	Y	Y	Y	Y
$\alpha_{\min}$	Y	Y	Y	Y	Y

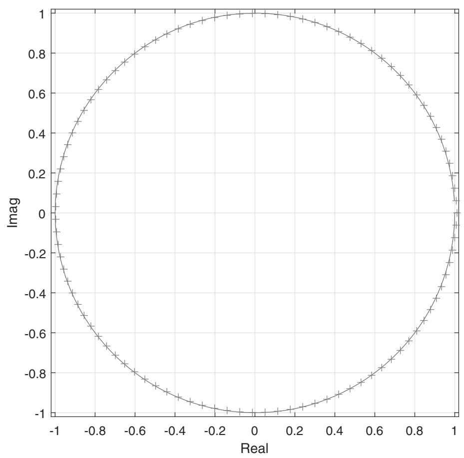

Fig. 2. Roots of the unstable characteristic polynomial (40) for $T_{100} = 4600\ [s^{\frac{1}{\alpha}}]$.

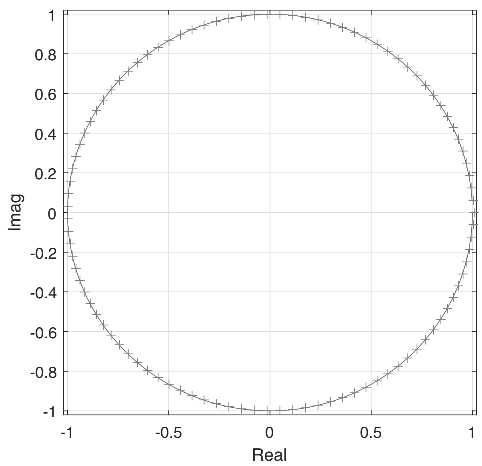

Fig. 3. Roots of the unstable characteristic polynomial (47) for $T_{100} = 4030\ [s^{\frac{1}{\alpha}}]$.

and (48)–(50) were employed. Results are given in Table 2. Here "Y" means that the stability condition is met, otherwise a suitable cell is marked as "N".

Table 2 shows that the necessary condtion is met for all tested scenarios.

Next, the unstability conditions for the transfer function $G_{VT}(\ldots)$ were examined. To do that, values of the time constants T_1 and T_L were calculated for $T_0 = 1[s^{\frac{1}{\alpha}}]$, $\alpha = 0.5$, $L = 100$ and $h = 0.1$. Using (44), (45) we obtain

$$T_1 \geq 263.2456\ [s^{\frac{1}{\alpha}}],$$

$$T_{100} \geq 4578.70\ [s^{\frac{1}{\alpha}}].$$

To verify these limit values T_1 and T_{100}, the roots of the characteristic polynomial (40) were calculated. This was done for $T_1 = 270\ [s^{\frac{1}{\alpha}}]$ using the MATLAB function *roots*. The polynomal has one unstable root with a module equal to 102.5668.

Next, for $T_{100} = 4600\ [s^{\frac{1}{\alpha}}]$, the characteristic polynomial has also unstable roots, shown in Fig. 2. In this case, the maximum module of an unstable root is equal to 1.0115.

The sufficient unstability conditions for the transfer function $G_{VTVO}(\ldots)$ were tested for the order α expressed by (55) and the quasi time constant T described by (63). The application of these parameters to (51) and (52) yields

$$T_1 \geq 2224.9\ [s^{\frac{1}{\alpha}}],$$

$$T_L \geq 4025.90\ [s^{\frac{1}{\alpha}}].$$

To test these results assume $T_1 = 2250$. This gives one unstable root with a module equal to 101.1273. For $T_{100} = 4030$, we also obtain an unstable polynomial, whose roots are illustrated in Fig. 3. The maximum module of an unstable root is equal to 1.0088.

4.3. Step responses of the model. Finally, step responses of the proposed models need to be given. They were prepared with the use of the MATLAB function *step* and the parameters described by (54)–(63). Examples are given in Figs. 4–6.

