Psychol. Belg.
1983, XXIII-2, 135-139.

SHORT NOTE

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ON SOME METRIC PROPERTIES OF DISSIMILARITIES SATISFYING RESTLE'S SYMMETRIC SET DIFFERENCE MODEL

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In this note some metric properties that are either always or sometimes satisfied by dissimilarities satisfying Restle's model, are brought together.

Let $S = \{a, b, c, \ldots\}$ be a finite set of stimuli and suppose that each stimulus is characterized as a set of features. The sets of features representing a, b, c will be written as A, B, C respectively. Restle (1959, 1961) proposed the following model accounting for the dissimilarity between two stimuli a and b

$$\delta(a, b) = m(A \Delta B) \tag{1}$$

where $A \Delta B$ denotes the symmetric set difference between A and B, and where m is a measure function defined on $\{A, B, C, \ldots\}$. Since the introduction of Tversky's (1977) contrast model, there has been a renewed interest in Restle's model, mainly because of an unpublished theorem, cited by Tversky (1977, p. 347) and attributed to Sattath, which shows that every symmetric contrast model is basically equivalent to (1).

This note brings together some metric properties, that are either always or sometimes satisfied by (S, δ). These properties are not new, in that they all can be found in the literature in some form or another, but not explicitly in reference to the Restle model. In this note, I only prove those properties for which no published proofs exist.

SOME DEFINITIONS

Some definitions need to be introduced. Let X be a set and f a mapping from X^2 into the non-negative reals. The following are some of the properties that can be satisfied by (X, f):

Minimality:
for all $x, y \in X$, $f(x,y) = 0$ if and only if $x = y$. (2)

I am indebded to Jan de Leeuw for bringing the Kelly (1970a, 1970b, 1975) papers to my attention.

Symmetry:
for all x, y ε X, $f(x,y) = f(y,x)$. (3)

Triangle inequality:
for all x, y, z ε X, $f(x,y) \leqslant f(x,z) + f(z,y)$. (4)

k-hypermetricity:
for all $x_1, \ldots, x_k, y_1, \ldots, y_{k+1}$ ε X,

$$\sum_{1 \leq i < j \leq k} f(x_i, x_j) + \sum_{1 \leq i < j \leq k+1} f(y_i, y_j) \leq \sum_{i=1}^{k} \sum_{j=1}^{k+1} f(x_i, y_j). \quad (5)$$

Four points condition (or *additive inequality*):
for all x, y, z, w ε X,
$f(x,y) + f(z,x) \leqslant \max[f(x,z) + f(y,w), f(x,w) + f(y,z)]$. (6)

Ultrametric inequality:
for all x, y, z ε X, $f(x,y) \leqslant \max[f(x,z), f(z,y)]$. (7)

Note that (6) is equivalent to saying that any four elements in X can be labeled x, y, z, w such that

$$f(x,y) + f(z,w) \leqslant f(x,z) + f(y,w) = f(x,w) + f(y,z).$$

Similarly, (7) is equivalent to saying that any three elements in X can be labeled x, y, z such that

$$f(x,y) \leqslant f(x,z) = f(z,y).$$

(X,f) is called a metric or a metric space whenever it satisfies (2), (3), and (4). If (X,f) satisfies (2), (3), and (5) for any positive integer k, then (X,f) is called a hypermetric. (X,f) is called a four points metric if it satisfies (2), (3), and (6). Finally, if (2), (3), and (7) are satisfied, (X,f) is called an ultrametric. Kelly (1970a, 1975) has shown the following irreversible implications:

(X,f) ultrametric $\Rightarrow$ (X,f) four points metric $\Rightarrow$
(X,f) hypermetric $\Rightarrow$ (X,f) metric.

METRIC PROPERTIES OF (S, δ)

In this section, it is shown that (S,δ) (with δ defined by eq. 1) always is a hypermetric. Moreover, in some cases (S,δ) is a four points metric or an ultrametric. Restle (1959) proved that (S,δ) is a metric. However, a stronger result can be derived from the work of Kelly (1970b).

Theorem 1. (S,δ) is a hypermetric.

Proof. The theorem follows immediately from Theorem 3.1 of Kelly (1970b, p. 196-197).

Theorems 2 and 3 state the conditions under which (S,δ) is respectively a four points metric and an ultrametric. These theorems have been informally stated before by Tversky (1977).

Theorem 2. If any three elements in S can be labeled a, b, c such that

$$A \cap B \supseteq A \cap C = B \cap C,$$
then (S,δ) is a four points metric.

