



A DISCRETE NON-CLASSICAL OPERATIONAL CALCULUS MODEL WITH THE HORADAM DIFFERENCE

ABSTRACT

In this paper, there has been constructed such a model of the non-classical Bittner operational calculus, in which the derivative S related to Horadam sequences is understood as a difference operation $S\{x(k)\} := \{x(k+2) - ax(k+1) - bx(k)\}$.

Key words:

operational calculus, derivative, integrals, limit conditions, Horadam difference.

INTRODUCTION

For any functions $\{f(t)\} \in C^0((\alpha, \beta), \mathbb{R})$, $\{x(t)\} \in C^1((\alpha, \beta), \mathbb{R})$ as well as for every $t_0 \in (\alpha, \beta) \subset \mathbb{R}$ and $t \in (\alpha, \beta)$ the fundamental theorems of the integral calculus apply [1]:

$$\frac{d}{dt} \int_{t_0}^t f(\tau) d\tau = f(t), \quad \int_{t_0}^t x'(\tau) d\tau = x(t) - x(t_0).$$

Using linear operations

$$S\{x(t)\} := \{x'(t)\}, \quad T_{t_0}\{f(t)\} := \left\{ \int_{t_0}^t f(\tau) d\tau \right\}, \quad s_{t_0}\{x(t)\} := \{x(t_0)\}, \quad (1)$$

we can present the above theorems as follows:

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$$ST_{t_0}f = f, \quad T_{t_0}Sx = x - s_{t_0}x, \quad (2)$$

where $f = \{f(t)\}$, $x = \{x(t)\}$ ¹.

Apart from the model (1) with the classical *ordinary derivative* $S = d/dt$, there exist other continuous and discrete models in which, for appropriately determined operations S, T_q, s_q , properties (2) hold. These models constitute particular cases (representations) of the so-called non-classical Bittner operational calculus [2–5].

Broadly speaking, the *Bittner operational calculus* is a system

$$CO(L^0, L^1, S, T_q, s_q, Q)^2, \quad (3)$$

in which L^0 and L^1 are linear spaces (over the same scalar field Γ) such that $L^1 \subset L^0$. The linear operation $S : L^1 \rightarrow L^0$ (denoted as $S \in \mathcal{L}(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathcal{L}(L^0, L^1)$ and $s_q \in \mathcal{L}(L^1, L^1)$ such that $ST_qf = f$, $f \in L^0$ and $s_qx = x - T_qSx$, $x \in L^1$. These operations are called *integrals* and *limit conditions*, respectively. The kernel of S , i.e. $\text{Ker } S$ is a set of elements understood as *constants* for the derivative S . The limit conditions $s_q, q \in Q$ are projections of L^1 on the subspace $\text{Ker } S$.

Beside the continuous model (2), we frequently use a classical discrete model with the derivative S understood as the *forward difference* Δ .

Let $\mathbb{N}_0$ and $\mathbb{C}$ mean the set of non-negative integers and the set of complexes, respectively. Moreover, let $L^0 := C(\mathbb{N}_0, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k)\}_{k \in \mathbb{N}_0}$ with usual sequences addition and sequences multiplication by complexes. In [2, 3, 5] Bittner considered a model with the derivative

$$Sx \equiv \Delta x := \{x(k+1) - x(k)\}$$

and its corresponding integral

$$T_0x := \begin{cases} 0 & \text{for } k = 0 \\ \sum_{i=0}^{k-1} x(i) & \text{for } k > 0 \end{cases}, \quad k \in \mathbb{N}_0$$

and limit condition

$$s_0x := \{x(0)\},$$

where $x = \{x(k)\} \in L^1 = L^0$.

¹ $\{f(t)\}$ stands for the symbol of the function f , i.e. $f = \{f(t)\}$, whereas $f(t)$ denotes the value of the function $\{f(t)\}$ at point t . This notation is derived from J. Mikusiński [15].

² CO stands for the French ‘calcul opératoire’ (operational calculus).

Later, in [6] there appeared a model with the forward difference $S \equiv \Delta$, integrals

$$T_{k_0}x := \begin{cases} -\sum_{i=k}^{k_0-1} x(i) & \text{for } k < k_0 \\ 0 & \text{for } k = k_0 \\ \sum_{i=k_0}^{k-1} x(i) & \text{for } k > k_0 \end{cases}, \quad k \in \mathbb{N}_0$$

and limit conditions

$$s_{k_0}x := \{x(k_0)\},$$

where $k_0 \equiv q \in Q := \mathbb{N}_0$.

