

A COMBINATION OF TWO CONJUGATE GRADIENT METHODS UNDER A NEW LINE SEARCH WITH ITS APPLICATION IN IMAGE RESTORATION PROBLEMS

ASMA MAIZA ^a, RAOUF ZIADI ^a, MOHAMMED A. SALEH ^{b,*}, ABDULGADER Z. ALMAYMUNI ^b

^aLaboratory of Fundamental and Numerical Mathematics (LMFN)
Department of Mathematics, University Setif-1-Ferhat Abbas, Setif, Algeria
e-mail: {asma.maiza.1996, ziadi.raouf}@gmail.com

^bDepartment of Cybersecurity
College of Computer, Qassim University, Saudi Arabia
e-mail: {m.saleh, almaymuni}@qu.edu.sa

A combined conjugate gradient algorithm is introduced for solving unconstrained optimization problems. In the suggested approach, the conjugate gradient parameter is defined as a combination of PRP (Polak-Ribière-Polyak) and BRB (Rahali-Belloufi-Benzine) conjugate gradient parameters. To improve the convergence properties, we have adopted a new inexact line search technique that fits in with the suggested approach. The proposed line search technique can be useful for other gradient descent methods. We have established the existence of a step length that meets the new line search conditions. The generated descent direction and the convergence properties of the suggested approach are studied under the new line search conditions and the proposed method converges globally under mild assumptions. Our approach is evaluated on various test functions, and a comparison with similar recent algorithms is carried out. Furthermore, the proposed algorithm is applied for restoring images with different noise levels.

Keywords: unconstrained optimization, conjugate gradient methods, inexact line search, global convergence, image processing.

1. Introduction

In this study, we are interested in the following unconstrained optimization problem:

$$f^* = \min_{x \in \mathbb{R}^n} f(x), \quad (\text{P})$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable. Numerous practical problems in real-life applications can be expressed as unconstrained optimization problems that involve differentiable cost functions (see, Ziadi and Bencherif-Madami, 2024; 2025; Yousif and Saleh, 2024). The solution of these problems becomes difficult when their dimensions are high. Scientists have explored various techniques, such as Newton's method, quasi-Newton methods, and conjugate gradient (CG) methods, to find the most efficient way to solve a particular problem. The conjugate gradient methods

(CG) have become one of the favorite approaches thanks to their straightforward iterative process and low memory requirements (see, Chen *et al.*, 2024; Sulaiman *et al.*, 2024); they are also widely employed in numerous applications, in particular in image processing (Khudhur and Halil, 2024; Souli *et al.*, 2025), neural networks (Saleh, 2023), grid computing (Collignon and Van Gijzen, 2010), molecular physics (Ziadi *et al.*, 2017) and statistical modeling (Mehamdia *et al.*, 2025). Starting from a point $x_0 \in \mathbb{R}^n$, a sequence of points $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ is generated by the following recursive scheme:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k \in \mathbb{N}, \quad (1)$$

where d_k is the descent direction and α_k is the step length that ensures that $f(x_{k+1}) \leq f(x_k)$.

The determination of the step length α_k is crucial for ensuring global convergence. Usually, it is determined using inexact line searches, which are guaranteed to take

*Corresponding author

steps that should be neither too long nor too short, such as the weak Wolfe line search

$$\begin{aligned} f(x_k + \alpha_k d_k) - f(x_k) &\leq \alpha_k \delta d_k^t g_k, \\ g(x_k + \alpha_k d_k)^t d_k &\geq \sigma d_k^t g_k, \end{aligned}$$

or the strong Wolfe line search

$$\begin{aligned} f(x_k + \alpha_k d_k) - f(x_k) &\leq \alpha_k \delta d_k^t g_k, \\ |g(x_k + \alpha_k d_k)^t d_k| &\leq -\sigma d_k^t g_k, \end{aligned}$$

where $0 < \delta < \sigma < 1$. The descent search direction d_k is typically computed by the following iterative formula:

$$d_0 = -g_0, \quad d_{k+1} = -g_{k+1} + \beta_k d_k, \quad k \in \mathbb{N}^*,$$

where $\beta_k \in \mathbb{R}$ is the conjugate parameter that characterizes the various conjugate gradient variants.

The most famous classical conjugate gradient methods include HS (Hestenes and Stiefel, 1952), FR (Fletcher and Reeves) (Fletcher, 1997), PRP (Polak-Ribière-Polyak) (Polyak, 1969; Polak and Ribière, 1969), CD (conjugate descent) (Fletcher, 1997), LS (Liu and Storey, 1991), and DY (Dai and Yuan) (Dai and Yuan, 2001), where their parameter β_k is given respectively as follows:

$$\begin{aligned} \beta_k^{\text{HS}} &= \frac{g_{k+1}^t y_k}{d_k^t y_k}, & \beta_k^{\text{PRP}} &= \frac{g_{k+1}^t y_k}{\|g_k\|^2}, \\ \beta_k^{\text{LS}} &= -\frac{g_{k+1}^t y_k}{g_k^t d_k}, & \beta_k^{\text{FR}} &= \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \\ \beta_k^{\text{CD}} &= -\frac{\|g_{k+1}\|^2}{g_k^t d_k}, & \beta_k^{\text{DY}} &= \frac{\|g_{k+1}\|^2}{d_k^t y_k}, \end{aligned}$$

where $y_k = g_{k+1} - g_k$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. The DY, CD and FR versions have better theoretical convergence properties, but practically, they are less effective. Conversely, the LS, HS and PRP methods are more efficient in practice, but they may not always be convergent.

Apart from the six classical methods mentioned above, other methods have proved their efficiency and obtained good theoretical convergence properties. Wei et al. (2006) proposed a competitive CG method that converges globally under weak Wolfe conditions and for which β_k is

$$\beta_k^{\text{WYL}} = \frac{g_{k+1}^t (g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k)}{\|g_k\|^2}.$$

Inspired by the WYL formula, Zhang (2009) suggested a modification of the β_k^{WYL} as follows

$$\beta_k^{\text{NHS}} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} |g_{k+1}^t g_k|}{d_k^t y_k}.$$

Table 1. Hybrid conjugate gradient methods with convex combinations.

Formula	References
$\beta^{\text{hLSDY}} = \lambda \beta^{\text{DY}} + (1 - \lambda) \beta^{\text{LS}}$	Liu and Li (2014)
$\beta^{\text{hHSDY}} = \lambda \beta^{\text{DY}} + (1 - \lambda) \beta^{\text{HS}}$	Andrei (2009)
$\beta^{\text{hHSCD}} = \lambda \beta^{\text{CD}} + (1 - \lambda) \beta^{\text{HS}}$	Zheng et al. (2020)
$\beta^{\text{hLSCD}} = \lambda \beta^{\text{CD}} + (1 - \lambda) \beta^{\text{LS}}$	Djordjevic (2017)

This modification is efficient and the algorithm converges globally under the strong Wolfe conditions. Also, Hamoda et al. (2016) proposed a CG variant, which has superior convergence characteristics and whose parameter β_k is defined as

$$\beta_k^{\text{HRM}} = \frac{g_{k+1}^t \left(g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k \right)}{\theta \|g_k\|^2 + (1 - \theta) \|d_k\|^2}$$

with $0 \leq \theta \leq 1$.