In order to prove Theorem 2, we need the following Lemma:

Lemma. If any three elements in S can be labeled a, b, c such that $A \cap B \supseteq A \cap C = B \cap C$, then any four elements in S can be labeled a, b, c, d such that

$$(A \cap B) \cup (C \cap D) \supseteq (A \cap C) \cup (B \cap D) = (A \cap D) \cup (B \cap C).$$

Proof. Any three stimuli in S can be labeled a, b, c such that (AB)C, where (AB)C, means $A \cap B \supseteq A \cap C = B \cap C$. If a fourth stimulus d is introduced, then either (AB)D, (AD)B, or (BD)A, and either (BC)D, (BD)C, or (CD)B, and either (AC)D, (AD)C, or (CD)A. This gives rise to $3^3 = 27$ different cases. For each case the lemma can be easily proven, as illustrated here for the first case in which (AB)C, (AB)D, (BC)D, and (AC)D. (AB)C, (AB)D, (BC)D, and (AC)D imply

$$A \cap B \supseteq A \cap C = B \cap C \supseteq A \cap D = B \cap D = C \cap D. \tag{8}$$

Since $C \cap D \subseteq A \cap B$ and $C \cap D = B \cap D$, it follows from (8) that

$$(A \cap B) \cup (C \cap D) \supseteq (A \cap C) \cup (B \cap D).$$

Moreover, since $A \cap C = B \cap C$ and $B \cap D = A \cap D$

$$(A \cap C) \cup (B \cap D) = (A \cap D) \cup (B \cap C),$$

which completes the proof for this case.

We now can proof Theorem 2.

Proof of Theorem 2. Since (S,δ) is a hypermetric, it suffices to show that the four points condition is satisfied whenever the condition of the theorem holds. Since

$$m(A \,\Delta\, B) = m(A) + m(B) - 2m(A \cap B)$$

we have

$$\delta(a,b) + \delta(c,d) = m(A) + m(B) + m(C) + m(D) - 2m(A \cap B) - 2m(C \cap D) \tag{9a}$$

$$\delta(a,c) + \delta(b,d) = m(A) + m(B) + m(C) + m(D) - 2m(A \cap C) - 2m(B \cap D) \tag{9b}$$

$$\delta(a,d) + \delta(b,c) = m(A) + m(B) + m(C) + m(D) - 2m(A \cap D) - 2m(B \cap C). \tag{9c}$$

From

$$(A \cap B) \cup (C \cap D) \supseteq (A \cap C) \cup (B \cap D) = (A \cap D) \cup (B \cap C)$$

and

$$m[(A \cap B) \cup (C \cap D)] = m(A \cap B) + m(C \cap D) - m(A \cap B \cap C \cap D)$$

$$m[(A \cap C) \cup (B \cap D)] = m(A \cap C) + m(B \cap D) - m(A \cap B \cap C \cap D)$$

$$m[(A \cap D) \cup (B \cap C)] = m(A \cap D) + m(B \cap C) - m(A \cap B \cap C \cap D),$$

if follows that

$$m(A \cap B) + m(C \cap D) \geqq m(A \cap C) + m(B \cap D) = m(A \cap D) + (B \cap C).$$

Combining this result with (9) gives

$$\delta(a,b) + \delta(c,d) \leqslant \delta(a,c) + \delta(b,d) = \delta(a,d) + \delta(b,c),$$

which completes the proof.

Theorem 3. If any three elements in S can be labeled a, b, c such that

$$A \cap B \supseteq A \cap C = B \cap C$$

and if

$$m(A) = \alpha \text{ for all } a \in S,$$

then (S,δ) is an ultrametric.

Proof. Because (S,δ) is a hypermetric, it suffices to show that (S,δ) satisfies the ultrametric inequality whenever the conditions of the theorem hold. Using the fact that $m(A) = m(B) = m(C) = \ldots = \alpha$, we obtain

$$\delta(a,b) = 2[\alpha - m(A \cap B)] \quad (10a)$$

$$\delta(a,c) = 2[\alpha - m(A \cap C)] \quad (10b)$$

$$\delta(b,d) = 2[\alpha - m(B \cap D)]. \quad (10c)$$

From (AB)C it follows that

$$m(A \cap B \geqq m(A \cap C) = m(B \cap C).$$

Combining this with (10) gives us the desired result:

$$\delta(a,b) \leqslant \delta(a,c) = \delta(b,c).$$

Q.E.D.

While Theorem 1 is mainly of theoretical interest, Theorems 2 and 3 have some important practical implications. Whenever the antecedent condition of Theorem 2 is fulfilled, the dissimilarities can be represented by an additive tree (see Buneman, 1971; Cunningham, 1978; De Soete, 1983; Sattath & Tversky, 1977). If, in addition, all stimuli are equally prominent (i.e., $m(A) = m(B) = m(C) = \ldots$), then the dissimilarities generated by Restle's model can be represented by an ultrametric tree (see Hartigan, 1967; Jardine, Jardine, & Sibson, 1967; Johnson, 1967).

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Received January 1983