Notice that the integrals T_{k_0} can be shown as follows

$$T_{k_0}x = \left\{ \sum_{i=0}^{k-1} x(i) - \sum_{i=0}^{k_0-1} x(i) \right\}^3.$$

In this paper, we shall discuss other discrete models of the Bittner operational calculus related to the operation

$$S\{x(k)\} := \{x(k+2) - ax(k+1) - bx(k)\}, \quad (4)$$

where $a, b \in \mathbb{C}$ and $b \neq 0$.

We will consider two cases:

$$D := a^2 + 4b \neq 0 \quad \text{and} \quad D = 0.$$

In literature (e.g. [7, 12, 14, 16]), each element c belonging to the kernel of the operation (4) is called a *Horadam sequence* [8, 9].

In particular, the Horadam sequence $c = \{c(k)\} \in \text{Ker } S$, i.e. a solution of the equation

$$c(k+2) = ac(k+1) + bc(k), \quad k \in \mathbb{N}_0, \quad (5)$$

can be [10]:

– the *Fibonacci sequence* $\{\mathcal{F}(k)\}$ (for $a = b = 1$, $\mathcal{F}(0) = 0$, $\mathcal{F}(1) = 1$)

$$\{\mathcal{F}(k)\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots\};$$

³ Given the definition of T_{k_0} we assume that $\sum_{i=0}^{-1} x(i) := 0$.

- the *Lucas sequence* $\{\mathcal{L}(k)\}$ (for $a = b = 1, \mathcal{L}(0) = 2, \mathcal{L}(1) = 1$)
 $\{\mathcal{L}(k)\} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \dots\};$
- the *Pell sequence* $\{\mathcal{P}(k)\}$ (for $a = 2, b = 1, \mathcal{P}(0) = 0, \mathcal{P}(1) = 1$)
 $\{\mathcal{P}(k)\} = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \dots\};$
- the *Pell-Lucas sequence* $\{p(k)\}$ (for $a = 2, b = 1, p(0) = 2, p(1) = 2$)
 $\{p(k)\} = \{2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, \dots\};$
- the *Jacobsthal sequence* $\{\mathcal{J}(k)\}$ (for $a = 1, b = 2, \mathcal{J}(0) = 0, \mathcal{J}(1) = 1$)
 $\{\mathcal{J}(k)\} = \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, \dots\};$
- the *Jacobsthal-Lucas sequence* $\{j(k)\}$ (for $a = 1, b = 2, j(0) = 2, j(1) = 1$)
 $\{j(k)\} = \{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, 4097, 8191, \dots\}.$

Another interesting example is also an *anti-forward Fibonacci sequence* $\{f(k)\}$, for which $a = -1, b = 1, f(0) = 0, f(1) = 1$. Then, we have

$$f(k+2) = -\underbrace{[f(k+1) - f(k)]}_{\Delta f(k)} \iff f(k) = f(k+1) + f(k+2), \quad k \in \mathbb{N}_0,$$

from which we obtain

$$\{f(k)\} = \{0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, \dots\}.$$

We also have

$$f(k) = (-1)^{k+1} \mathcal{F}(k), \quad k \in \mathbb{N}_0. \quad (6)$$

In [13] Kalman and Mena presented two-term recurrences (5) and related Horadam sequences in the operational approach, using classical difference operations.

A MODEL WITH THE HORADAM DIFFERENCE, WHEN $D \neq 0$

In what follows, we shall call the operation (4) a *Horadam derivative* or *difference*.

Let

$$\Phi_{a,b} := \frac{a + \sqrt{D}}{2}, \quad \varphi_{a,b} := \frac{a - \sqrt{D}}{2}, \quad D \neq 0.$$

Then, we have

$$\Phi_{a,b}^2 - a \Phi_{a,b} - b = 0, \quad \varphi_{a,b}^2 - a \varphi_{a,b} - b = 0$$

and

$$\Phi_{a,b} + \varphi_{a,b} = a, \quad \Phi_{a,b} - \varphi_{a,b} = \sqrt{D}, \quad \Phi_{a,b} \varphi_{a,b} = -b.$$