Recently, there have been suggestions for new, effective conjugate gradient formulas. Rahali et al. (2021) have introduced a quite efficient CG method where the conjugate coefficient β_k is defined as

$$\beta_k^{\text{BRB}} = \frac{\|g_{k+1}\|^2}{\|d_k\|^2}, \quad (2)$$

In order to achieve effective performance and good convergence properties, numerous combinations of CG methods have been proposed. We gather some famous hybrid conjugate coefficients based on a convex combination in Table 1.

Building on the above discussion, in order to combine a good practical performance and powerful global convergence properties of both of BRB and PRP methods, we propose to combine them into a new hybrid method named hPB (hybrid PRP-BRB). To achieve good convergence properties, we adopt a new inexact line search technique where the step length α_k meets the following conditions:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \alpha_k \delta d_k^t g_k \frac{\|g_k\|^2}{\|d_k\|^2}, \quad (3)$$

$$|g(x_k + \alpha_k d_k)^t d_k| \leq -\sigma d_k^t g_k \frac{\|g_k\|^2}{\|d_k\|^2}, \quad (4)$$

where

$$\sigma \in \left(0, \frac{\mu - 1}{\mu^2 (\mu^2 + 1.2)} \right],$$

with $\mu > 1$ and $0 < \delta < \sigma$. Since $\mu > 1$, it follows that $0 < \delta < \sigma < 1$.

The new step length matches the combined algorithm and the approach converges globally under mild assumptions. The proposed method inherits the good practical performance characteristics of BRB and PRP methods and the algorithm is successfully applied to a broad set of test functions (with varied analytical expressions and structures) that range from the simplest to the hardest, as well as to image processing.

The new hybrid conjugate parameter β_k^{hPB} and the pseudocode of the hPB algorithm are described in the next section. Then, in Section 3 we study the descent direction properties. The convergence analysis and the global convergence are established in Section 5. The performance of the suggested approach is presented in the last section with some conclusions.

2. Proposed hPB (hybrid PRP-BRB) method

The new hybrid conjugate gradient parameter, β_k^{hPB} , is computed as follows:

$$\beta_k^{\text{hPB}} = \begin{cases} \beta_k^{\text{PRP}} & \text{if } \beta_k^{\text{PRP}} > 0, \\ \lambda_k \beta_k^{\text{BRB}} + (1 - \lambda_k) \beta_k^{\text{PRP}} & \text{if } \lambda_k \beta_k^{\text{BRB}} + (1 - \lambda_k) \beta_k^{\text{PRP}} > 0 \\ & \text{and } \beta_k^{\text{PRP}} \leq 0, \\ \beta_k^{\text{BRB}} & \text{otherwise.} \end{cases} \quad (5)$$

It is chosen positive, where the hybridization parameter λ_k is defined in such a way that the descent direction d_{k+1} fulfills the conjugacy condition, that is, $d_{k+1}^t y_k = 0$. Indeed, in the case where

$$\beta_k^{\text{hPB}} = \lambda_k \beta_k^{\text{BRB}} + (1 - \lambda_k) \beta_k^{\text{PRP}},$$

it follows that

$$d_{k+1} = -g_{k+1} + \beta_k^{\text{hPB}} d_k, \quad (6)$$

$$\begin{aligned} &= -g_{k+1} + (\lambda_k \beta_k^{\text{PRP}} + (1 - \lambda_k) \beta_k^{\text{BRB}}) d_k, \\ &= -g_{k+1} + \left(\lambda_k \frac{g_{k+1}^t y_k}{\|g_k\|^2} + (1 - \lambda_k) \frac{\|g_{k+1}\|^2}{\|d_k\|^2} \right) d_k. \end{aligned} \quad (7)$$

By pre-multiplying both the sides of (7) by y_k^t , it follows that

$$\begin{aligned} y_k^t d_{k+1} &= -y_k^t g_{k+1} + \left(\lambda_k \frac{g_{k+1}^t y_k}{\|g_k\|^2} + (1 - \lambda_k) \frac{\|g_{k+1}\|^2}{\|d_k\|^2} \right) y_k^t d_k. \end{aligned}$$

Using the conjugacy condition, it follows that

$$\lambda_k = \frac{\vartheta_k - \psi_k}{\phi_k}, \quad (8)$$

where

$$\begin{aligned} \psi_k &= \|g_{k+1}\|^2 \|g_k\|^2 y_k^t d_k, \\ \vartheta_k &= y_k^t g_{k+1} \|g_k\|^2 \|d_k\|^2, \\ \phi_k &= y_k^t g_{k+1} y_k^t d_k \|d_k\|^2 - \psi_k. \end{aligned}$$

For each iteration k , we set

$$\lambda_k = \max \left\{ 0, \min \left\{ \frac{\vartheta_k - \psi_k}{\phi_k}, 1 \right\} \right\},$$

and as the search progresses, if for an iteration $k \in \mathbb{N}$ we have $\phi_k = 0$, we set $\beta_k^{\text{hPB}} = \beta_k^{\text{BRB}}$. The main steps of the hPB method are sketched in Algorithm 1.

3. Sufficient descent property of hPB algorithm

Before proving the intended property, we first prove the following auxiliary result.

Theorem 1. *The sequences $\{g_k\}_{k \in \mathbb{N}}$ and $\{d_k\}_{k \in \mathbb{N}}$, generated by the hPB algorithm under the new line search conditions (3) and (4) with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}\right]$ and $\mu > 1$, satisfy*

$$\frac{\|g_k\|}{\|d_k\|} \leq \mu, \quad \forall k \in \mathbb{N}. \quad (9)$$

Proof. It is clear that in the case where Powell's restart criterion holds (i.e., $|g_k^t g_{k+1}| \geq 0.2 \|g_{k+1}\|^2$), the descent direction d_k is defined as $d_k = -g_k$ and the relation (9) holds.

Now, if the Powell condition does not hold, we prove the above relation by induction. For $k = 0$, the condition (9) holds since $d_0 = -g_0$. Assume that the relation (9) holds for $k \geq 1$, and let us prove it for $k + 1$. Since

$$d_{k+1} = -g_{k+1} + \beta_k^{\text{hPB}} d_k, \quad (10)$$

by multiplying both the sides of (10) by g_{k+1}^t , we obtain

$$d_{k+1}^t g_{k+1} = -\|g_{k+1}\|^2 + \beta_k^{\text{hPB}} d_k^t g_{k+1}. \quad (11)$$

On the other hand, we have

$$|\beta_k^{\text{hPB}}| \leq \frac{\|g_{k+1}\|^2}{\|d_k\|^2} + 1.2 \frac{\|g_{k+1}\|^2}{\|g_k\|^2}. \quad (12)$$

Indeed,

$$\begin{aligned} |\beta_k^{\text{hPB}}| &\leq |\beta_k^{\text{BRB}}| + |\beta_k^{\text{PRP}}| \\ &\leq \frac{\|g_{k+1}\|^2}{\|d_k\|^2} + \frac{\|g_{k+1}\|^2 + |g_{k+1}^t g_k|}{\|g_k\|^2} \\ &\leq \frac{\|g_{k+1}\|^2}{\|d_k\|^2} + 1.2 \frac{\|g_{k+1}\|^2}{\|g_k\|^2}. \end{aligned}$$

Algorithm 1. The hPB algorithm.