Those figures show that the shape of the step response is determined by the variability of both parameters. This was to be expected and is the advantage of the proposed approach, because it gives us an additional tool to precisely fit the model to experimental data.

5. Conclusions

The main conclusion of the paper is that the proposed, time variant, transfer function models are simple in analysis with the use of typical tools and easy to implement on typical digital platforms.

The proposed models expand the usefulness of the fractional approach to various specific cases. For example, they allow us to describe variable-structure physical systems and processes.

Another idea for the use of the proposed models are control algorithms with time-variable parameters, for example, model-based control for time-varying systems.

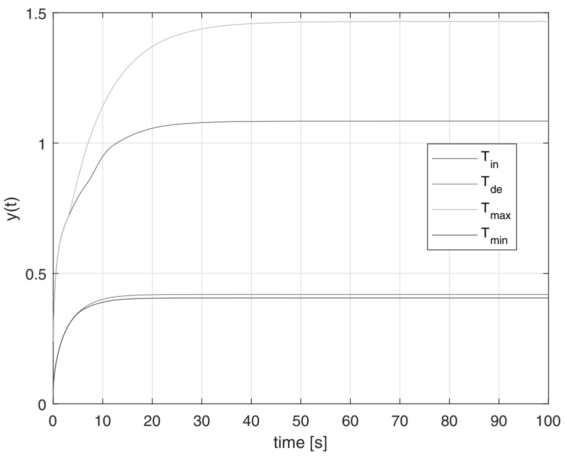

Fig. 4. Step responses for $\alpha = 0.5$ and T described by (60)–(63).

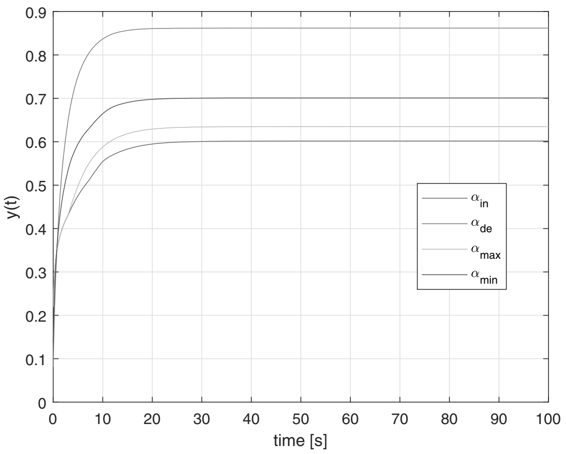

Fig. 5. Step responses for $T = 2$ and α described by (55)–(58).

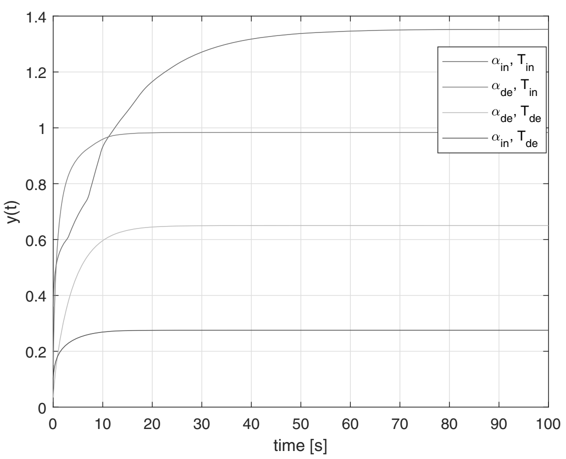

Fig. 6. Step responses for increasing and decreasing T and α, expressed by (60), (61), (55) and (56).

The area of further exploration covers stronger justification of their stability and expanding the proposed approach to state space models.

Another interesting issue is effective identification of variable parameters of the proposed models with the use of real, experimental data. It is important to note that each discrete value α_l or T_l can be identified independently of others. On the one hand, this greatly complicates identification, but on the other, it allows achieving very high model accuracy. To do that, biologically inspired methods can be applied.

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