We will prove the following

Theorem 1. *The system (3), where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $k_0 \equiv q \in Q := \mathbb{N}_0$ and*

$$Sx := \{x(k+2) - ax(k+1) - bx(k)\}, \quad (7)$$

$$T_{k_0}x := \frac{1}{\sqrt{D}} \begin{cases} -\sum_{i=k-1}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) & \text{for } k < k_0 \\ 0 & \text{for } k = k_0, k_0 + 1, \quad k \in \mathbb{N}_0, \\ \sum_{i=k_0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) & \text{for } k > k_0 + 1 \end{cases} \quad (8)$$

$$s_{k_0}x := \left\{ \frac{1}{\sqrt{D}} [b(\Phi_{a,b}^{k-1-k_0} - \varphi_{a,b}^{k-1-k_0})x(k_0) + (\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0})x(k_0+1)] \right\}, \quad (9)$$

forms a discrete model of the Bittner operational calculus with the Horadam difference (7), when $D \neq 0$.

Proof. It is obvious that (7) – (9) are linear operations. It is also easy to verify that T_{k_0} can be presented in the form of

$$T_{k_0}x := \left\{ \frac{1}{\sqrt{D}} \left[\sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \right] \right\},$$

where $\sum_{i=0}^j x(i) := 0$ for $j = -2, -1$.

Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Hence

$$\begin{aligned} S\{y(k)\} &= \{y(k+2) - ay(k+1) - by(k)\} \\ &= \left\{ \frac{1}{\sqrt{D}} \left(\left[\sum_{i=0}^k (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) \right] \right. \right. \\ &\quad \left. - a \left[\sum_{i=0}^{k-1} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i})x(i) \right] \right. \\ &\quad \left. \left. - b \left[\sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \right] \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{1}{\sqrt{D}} \left[(\Phi_{a,b} - \varphi_{a,b})x(k) + (\Phi_{a,b}^2 - \varphi_{a,b}^2 - a(\Phi_{a,b} - \varphi_{a,b}))x(k-1) \right. \right. \\
&\quad + \sum_{i=0}^{k-2} [(\Phi_{a,b}^2 - a\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}]x(i) \\
&\quad \left. \left. - \sum_{i=0}^{k_0-1} [(\Phi_{a,b}^2 - a\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}]x(i) \right] \right\} = \{x(k)\},
\end{aligned}$$

so $ST_{k_0}\{x(k)\} = \{x(k)\}$ holds.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+2) - ax(k+1) - bx(k)\}$. Then

$$\begin{aligned}
&T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} \\
&= \left\{ \frac{1}{\sqrt{D}} \left[\sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})f(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})f(i) \right] \right\} \\
&= \left\{ \frac{1}{\sqrt{D}} \left[\sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i+2) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i+2) \right. \right. \\
&\quad - a \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i+1) + a \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i+1) \\
&\quad \left. \left. - b \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) + b \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \right] \right\} \\
&= \left\{ \frac{1}{\sqrt{D}} \left[\sum_{i=2}^k (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) - \sum_{i=2}^{k_0+1} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) \right. \right. \\
&\quad - a \sum_{i=1}^{k-1} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i})x(i) + a \sum_{i=1}^{k_0} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i})x(i) \\
&\quad \left. \left. - b \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) + b \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \right] \right\} \\
&= \left\{ \frac{1}{\sqrt{D}} \left[(\Phi_{a,b} - \varphi_{a,b})x(k) + (\Phi_{a,b}^2 - \varphi_{a,b}^2 - a(\Phi_{a,b} - \varphi_{a,b}))x(k-1) \right. \right. \\
&\quad + \sum_{i=2}^{k-2} [(\Phi_{a,b}^2 - a\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}]x(i) \\
&\quad \left. \left. - (\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0})x(k_0+1) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=2}^{k_0} [(\Phi_{a,b}^2 - a\Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a\varphi_{a,b} - b)\varphi_{a,b}^{k-1-i}]x(i) \\
& - b(\Phi_{a,b}^{k-1-k_0} - \varphi_{a,b}^{k-1-k_0})x(k_0) \Big\} \\
& = \{x(k)\} - \left\{ \frac{1}{\sqrt{D}} [b(\Phi_{a,b}^{k-1-k_0} - \varphi_{a,b}^{k-1-k_0})x(k_0) + (\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0})x(k_0+1)] \right\}.
\end{aligned}$$