Require: Choose a scalar $\mu > 1$ and the parameters δ and σ such that $0 < \delta < \sigma < \frac{\mu-1}{\mu^2(\mu^2+1.2)}$. Choose a scalar $\epsilon > 0$ sufficiently small to stop the algorithm.

Initialization

Set $k = 0$ and select a starting point $x_0 \in \mathbb{R}^n$.

Set $g_0 = \nabla f(x_0)$ and $d_0 = -g_0$.

```

1: while  $\|g_k\| \geq \epsilon$  do
2:   Compute the step length  $\alpha_k$  using the new inexact
   line search technique (3) and (4).
3:   Compute  $x_{k+1} = x_k + \alpha_k d_k$ ,  $g_{k+1} = \nabla f(x_{k+1})$ .
4:   if  $|g_k^t g_{k+1}| \geq 0.2 \|g_{k+1}\|^2$  then
5:     Set  $d_{k+1} = -g_{k+1}$  (The restart criterion of
     Powell holds).
6:   else
7:     Set  $y_k = g_{k+1} - g_k$ .
8:     if  $\beta_k^{\text{PRP}} > 0$  then
9:       Set  $\beta_k^{\text{hPB}} = \beta_k^{\text{PRP}}$ .
10:    else
11:      Compute  $\phi_k = y_k^t g_{k+1} y_k^t d_k \|d_k\|^2 -$ 
         $\|g_{k+1}\|^2 \|g_k\|^2 y_k^t d_k$ .
12:      if  $\phi_k = 0$  then
13:        Set  $\beta_k^{\text{hPB}} = \beta_k^{\text{BRB}}$ .
14:      else
15:        Compute the hybridization parameter  $\lambda_k$ 
        following (8).
16:        Set  $\lambda_k = \max \left\{ 0, \min \left\{ \frac{\vartheta_k - \psi_k}{\phi_k}, 1 \right\} \right\}$ .
17:        if  $\lambda_k \beta_k^{\text{BRB}} + (1 - \lambda_k) \beta_k^{\text{PRP}} > 0$  then
18:          Set  $\beta_k^{\text{hPB}} = \lambda \beta_k^{\text{BRB}} + (1 - \lambda_k) \beta_k^{\text{PRP}}$ .
19:        else
20:          Set  $\beta_k^{\text{hPB}} = \beta_k^{\text{BRB}}$ .
21:        end if
22:      end if
23:    end if
24:    Compute the descent direction  $d_{k+1} = -g_{k+1} +$ 
         $\beta_k d_k$ .
25:  end if
26:  Set  $k = k + 1$ .
27: end while

```

Hence, from (11) and condition (4) it follows that

$$\begin{aligned}
\|g_{k+1}\|^2 &\leq |d_{k+1}^t g_{k+1}| + |\beta_k^{\text{hPB}}| |d_k^t g_{k+1}| \\
&\leq |d_{k+1}^t g_{k+1}| - \sigma |\beta_k^{\text{hPB}}| |d_k^t g_k| \frac{\|g_k\|^2}{\|d_k\|^2} \\
&\leq |d_{k+1}^t g_{k+1}| + \sigma |\beta_k^{\text{hPB}}| |d_k^t g_k| \frac{\|g_k\|^2}{\|d_k\|^2} \\
&\leq |d_{k+1}^t g_{k+1}| + \sigma |\beta_k^{\text{hPB}}| \|d_k\| \|g_k\| \frac{\|g_k\|^2}{\|d_k\|^2} \\
&\leq \|d_{k+1}\| \|g_{k+1}\| + \sigma |\beta_k^{\text{hPB}}| \frac{\|g_k\|^3}{\|d_k\|} \\
&\leq \|d_{k+1}\| \|g_{k+1}\| \\
&\quad + \sigma \left(1.2 \frac{\|g_k\|}{\|d_k\|} + \frac{\|g_k\|^3}{\|d_k\|^3} \right) \|g_{k+1}\|^2. \quad (13)
\end{aligned}$$

Furthermore, since

$$\sigma \leq \frac{\mu - 1}{\mu^2(\mu^2 + 1.2)},$$

we get

$$\begin{aligned}
1 - \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|} \right) \\
\geq 1 - \sigma(\mu^3 + 1.2\mu) \geq 1 - \frac{\mu - 1}{\mu} = \frac{1}{\mu} > 0.
\end{aligned} \quad (14)$$

Dividing both the sides of (13) by $\|g_{k+1}\| \cdot \|d_{k+1}\|$, we get

$$\frac{\|g_{k+1}\|}{\|d_{k+1}\|} \leq 1 + \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|} \right) \frac{\|g_{k+1}\|}{\|d_{k+1}\|},$$

which yields

$$\frac{\|g_{k+1}\|}{\|d_{k+1}\|} \left(1 - \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|} \right) \right) \leq 1.$$

Then from (14) it results that

$$\frac{\|g_{k+1}\|}{\|d_{k+1}\|} \leq \left(1 - \sigma \left(\frac{\|g_k\|^3}{\|d_k\|^3} + 1.2 \frac{\|g_k\|}{\|d_k\|} \right) \right)^{-1} \leq \mu.$$

The proof is complete. $\blacksquare$

Now, we are in a position to prove the sufficient descent property.

Theorem 2. The sequences $\{g_k\}_{k \in \mathbb{N}}$ and $\{d_k\}_{k \in \mathbb{N}}$, generated by the hPB algorithm under the new line search conditions (3) and (4) with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)} \right]$ and $\mu > 1$ satisfy

$$g_k^t d_k \leq -\xi \|g_k\|^2, \quad \forall k \in \mathbb{N}. \quad (15)$$

where $\xi > 0$.

Proof. In the case where the Powell restart criterion holds (i.e., $|g_k^t g_{k+1}| \geq 0.2 \|g_{k+1}\|^2$), it is evident the descent direction d_k is given by $d_k = -g_k$ and the relation (15) holds.

In the case where the Powell condition does not hold, we prove the above relation by induction. Indeed, for $k = 0$, the search direction is $d_0 = -g_0$. This implies $d_0^t g_0 = -\|g_0\|^2$, and relation (15) holds.

