Psychol. Belg.
1984, XXIV-1, 83-86.

SHORT NOTE

University of Ghent

ON THE VALIDITY OF LUCE'S SIMILARITY CHOICE MODEL FOR CONFUSION DATA

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In this note it is indicated on which assumptions Luce's similarity choice model for confusion data is based and how they can be tested. Application of the tests on ten published confusion matrices reveals that Luce's model is not universally applicable due to systematic violation of the quasi-symmetry assumption.

One of the most prominent models for representing confusion data is Luce's (1963) similarity choice model. Not unlike Thurstone's Law of Comparative Judgment Case V, Luce's similarity choice model has often been used without explicitly testing its validity (e.g., Lupker, 1979). In this paper we first indicate on which assumptions the model is based and how they can be verified. Next, we apply the tests on ten published data sets.

The following notation is used throughout the paper. $\mathrm{X} = (x_{ij})_{n \times n}$ denotes an empirical confusion matrix where x_{ij} is the number of times the subject(s) responded with stimulus j after being presented stimulus i. $\pi = (\pi_{ij})_{n \times n}$ denotes a theoretical confusion matrix. π_{ij} is the probability of confusing j for i. The expected count in the (i,j)'th cell of an empirical confusion matrix is written as m_{ij}. Obviously,

$$m_{ij} = \pi_{ij} \sum_{k}^{n} x_{ik}.$$

In Luce's similarity choice model, the probability of confusing j for i is defined as

$$\pi_{ij} = \frac{\eta_{ij}\beta_j}{\sum_{k=1}^{n} \eta_{ik}\beta_k}$$

I am indebted to Professor W. De Coster for providing computer facilities at the Laboratory of Experimental, Differential, and Developmental Psychology of the University of Ghent.

with (1)

$$0 \leqslant \beta_j \leqslant 1, \tag{2a}$$
$$0 \leqslant \eta_{ij} \leqslant 1, \tag{2b}$$
$$\eta_{ij} = \eta_{ji}, \tag{2c}$$
$$\eta_{ii} = 1. \tag{2d}$$

The β scale can be interpreted as a response bias measure while the η scale can be interpreted as a similarity measure. More specifically, $-\log \eta$ can be considered as a psychological distance scale. Restrictions (2a) and (2b) are identification constraints. Restriction (2c) states that the similarity scale is symmetric while restriction (2d) requires all self-similarities to be equal to each other and maximal.

Townsend and Landon (1982, Theorem 2) proved that a theoretical confusion matrix π satisfies Luce's similarity choice model if and only if the following two properties hold:

1) quasi-symmetry: $\pi_{ij}\pi_{jk}\pi_{ki} = \pi_{ik}\pi_{kj}\pi_{ji}$ $(i \neq j \neq k)$ (3)

2) diagonal maximization: $\pi_{ii}\pi_{jj} \geqslant \pi_{ij}\pi_{ji}$. (4)

Caussinus (1965) showed that quasi-symmetry is a necessary and sufficient condition for the following expected count model

$$m_{ij} = c_{ij} b_j. \tag{5}$$

It is easy to prove that under the product multinomial sampling model appropriate for confusion data, model (5) is equivalent to Luce's similarity choice model *without* restriction (2d). Consequently, the validity of the quasi-symmetry assumption can be tested statistically through model (5).

Caussinus (1965) developed a globally convergent iterative proportional fitting algorithm for obtaining maximum likelihood estimates of m_{ij} according to (5). The $(t+1)$'th cycle of the algorithm is defined as

$$m_{ij}^{(3t+1)} = m_{ij}^{(3t)} \frac{\sum\limits_{\substack{k=1\\k\neq i}}^{n} x_{ik}}{\sum\limits_{\substack{k=1\\k\neq i}}^{n} m_{ik}^{(3t)}} \qquad (i \neq j)$$

$$m_{ij}^{(3t+2)} + m_{ij}^{(3t+1)} \frac{\sum\limits_{\substack{k=1\\k\neq j}}^{n} x_{kj}}{\sum\limits_{\substack{k=1\\k\neq j}}^{n} m_{kj}^{(3t+1)}}$$

$$m_{ij}^{(3t+3)} = m_{ij}^{(3t+1)} \frac{(x_{ij} + x_{ji})}{(m_{ij}^{(3t+2)} + m_{ji}^{(3t+2)})}$$

starting from $m_{ij}^{(0)} = 1$ $(i \neq j)$. The maximum likelihood estimates of the diagonal elements are

$$\hat{m}_{ii} = x_{ii}.$$

Bishop, Fienberg, and Holland (1975) discuss how standard computer programs for contingency table analysis (such as ECTA) can be used for fitting (5) through the above algorithm (see also Haberman, 1979).

Once maximum likelihood estimates $\hat{m}_{ij}$ are obtained, the goodness-of-fit of the model can be evaluated by means of a likelihood ratio test. The statistic

$$G = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \log (x_{ij}/\hat{m}_{ij})$$

is under the null hypothesis asymptotically distributed as a χ^2 with $(n-1)(n-2)/2$ degrees of freedom. If G exceeds the $100(1-\alpha)$'th percentile of the χ^2 distribution with $(n-1)(n-2)/2$ degrees of freedom, the quasi-symmetry model can be rejected at significance level α.

If quasi-symmetry holds, one can check whether the estimated cell probabilities

$$\hat{\pi}_{ij} = \hat{m}_{ij} / \sum_{k=1}^{n} x_{ik} \tag{6}$$

TAB. 1. RESULTS OF THE QUASI-SYMMETRY ANALYSES

No.	reference	stimuli	a	b	G	d.f.	p
1	Hodge & Pollack (1962, Exp. VIII)	8 tones	6	576	50.56	21	0.0003
2	Shepard (1958, Exp. I)	9 color chips	36	800	53.66	28	0.0025
3	McGuire (see Shepard, 1958)	9 circles with varying diameter	10	800	156.11	28	< 0.0001
4	Morgan et al. (1973, Table 1)	9 digits	28	534	76.26	28	< 0.0001
5	Morgan et al. (1973, Table 2)	9 digits	67	1556	174.99	28	< 0.0001
6	Morgan et al. (1973, Table 3)	9 digits	230	2628	32.81	28	0.2429
7	Morgan et al. (1973, Table 4)	9 digits	328	2750	49.25	28	0.0078
8	Miller & Nicely (1955, Tab. I-VI)	16 consonants followed by /a/	5	1500	293.23	105	< 0.0001
9	Miller & Nicely (1955, Tab. VII-XII)	16 consonants followed by /a/	5	1500	519.41	105	< 0.0001
10	Miller & Nicely (1955, Tab. XIII-XVIII)	16 consonants followed by /a/	5	1250	230.67	105	< 0.0001

[a] Number of subjects.
[b] Average number of responses to each stimulus.

satisfy the diagonal maximization property. Up to now, no statistical test exists for doing this.

The quasi-symmetry test was performed on ten published empirical confusion matrices. The results are summarized in Table 1. As can be inferred from the table, only data set number 6 satisfies quasi-symmetry. For all data sets, the estimated probabilities (6) satisfied the diagonal maximization property perfectly! Consequently, it can be concluded that Luce's similarity choice model is not universally applicable due to violation of the quasi-symmetry assumption. Future models for confusion data may, in order to be realistic, not require quasi-symmetry.

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Received June 1983