Therefore, $T_{k_0}S\{x(k)\} = \{x(k)\} - s_{k_0}\{x(k)\}$ also holds, which completes the proof. $\square$

Example 1. It is not difficult to check (see Th. 3[5]) that an abstract differential equation

$$Sx = f, \quad f \in L^0, x \in L^1$$

with the limit condition

$$s_q x = c_{0,q}, \quad c_{0,q} \in \text{Ker } S$$

has exactly one solution

$$x = c_{0,q} + T_q f. \quad (10)$$

A. In particular,

$$x(k) = \frac{1}{\sqrt{5}} \left[\left(\frac{-1 + \sqrt{5}}{2} \right)^k - \left(\frac{-1 - \sqrt{5}}{2} \right)^k \right], \quad k \in \mathbb{N}_0$$

is the solution of the homogeneous difference equation

$$x(k+2) + x(k+1) - x(k) = 0, \quad k \in \mathbb{N}_0$$

with initial conditions

$$x(0) = 0, x(1) = 1,$$

which results from the limit condition form (9) for $a = -1, b = 1$ and $k_0 = 0$.

Hence we have

$$x(k) = (-1)^{k+1} \cdot \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right], \quad k \in \mathbb{N}_0.$$

It is a form (6) of the anti-forward Fibonacci sequence $\{f(k)\}$ general term, where

$$\mathcal{F}(k) = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right], \quad k \in \mathbb{N}_0$$

is a well-known Binet formula of the Fibonacci sequence $\{\mathcal{F}(k)\}$ general term.

B. For the Cauchy problem

$$\begin{aligned}\mathcal{T}(k+2) - \mathcal{T}(k+1) - 2\mathcal{T}(k) &= 3, \quad k \in \mathbb{N}_0 \\ \mathcal{T}(0) &= 0, \mathcal{T}(1) = 1\end{aligned}\tag{11}$$

we have $a = 1, b = 2$ and for $k_0 = 0$, on the basis of (10), we obtain

$$\mathcal{T}(k) = \frac{1}{3}[2^k - (-1)^k + 3 \sum_{i=0}^{k-2} (2^{k-1-i} - (-1)^{k-1-i})] = \frac{1}{6}(2^{k+3} + (-1)^k - 9), \quad k \in \mathbb{N}_0.$$

Similarly, if

$$\begin{aligned}\mathcal{T}(k+2) - \mathcal{T}(k) &= 2^{k+2}, \quad k \in \mathbb{N}_0 \\ \mathcal{T}(0) &= 0, \mathcal{T}(1) = 1,\end{aligned}\tag{12}$$

then $a = 0, b = 1$ and for $k_0 = 0$, we get

$$\mathcal{T}(k) = \frac{1}{2}[1 - (-1)^k + \sum_{i=0}^{k-2} (1 - (-1)^{k-1-i}) \cdot 2^{i+2}] = \frac{1}{6}(2^{k+3} + (-1)^k - 9), \quad k \in \mathbb{N}_0.$$

The sequence $\{\mathcal{T}(k)\}$, defined with the use of Jacobsthal numbers as

$$\mathcal{T}(k) := \begin{cases} \mathcal{J}(0) & \text{for } k = 0 \\ \mathcal{J}(1) & \text{for } k = 1 \\ \sum_{i=2}^{k+1} \mathcal{J}(i) & \text{for } k \in \mathbb{N}_0 \setminus \{0, 1\} \end{cases},$$

was introduced in [11], where Horadam gave a number of its properties, including (11) and (12).

A MODEL WITH THE HORADAM DIFFERENCE, WHEN $D = 0$

If the Horadam difference (4) takes a particular form of

$$S\{x(k)\} := \left\{ x(k+2) - a x(k+1) + \frac{1}{4}a^2 x(k) \right\},$$

then $D = 0$. For this case, we will prove.

Theorem 2. *The system (3), where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $k_0 \equiv q \in Q := \mathbb{N}_0$ and*

$$S\{x(k)\} := \left\{ x(k+2) - a x(k+1) + \frac{1}{4}a^2 x(k) \right\},\tag{13}$$

$$T_{k_0} x := \left\{ \left(\frac{a}{2} \right)^{k-2} \left[\sum_{i=0}^{k-2} \left(\frac{2}{a} \right)^i (k-1-i) x(i) - \sum_{i=0}^{k_0-1} \left(\frac{2}{a} \right)^i (k-1-i) x(i) \right] \right\}, \quad (14)$$

$$s_{k_0} x := \left\{ - \left(\frac{a}{2} \right)^{k-k_0} \left[(k-1-k_0) x(k_0) - \frac{2}{a} (k-k_0) x(k_0+1) \right] \right\}, \quad (15)$$

forms a discrete model of the Bittner operational calculus with the Horadam difference (13).