Suppose that (15) is true for $k \geq 1$. For $k + 1$, by multiplying the two sides of (10) by g_{k+1}^t , we get

$$\begin{aligned}
d_{k+1}^t g_{k+1} &= -\|g_{k+1}\|^2 + \beta_k^{\text{hPB}} d_k^t g_{k+1} \\
&\leq -\|g_{k+1}\|^2 + |\beta^{\text{PowellhPB}}| |d_k^t g_{k+1}| \\
&\leq -\|g_{k+1}\|^2 + \sigma \left(\frac{1.2 \|g_{k+1}\|^2}{\|g_k\|^2} + \frac{\|g_{k+1}\|^2}{\|d_k\|^2} \right) \\
&\quad \cdot \|g_k\| \|d_k\| \frac{\|g_k\|^2}{\|d_k\|^2}, \\
&\quad \text{(using relations (4) and (12))} \\
&\leq -\|g_{k+1}\|^2 + \sigma \left(1.2 \frac{\|g_k\|}{\|d_k\|} + \frac{\|g_k\|^3}{\|d_k\|^3} \right) \|g_{k+1}\|^2 \\
&\leq -\|g_{k+1}\|^2 (1 - \sigma(\mu^3 + 1.2\mu)) \\
&\leq -\frac{1}{\mu} \|g_{k+1}\|^2, \\
&\quad \text{(using relations (9) and (14)).}
\end{aligned}$$

Then the proof is complete for $\xi = 1/\mu$. $\blacksquare$

4. Convergence analysis

Before analyzing the convergence of the proposed approach, we first show that it is well defined. In the next theorem, we prove the existence of a step length α ($0 < \alpha < \infty$) that meets the conditions (3) and (4), where $0 < \delta < \sigma$, with $\sigma \in \left(0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}\right]$ and $\mu > 1$.

Theorem 3. *Let f be a twice continuously differentiable function that is bounded below. If $g_k^t d_k < 0$, then there exists a strictly positive real value α that meets the conditions (3) and (4).*

Proof. Define the following function:

$$h(\alpha) = f(x_k + \alpha d_k) - f(x_k) - \delta \alpha g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}.$$

Using a standard Taylor expansion, from (9) it results that

$$\begin{aligned}
h(\alpha) &= f(x_k + \alpha d_k) - f(x_k) - \delta \alpha g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} \\
&= (f(x_k) + \alpha g_k^t d_k + o(\alpha)) \\
&\quad - f(x_k) - \delta \alpha g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}, \\
&\leq \alpha (1 - \delta \mu^2) g_k^t d_k + o(\alpha) \\
&\leq \alpha \left(1 - \frac{\mu-1}{\mu^2+1.2} \right) g_k^t d_k + o(\alpha) < 0
\end{aligned}$$

Furthermore, since f is bounded from below, we get $\lim_{\alpha \rightarrow +\infty} h(\alpha) = +\infty$ and $h(0) = 0$. Hence, the function $h(\cdot)$ changes its sign and then there exists a real

value $\tau > 0$ with $h(\tau) = 0$. It is clear that $h(\alpha)$ has a negative sign over the interval $[0, \tau]$, and its global minimum cannot occur at the endpoints, since $h(0) = h(\tau) = 0$. Therefore, there exists $\alpha^* \in (0, \tau)$, such that $h(\alpha^*) < 0$ and $h'(\alpha^*) = 0$. Hence

$$\begin{aligned}
h(\alpha^*) &= f(x_k + \alpha^* d_k) - f(x_k) \\
&\quad - \delta \alpha^* g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} < 0,
\end{aligned}$$

so that

$$f(x_k + \alpha^* d_k) < f(x_k) + \delta \alpha^* g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2},$$

and the first condition (3) holds.

On the other hand, we have

$$h'(\alpha) = g_{k+1}^t d_k - \delta g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}.$$

Since $h'(\alpha^*) = 0$, we deduce that

$$\sigma g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} \leq \delta g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} = g_{k+1}^t d_k < 0.$$

Therefore,

$$|g_{k+1}^t d_k| \leq -\sigma g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2}.$$

Thus, the second condition (4) also holds. $\blacksquare$

The global convergence property is crucial for any conjugate gradient method. To establish it, we make the following assumptions:

Assumption 1. The level set $\Omega = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$ is bounded for any starting point x_0 .

Assumption 2. The function f is continuously differentiable and its gradient g satisfies the Lipschitz condition in some closed neighborhood $\mathcal{N}$ of Ω , i.e., $\exists L > 0$ such that

$$\|g(x) - g(y)\| \leq L \|x - y\|, \quad \forall x, y \in \mathcal{N}. \quad (16)$$

The above assumptions imply the existence of a real number $\Gamma \geq 0$ such that

$$\|g(x)\| \leq \Gamma, \quad \forall x \in \Omega. \quad (17)$$

To establish that the hPB algorithm converges globally (see Theorem 4 below), we need to prove the following results, which will be needed below.

Lemma 1. Consider the sequences $\{g_k\}_{k \in \mathbb{N}}$, $\{\alpha_k\}_{k \in \mathbb{N}}$ and $\{d_k\}_{k \in \mathbb{N}}$, generated by the hPB algorithm. For some $\varpi > 0$, we have

$$g_k^t d_k \geq -\varpi \|g_k\|^2, \quad \forall k \in \mathbb{N}. \quad (18)$$

Proof. Multiplying (6) by g_{k+1} , we get

$$|d_{k+1}^t g_{k+1}| \leq \|g_{k+1}\|^2 + |\beta_k^{\text{hPB}}| |d_k^t g_{k+1}|.$$

Then, from (9) it follows that

$$\begin{aligned} |d_{k+1}^t g_{k+1}| &\leq \|g_{k+1}\|^2 + \sigma \left(1.2 \frac{\|g_k\|}{\|d_k\|} \right. \\ &\quad \left. + \frac{\|g_k\|^3}{\|d_k\|^3} \right) \|g_{k+1}\|^2, \\ &\leq (1 + \sigma (1.2\mu + \mu^3)) \|g_{k+1}\|^2, \end{aligned}$$

which means that

$$-\varpi \|g_{k+1}\|^2 \leq d_{k+1}^t g_{k+1} \leq \varpi \|g_{k+1}\|^2,$$

where $\varpi = 1 + \sigma (1.2\mu + \mu^3)$. This completes the proof. ■

Next, we state the following important result:

Lemma 2. *Under the above assumptions, the sequence of steplengths $\{\alpha_k\}_{k \in \mathbb{N}}$ generated by the hPB algorithm under the new line search conditions (3) and (4) with $\sigma \in (0, \frac{\mu-1}{\mu^2(\mu^2+1.2)}]$ and $\mu > 1$, satisfies*

$$\alpha_k \geq \frac{\sigma \frac{\|g_k\|^2}{\|d_k\|^2} - 1}{L \|d_k\|^2} d_k^t g_k > 0, \quad \forall k \in \mathbb{N}.$$

Proof. From Theorem 1, it results that

$$\sigma \frac{\|g_k\|^2}{\|d_k\|^2} < \frac{\mu - 1}{\mu^2 + 1.2} < 1.$$

Hence, from condition (4) it follows that

$$\begin{aligned} 0 &< \left(\sigma \frac{\|g_k\|^2}{\|d_k\|^2} - 1 \right) d_k^t g_k \\ &< d_k^t (g_{k+1} - g_k) \leq L \alpha_k \|d_k\|^2, \end{aligned}$$

which completes the proof. ■

Lemma 3. *Under Assumption 1, the sequences $\{g_k\}_{k \in \mathbb{N}}$ and $\{d_k\}_{k \in \mathbb{N}}$ produced by the hPB algorithm satisfy*

$$\sum_{k \geq 0} \frac{(g_k^t d_k)^2}{\|d_k\|^2} < +\infty. \quad (19)$$

Proof. Using the condition (3) and relation (18), we have

$$\begin{aligned} f(x_k) - f(x_k + \alpha_k d_k) &\geq -\delta \alpha_k g_k^t d_k \frac{\|g_k\|^2}{\|d_k\|^2} \\ &\geq \frac{\delta \alpha_k (g_k^t d_k)^2}{\varpi \|d_k\|^2}. \end{aligned}$$

Then we get

$$\frac{\delta \alpha_k (g_k^t d_k)^2}{\varpi \|d_k\|^2} \leq f(x_k) - f(x_{k+1}).$$

Let $m = \min\{\alpha_k : k \in \mathbb{N}\}$. Under Assumption 1, by summing these inequalities from $k = 0$ to $+\infty$, we get

$$\frac{\delta m}{\varpi} \sum_{k \geq 0} \frac{(g_k^t d_k)^2}{\|d_k\|^2} \leq \sum_{k \geq 0} \frac{\delta \alpha_k (g_k^t d_k)^2}{\varpi \|d_k\|^2} < +\infty.$$

and thus (19) holds. ■

Now, we are in a position to address the global convergence of the hPB algorithm.

Theorem 4. *Under the above assumptions, the hPB algorithm converges globally in the sense that,*

$$\liminf_{k \rightarrow +\infty} \|g_k\| = 0. \quad (20)$$

Proof. Assume that the assertion (20) does not hold. Then there exists a positive value $r > 0$ such that

$$\|g_k\| > r, \quad \forall k \in \mathbb{N}. \quad (21)$$

Let $m = \min\{\alpha_k : k \in \mathbb{N}\}$ and $D = \max\{\|x - y\| : x, y \in \Omega\}$. From relation (1), it results that

$$\|d_k\|^2 = \frac{\|x_{k+1} - x_k\|^2}{\alpha_k^2} \leq \frac{D^2}{m^2}.$$

On the other hand, from (12), (21), (17) and (9) we get

$$\begin{aligned} d_{k+1} &\leq \|g_{k+1}\| + |\beta_k^{\text{hPB}}| \|d_k\| \\ &\leq \Gamma + \left(\frac{1.2\Gamma^2}{r^2} + \frac{\mu^2\Gamma^2}{r^2} \right) \frac{D}{m} = M. \end{aligned}$$

Hence

$$\sum_{k \geq 0} \frac{1}{\|d_k\|^2} = +\infty.$$

But, from (21), (15) and (19) it results that

$$\begin{aligned} C^2 r^4 \sum_{k \geq 0} \frac{1}{\|d_k\|^2} &\leq \sum_{k \geq 0} \frac{C^2 \|g_k\|^4}{\|d_k\|^2} \\ &\leq \sum_{k \geq 0} \frac{(g_k^t d_k)^2}{\|d_k\|^2} < +\infty. \end{aligned}$$

Therefore,

$$\sum_{k \geq 0} \frac{1}{\|d_k\|^2} < +\infty,$$

which contradicts the claim (21). Thus, the assertion (20) is true. ■

5. Computational experiments

Here, we present a series of computational performances concerning the hPB algorithm, applied to 40 test functions with 270 test problems, as listed in Table 2 and taken from (Andrei, 2008), with dimensions ranging from 2 to 100000, as well as on restoring four images with different noise levels. All the codes are written and implemented in Matlab version R2018a. In all experiments, the hPB algorithm is implemented with the setting parameters $\mu = 1.6$, $\delta = 10^{-4}$ and $\sigma = 10^{-3}$.

In order to illustrate the fact that the hPB method performs better by using the proposed inexact line search (3) and (4), we compare it with hPB-SW (hPB method using strong Wolfe conditions with $\delta = 10^{-4}$ and $\sigma = 10^{-3}$). Also, to demonstrate the effectiveness of the suggested approach, we compared it with HRM (Hamoda *et al.*, 2016) with $\theta = 0.4$, PRP⁺ (Gilbert and Nocedal, 1992), PRP (Polyak, 1969), BRB (Rahali *et al.*, 2021) and NHS (Zhang, 2009). For this comparison, the same starting point is assigned for each test problem, and each implementation is considered successful if a point x_k where $\|g(x_k)\|_\infty \leq 10^{-6}$ is reached within 2000 iterations, with CPU-time less than 500 seconds; otherwise, the implementation is assigned as failure.

To examine and compare the performances of the implemented algorithms, we will use a graphical comparison. Figures 1–4 illustrate the performances of the methods (according to their CPU times, numbers of iterations, function and gradient evaluations needed to reach the stopping criterion) using the performance profile of Dolan and Moré (2002). The performance of each method is plotted by a curve that refers to the percentage of solved problems within a factor of τ . The curve that is shaped on top corresponds to the code that solves a majority of the test problems within the given factor τ ; for more details, see (Dolan and Moré, 2002).

Figures 1–4 illustrate the fact that the hPB outperforms the others. Specifically, it is faster for approximately 48% of the test problems and successfully solves around 96% of them, followed by hPB-sw and BRB with 95% and 93%, respectively. These outcomes prove the competitiveness and rapid convergence of the hPB algorithm in the majority of testing problems.

5.1. Image restoration problems. In optimization fields, image restoration problems are considered among the most difficult ones. They aim to restore the original image from one that has been corrupted by impulse noise. For this comparison, four test images (Man.png, Boat.png, Hill.jpg and Bridge.bmp) of size 512×512 are chosen to evaluate the effectiveness of the hPB algorithm against the same variants used in the previous comparisons. The image quality is assessed by

two factors: the peak signal-to-noise ratio (PSNR) and its relative error (Err),

$$\text{PSNR} = 10 \log_{10} \frac{M \times N \times 255^2}{\sum_{i,j} (x_{i,j}^r - x_{i,j}^*)^2},$$

$$\text{Err} = \frac{\|x^r - x^*\|}{\|x^*\|},$$

where M and N are the sizes of the image, $x_{i,j}^r$ represents the pixel values of the restored image and $x_{i,j}^*$ denotes the original pixel values. The setting parameters of the proponent algorithms are set similarly to the previous test, and each computation will stop if any of the following criteria is fulfilled,

$$\text{Iter} > 300 \quad \text{or} \quad \frac{|f(x_{k+1}) - f(x_k)|}{|f(x_k)|} < 10^{-4}.$$

Figures 5–7 show the images restored for 30 %, 50% and 70 % of noise. The performance of each algorithm is measured by the restored image quality, the elapsed time and the number of iterations. The numerical outcomes are reported in Tables 3–5. The algorithm with a high PSNR, minimal error, with less CPU time, is considered best.