Proof. Similarly as before, the operations (13)–(15) are linear. Moreover, if $\{y(k)\} := T_{k_0}\{x(k)\}$, then

$$\begin{aligned} S\{y(k)\} &= \left\{ y(k+2) - a y(k+1) + \frac{1}{4} a^2 y(k) \right\} \\ &= \left\{ \sum_{i=0}^k \left(\frac{a}{2} \right)^{k-i} (k+1-i) x(i) - \sum_{i=0}^{k_0-1} \left(\frac{a}{2} \right)^{k-i} (k+1-i) x(i) \right. \\ &\quad - 2 \sum_{i=0}^{k-1} \left(\frac{a}{2} \right)^{k-i} (k-i) x(i) + 2 \sum_{i=0}^{k_0-1} \left(\frac{a}{2} \right)^{k-i} (k-i) x(i) \\ &\quad \left. + \sum_{i=0}^{k-2} \left(\frac{a}{2} \right)^{k-i} (k-1-i) x(i) - \sum_{i=0}^{k_0-1} \left(\frac{a}{2} \right)^{k-i} (k-1-i) x(i) \right\} \\ &= \left\{ x(k) + \sum_{i=0}^{k-2} \left(\frac{a}{2} \right)^{k-i} \left[(k+1-i) - 2(k-i) + (k-1-i) \right] x(i) \right. \\ &\quad \left. - \sum_{i=0}^{k_0-1} \left(\frac{a}{2} \right)^{k-i} \left[(k+1-i) - 2(k-i) + (k-1-i) \right] x(i) \right\} = \{x(k)\}. \end{aligned}$$

If, in turn, $\{f(k)\} := S\{x(k)\} = \{x(k+2) - a x(k+1) + \frac{1}{4} a^2 x(k)\}$, then

$$\begin{aligned} T_{k_0} S\{x(k)\} &= T_{k_0}\{f(k)\} \\ &= \left\{ \left(\frac{a}{2} \right)^{k-2} \left[\sum_{i=0}^{k-2} \left(\frac{2}{a} \right)^i (k-1-i) f(i) - \sum_{i=0}^{k_0-1} \left(\frac{2}{a} \right)^i (k-1-i) f(i) \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ \left(\frac{a}{2} \right)^{k-2} \left[\sum_{i=0}^{k-2} \left(\frac{2}{a} \right)^i (k-1-i) x(i+2) - \sum_{i=0}^{k_0-1} \left(\frac{2}{a} \right)^i (k-1-i) x(i+2) \right. \right. \\
&\quad \left. \left. - a \sum_{i=0}^{k-2} \left(\frac{2}{a} \right)^i (k-1-i) x(i+1) + a \sum_{i=0}^{k_0-1} \left(\frac{2}{a} \right)^i (k-1-i) x(i+1) \right. \right. \\
&\quad \left. \left. + \left(\frac{a}{2} \right)^2 \sum_{i=0}^{k-2} \left(\frac{2}{a} \right)^i (k-1-i) x(i) - \left(\frac{a}{2} \right)^2 \sum_{i=0}^{k_0-1} \left(\frac{2}{a} \right)^i (k-1-i) x(i) \right] \right\} \\
&= \left\{ \sum_{i=2}^k \left(\frac{a}{2} \right)^{k-i} (k+1-i) x(i) - \sum_{i=2}^{k_0+1} \left(\frac{a}{2} \right)^{k-i} (k+1-i) x(i) \right. \\
&\quad \left. - 2 \sum_{i=1}^{k-1} \left(\frac{a}{2} \right)^{k-i} (k-i) x(i) + 2 \sum_{i=1}^{k_0} \left(\frac{a}{2} \right)^{k-i} (k-i) x(i) \right. \\
&\quad \left. + \sum_{i=0}^{k-2} \left(\frac{a}{2} \right)^{k-i} (k-1-i) x(i) - \sum_{i=0}^{k_0-1} \left(\frac{a}{2} \right)^{k-i} (k-1-i) x(i) \right\} \\
&= \left\{ x(k) + \sum_{i=2}^{k-2} \left(\frac{a}{2} \right)^{k-i} \left[(k+1-i) - 2(k-i) + (k-1-i) \right] x(i) \right. \\
&\quad \left. - \left(\frac{a}{2} \right)^{k-k_0-1} (k-k_0) x(k_0+1) - \left(\frac{a}{2} \right)^{k-k_0} (k+1-k_0) x(k_0) + 2 \left(\frac{a}{2} \right)^{k-k_0} (k-k_0) x(k_0) \right. \\
&\quad \left. - \sum_{i=2}^{k_0-1} \left(\frac{a}{2} \right)^{k-i} \left[(k+1-i) - 2(k-i) + (k-1-i) \right] x(i) \right\} \\
&= \{x(k)\} - \left\{ - \left(\frac{a}{2} \right)^{k-k_0} \left[(k-1-k_0) x(k_0) - \frac{2}{a} (k-k_0) x(k_0+1) \right] \right\} \\
&= \{x(k)\} - s_{k_0} \{x(k)\}.
\end{aligned}$$