Upon examining the results in Tables 3–5 and Figures 5–7, it becomes evident that the hPB algorithm delivers good performance. In fact, as illustrated in Table 3 and Fig. 5, we can observe that the NHS method failed to restore images with 30% of noise, whereas the other methods were successful with a PSNR greater than 25, and the hPB method has the highest PSNR value for the Man and Boat images. On the other hand, the visual outcomes of Fig. 6 with 50% of noise show that the NHS method also failed to remove the noise with a PSNR less than 25, whereas the PRP⁺ method failed to restore the image of Man, while the HRM, hPB, BRB, PRP and hPB-sw methods succeeded for restoring all the images and the bold values in Table 4 indicate the superiority of the hPB method for restoring the images of Man, Boat and Hill. For 70% of noise, as shown in Fig. 7 and Table 5, the NHS method failed to restore the original images; the HRM, PRP⁺, BRB and hPB-sw methods succeeded to remove the noise from two images, while the hPB and PRP methods succeeded in restoring the images of Man, Boat and Hill, where the highest PSNR value corresponds to the hPB method.

On the whole, the numerical and visual outcomes of removing 30%, 50%, and 70% of noise show a satisfactory performance of the hPB algorithm. Notably, the bold values in Tables 3–5 highlight the efficiency of the proposed algorithm, as it produces higher PSNR values and takes a short time to restore a majority of the test images.

Table 2. List of test problems.

Function	Dimension n
Extended Maratos	500, 700, 1000, 1500, 2000, 4000, 9000, 9500, 60000, 10000, 15000
Arwhead	50, 60, 70, 80, 100, 150, 200, 800, 1000, 5000
ENGVAL1	100, 600, 700, 800, 1000, 1500, 1600, 1800, 10000
Diagonal 1	2, 4, 10, 100, 1000, 10000, 100000
FLETCHCR	2, 4
Generalized Tridiagonal 1	2, 10, 20, 30, 40, 300, 500, 700, 10000
Diagonal 2	2, 4, 10, 800, 1000, 80000
Extended White and Holst	1000, 2000, 3000, 4000, 5000, 6000
Diagonal 7	500, 700, 1000, 1500, 2000, 7000, 8000
Extended Rosenbrock	10, 20, 30, 100, 1200, 3000, 4000, 5000
Quadratic QF2	100, 200, 1000, 5000, 7000, 9000, 10000, 50000,
Diagonal 8	100, 200, 300, 400, 500, 1000, 1500, 2000
Extended Freudenstein and Roth	10, 100, 1000, 4000, 9000, 10000, 20000, 50000, 60000, 80000
Diagonal 3	2, 4, 6, 10, 50, 100, 200, 400, 700
Extended DENSCHNF	10, 100, 10000, 25000, 30000, 50000, 70000, 80000, 90000
Perturbed Quadratic	2
POWER	2
QUARTC	2
Extended Himmelblau	4, 6, 8, 10, 9000, 10000
Raydan 1	2
Raydan 2	1000, 4000, 50000, 80000
Perturbed quadratic diagonal	2
Extended DENSCHNB	10, 90, 100, 2000, 3000, 4000, 5000, 6000, 7000, 9000
HIMMELBG	2000, 2500, 2700, 3000, 6000, 30000, 50000, 80000
LIARWHD	10, 20, 40, 50, 4000, 5000, 5500, 10000, 20000, 80000
Extended quadratic exponential EP1	40000, 50000, 60000, 70000
Diagonal 5	100, 200, 700, 1000, 1500, 2000, 2200, 2500
Generalized Rosenbrock	2, 50, 800, 1000
Extended BD1	4, 800, 900, 2000, 3000, 5000, 20000, 40000, 60000, 70000, 80000
Hager	2, 4, 6, 10, 50, 80, 150, 300
NONSCOMP	2, 4, 1000, 5000, 70000
Extended quadratic penalty QP1	4, 6, 8, 10, 50, 100, 700, 1000, 1500
Quadratic QF1	5000, 6000, 8000, 9000, 20000, 50000, 70000, 80000
HIMMELLH	10, 50, 300, 500, 10000, 50000, 60000, 80000, 10000
Extended quadratic penalty QP2	40, 60, 70, 200
DIXON3DQ	2, 4, 6, 600
Diagonal 4	20000, 30000, 40000, 50000, 60000, 70000
Extended PSC1	2, 4, 6, 8, 10, 100, 1000, 10000
Almost Perturbed Quadratic	2, 4, 6
Extended Tridiagonal 1	6, 10, 20, 30, 80, 90, 100, 150, 300, 500, 700, 1000, 5000, 6000

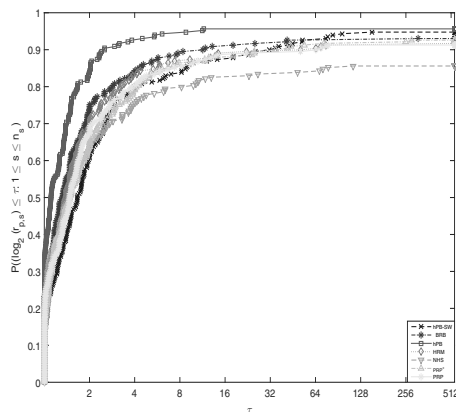


Fig. 1. Performance profiles based on CPU time.

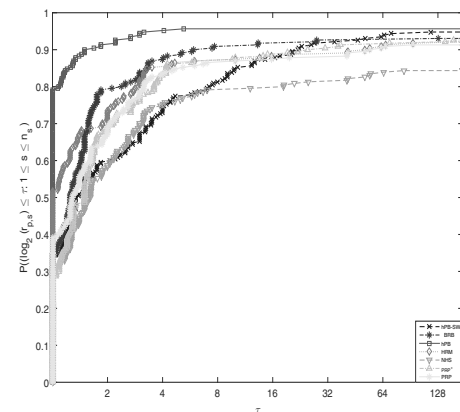


Fig. 2. Performance profiles based on the number of iterations.

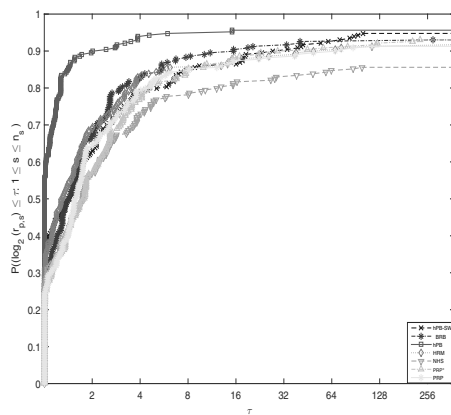


Fig. 3. Performance profiles based on the number of function evaluations.