□

REFERENCES

- [1] Apostol T. M., *Calculus*, Vol. 1: *One-Variable Calculus, with an Introduction to Linear Algebra*, John Wiley and Sons, New York — London 1967.
- [2] Bittner R., *On certain axiomatics for the operational calculus*, 'Bull. Acad. Polon. Sci.' Cl. III, 1959, Vol. 7, No. 1, pp. 1–9.

- [3] Bittner R., *Operational calculus in linear spaces*, 'Studia Math.', 1961, 20, pp. 1–18.
- [4] Bittner R., *Algebraic and analytic properties of solutions of abstract differential equations*, 'Rozprawy Matematyczne' ['Dissertationes Math.'], 41, PWN, Warszawa 1964.
- [5] Bittner R., *Rachunek operatorów w przestrzeniach liniowych*, PWN, Warszawa 1974 [*Operational Calculus in Linear Spaces* — available in Polish].
- [6] Bittner R., Mieloszyk E., *About eigenvalues of differential equations in the operational calculus*, 'Zeszyty Naukowe Politechniki Gdańskiej', Matematyka XI, 1978, 285, pp. 87–99.
- [7] Čerin Z., *On sums of products of Horadam numbers*, 'Kyungpook Math. J.', 2009, 49, pp. 483–492.
- [8] Horadam A. F., *Generating functions for powers of a certain generalised sequence of numbers*, 'Duke Math. J.', 1965, 32 (3), pp. 437–446.
- [9] Horadam A. F., *Basic properties of a certain generalized sequence of numbers*, 'Fibonacci Quart.', 1965, 3 (3), pp. 161–176.
- [10] Horadam A. F., *Jacobsthal and Pell curves*, 'Fibonacci Quart.', 1988, 26 (1), pp. 77–83.
- [11] Horadam A. F., *Jacobsthal representation numbers*, 'Fibonacci Quart.', 1996, 34 (1), pp. 40–54.
- [12] Horzum T., Kocer E. G., *On some properties of Horadam polynomials*, 'Int. Math. Forum', 2009, 4 (25), pp. 1243–1252.
- [13] Kalman D., Mena R., *The Fibonacci numbers — exposed*, 'Math. Magazine', 2003, 76 (3), pp. 167–181.
- [14] Mansour T., *A formula for the generating functions of powers of Horadam's sequence*, 'Australas. J. Combin.', 2004, 30, pp. 207–212.
- [15] Mikusiński J., *Operational Calculus*, Pergamon Press, London 1959.
- [16] Weisstein E. W., 'Horadam Sequence', From *MathWorld* — A Wolfram Web Resource, [online], <http://mathworld.wolfram.com/HoradamSequence.html> [access 12.04.2016].

MODEL DYSKRETNY NIEKLASYCZNEGO RACHUNKU OPERATORÓW Z RÓŻNICĄ HORADAMA

STRESZCZENIE

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna S , związana z ciągami Horadama, rozumiana jest jako operacja różnicowa $S\{x(k)\} := \{x(k+2) - a x(k+1) - b x(k)\}$.

Słowa kluczowe:

rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica Horadama.