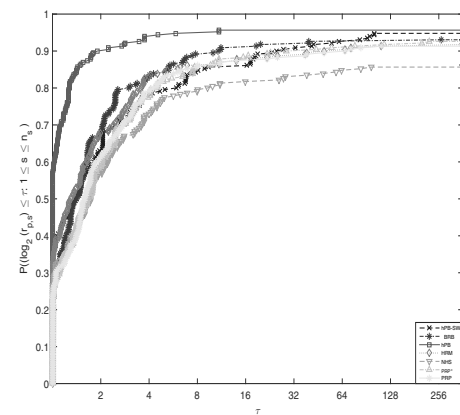


Fig. 4. Performance profiles based on the number of gradient evaluations.

6. Conclusions

In this work, a new hybrid conjugate gradient method, named hPB (hybrid PRP-BRB), is introduced for unconstrained nonconvex minimization. To achieve good convergence properties, we adopt a new inexact line search technique to determine the step length. Mathematical results concerning the convergence of the hPB method are established and the algorithm converges globally under mild assumptions. The performance of the algorithm is examined on various test functions and applied for restoring images with different noise levels. Moreover, a comparison with similar and recent algorithms is carried out, and the preliminary results indicate the robustness and competitiveness of the suggested approach.

Acknowledgment

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

References

- Andrei, N. (2008). An unconstrained optimization test functions, *Advanced Modeling and Optimization* **10**(1): 147–161.
- Andrei, N. (2009). Another nonlinear conjugate gradient algorithm for unconstrained optimization, *Optimization Methods & Software* **24**(1): 89–104.
- Chen, Z., Shao, H., Liu, P., Li, G. and Rong, X. (2024). An efficient hybrid conjugate gradient method with an adaptive strategy and applications in image restoration problems, *Applied Numerical Mathematics* **204**: 362–379.
- Collignon, T., P. and Van Gijzen, M., B. (2010). Two implementations of the preconditioned conjugate gradient method on heterogeneous computing grids, *International Journal of Applied Mathematics and Computer Science* **20**(1): 109–121, DOI: 10.2478/v10006-010-0008-4.
- Dai, Y.H. and Yuan, Y. (2001). An efficient hybrid conjugate gradient method for unconstrained optimization, *Annals of Operations Research* **103**: 33–47.
- Djordjevic, S.S. (2017). New hybrid conjugate gradient method as a convex combination of LS and CD methods, *Filomat* **31**(6): 1813–1825.



Fig. 5. Images restored by HRM, NHS, hPB, PRP⁺, PRP, BRB and hPB-SW subject to 30% salt-and-pepper noise.



Fig. 6. Images restored by HRM, NHS, hPB, PRP⁺, PRP, BRB and hPB-SW subject to 50% salt-and-pepper noise.



Fig. 7. Images restored by HRM, NHS, hPB, PRP⁺, PRP, BRB and hPB-SW subject to 70% salt-and-pepper noise.

Table 3. Results of restoring images with 30% salt-and-pepper noise.

Images		Man	Boat	Hill	Bridge
Methods	Iter	14	14	18	19
HRM	CPU	14.4333	16.1088	19.6281	18.3902
	PSNR	31.4316	33.5530	34.8439	28.5705
	Err	0.055800	0.038858	0.0376	0.075280
NHS	Iter	26	26	24	23
	CPU	15.7869	14.9272	14.8145	14.4291
	PSNR	15.7231	17.5183	17.0234	16.2491
	Err	0.3405	0.2462	0.2931	0.3110
hPB	Iter	13	17	17	14
	CPU	16.5572	11.2373	16.3538	16.4848
	PSNR	31.5644	33.7000	34.98430	28.4354
	Err	0.054953	0.038206	0.0370	0.076460
PRP ⁺	Iter	9	14	17	13
	CPU	11.7255	17.2765	17.4224	12.5381
	PSNR	29.8526	32.7850	34.9240	28.3708
	Err	0.066923	0.042450	0.0373	0.077031
PRP	Iter	11	11	16	14
	CPU	16.0896	14.6281	16.4851	15.5467
	PSNR	31.3374	33.1674	34.7344	28.5131
	Err	0.0564	0.0406	0.0381	0.0757
BRB	Iter	13	15	15	14
	CPU	17.8876	19.6420	16.0520	17.0935
	PSNR	31.4483	32.7617	34.8471	28.5141
	Err	0.055693	0.042564	0.0376	0.075770
hPB-SW	Iter	13	15	14	15
	CPU	18.3206	17.9993	16.6398	19.4814
	PSNR	28.5141	31.4483	34.8471	32.7617
	Err	0.07577	0.055693	0.0376	0.042564

Table 4. Results of restoring images with 50% salt-and-pepper noise.

Images		Man	Boat	Hill	Bridge
Methods	Iter	19	14	21	14
HRM	CPU	31.6329	22.9351	29.2036	21.4267
	PSNR	29.0482	30.0439	32.5694	26.2812
	Err	0.073418	0.058202	0.0489	0.097982
NHS	Iter	31	34	30	30
	CPU	24.3502	24.8263	21.1211	23.0874
	PSNR	13.4363	15.2070	14.7059	14.1568
	Err	0.4430	0.3212	0.3828	0.3957
hPB	Iter	19	24	15	18
	CPU	26.5356	22.2738	22.325	23.2843
	PSNR	29.1751	31.1843	32.6828	26.7130
	Err	0.072353	0.051041	0.0483	0.093230
PRP ⁺	Iter	5	17	20	16
	CPU	16.5329	22.4427	29.8420	21.0229
	PSNR	15.3421	30.9165	32.4165	26.5201
	Err	0.355716	0.052639	0.0498	0.095324
PRP	Iter	20	17	21	24
	CPU	36.9955	24.7400	30.9204	38.2190
	PSNR	28.9228	30.8359	32.5124	26.4954
	Err	0.0744	0.0531	0.0492	0.0955
BRB	Iter	24	17	15	22
	CPU	29.8683	21.7964	22.5881	27.3311
	PSNR	29.1394	31.1089	30.7900	26.6435
	Err	0.072651	0.051485	0.0600	0.093978
hPB-sw	Iter	19	24	15	22
	CPU	24.5807	22.4704	22.6654	19.7673
	PSNR	26.6435	29.1394	30.7900	31.1089
	Err	0.093978	0.072651	0.0600	0.051485

Dolan, E.D. and Moré, J.J. (2002). Benchmarking optimization software with performance profiles, *Mathematical Programming* **91**: 201–213.

Fletcher, R. (1997). *Practical Method of Optimization*, 2nd Ed., Wiley, New York.

Table 5. Results of restoring images with 70% salt-and-pepper noise.

Images		Man	Boat	Hill	Bridge
Methods	Iter	10	26	19	25
HRM	CPU	28.8301	49.0126	44.8694	48.7749
	PSNR	22.3136	28.1401	27.8223	24.2470
	Err	0.159416	0.072465	0.0845	0.123838
NHS	Iter	34	37	34	31
	CPU	29.7937	30.6846	29.0856	22.3260
	PSNR	11.3935	12.9901	12.5343	12.1899
	Err	0.5604	0.4146	0.4915	0.4963
hPB	Iter	40	34	33	37
	CUP	30.8349	28.3926	31.1979	30.1883
	PNSR	26.3222	28.2927	29.8827	24.5192
	Err	0.100484	0.071202	0.0667	0.120017
PRP ⁺	Iter	29	24	16	21
	CPU	50.3015	35.5985	29.1123	35.9517
	PSNR	26.1898	28.2480	29.3224	24.1335
	Err	0.102028	0.071570	0.0711	0.125468
PRP	Iter	20	15	14	15
	CPU	34.7941	28.4139	32.8371	41.2126
	PSNR	26.1118	27.5034	25.6558	22.0517
	Err	0.1029	0.0779	0.1085	0.1594
BRB	Iter	5	21	30	20
	CPU	20.0740	27.3770	38.0558	31.7151
	PSNR	13.1600	28.2074	29.7744	24.1351
	Err	0.457310	0.071905	0.0675	0.125444
hPB-sw	Iter	20	5	30	21
	CPU	32.7535	20.4136	38.0359	29.5325
	PSNR	24.1351	13.16	29.7744	28.2074
	Err	0.125444	0.45731	0.0675	0.071905

Gilbert, J.C. and Nocedal, J. (1992). Global convergence properties of conjugate gradient methods for optimization, *SIAM Journal on Optimization* **2**(1): 21–42.

Hamoda, M., Mamat, M., Rivaie, M. and Salleh, Z. (2016). A conjugate gradient method with strong Wolfe-Powell line search for unconstrained optimization, *Applied Mathematical Sciences* **10**(15): 721–734.

Hestenes, M.R. and Stiefel, E. (1952). Methods of conjugate gradients for solving linear systems, *Journal of Research of the National Bureau of Standards* **49**(6): 409–436.

Khudhur, H.M. and Halil, I.H. (2024). Noise removal from images using the proposed three-term conjugate gradient algorithm, *Computer Research and Modeling* **16**(4): 841–853.

Liu, J.K. and Li, S.J. (2014). New hybrid conjugate gradient method for unconstrained optimization, *Applied Mathematics and Computation* **245**: 36–43.

Liu, Y. and Storey, C. (1991). Efficient generalized conjugate gradient algorithms, Part 1: Theory, *Journal of Optimization Theory and Applications* **69**: 129–137.

Mehamdia, A. E., Chaib, Y. and Bechouat, T. (2025). Two improved nonlinear conjugate gradient methods with application in conditional model regression function, *Journal of Industrial and Management Optimization* **21**(2): 658–675.

Polak, E. and Ribière, G. (1969). Note sur la convergence de méthodes des directions conjuguées, *Revue Française d'Informatique et Recherche, Opérationnelle* **3**(16): 35–43.

Polyak, B.T. (1969). The conjugate gradient method in extremal problems, *USSR Computational Mathematics and Mathematical Physics* **9**(4): 94–112.

- Rahali, N., Belloufi, M. and Benzine, R. (2021). A new conjugate gradient method for acceleration of gradient descent algorithms, *Moroccan Journal of Pure and Applied Analysis* **7**(1): 1–11.
- Saleh, M.A. (2023). Enhancing deep learning optimizers for detecting malware using line search method under strong wolfe conditions, *2023 3rd International Conference on Computing and Information Technology (ICCIT)*, Tabuk, Saudi Arabia, pp. 222–226.
- Souli, C., Ziadi, R., Lakhdari, I. and Leulmi, A. (2025). An efficient hybrid conjugate gradient method for unconstrained optimization and image restoration problems, *Iranian Journal of Numerical Analysis and Optimization* **15**(1): 99–123.
- Sulaiman, I.M., Kaelo, P., Khalid, R. and Nawawi, M.K.M. (2024). A descent generalized rml spectral gradient algorithm for optimization problems, *International Journal of Applied Mathematics and Computer Science* **34**(2): 225–233, DOI: 10.61822/amcs-2024-0016.
- Wei, Z., Yao, S. and Liu, L. (2006). The convergence properties of some new conjugate gradient methods, *Applied Mathematics and Computation* **183**(2): 1341–1350.
- Yousif, O.O.O. and Saleh, M.A. (2024). Another modified version of rml conjugate gradient method, *Applied Numerical Mathematics* **202**: 120–126.
- Zhang, L. (2009). An improved Wei–Yao–Liu nonlinear conjugate gradient method for optimization computation, *Applied Mathematics and Computation* **215**(6): 2269–2274.
- Zheng, X.Y., Dong, X.L., Shiand, J.R. and Yang, W. (2020). Further comment on another hybrid conjugate gradient algorithm for unconstrained optimization by Andrei, *Numerical Algorithms* **84**: 603–608.
- Ziadi, R. and Bencherif-Madani, A. (2024). A mixed algorithm for smooth global optimization, *Journal of Mathematical Modeling* **11**(2): 207–228.
- Ziadi, R. and Bencherif-Madani, A. (2025). A perturbed quasi-newton algorithm for bound-constrained global optimization, *Journal of Computational Mathematics* **43**: 143–173.
- Ziadi, R., Ellaia, R. and Bencherif-Madani, A. (2017). Global optimization through a stochastic perturbation of the Polak–Ribière conjugate gradient method, *Journal of Computational and Applied Mathematics* **317**: 672–684.

Asma Maiza received her MS degree in optimization and control from University Setif-1, Ferhat Abbas, Algeria in 2019. Currently, she is a PhD candidate at the same university. Her research interests are in optimization and related fields.

Raouf Ziadi is an assistant professor at the Department of Mathematics, University Setif 1 Ferhat Abbas, Algeria. He received his PhD degree in applied mathematics from Ferhat Abbas University in 2017. His research interests are in optimization and related fields.

Mohammed A. Saleh is an associate professor at the Department of Cybersecurity, College of Computer, Qassim University, Saudi Arabia. He received his BS in mathematical and computer science from the Faculty of mathematical and computer science, University of Gezira, Sudan. He obtained his MS in information security from the Faculty of Computer Science and Information System, University of Technology (UTM), Malaysia, and earned his PhD in information security from the Faculty of Computing, University of Technology (UTM), Malaysia. His research interests include malware analysis and artificial intelligence in cybersecurity.

Abdulgader Z. Almaymuni is an associate professor at the Department of Cybersecurity, College of Computer, Qassim University, Saudi Arabia. He received the Ph.D. from Demont Fort University (UK) in 2013 with specialization in computer security and smart systems. His research interests include cybersecurity, formal method and smart systems.

Received: 3 October 2024

Revised: 4 December 2024

Re-revised: 24 December 2024

Accepted: 27 December 2024