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Research Article

Analytycal study on the applicability of generalized Yukawa potential to predict the mass spectra of the heavy quarkonium

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Abstract

We solved the radial Schrödinger equation analytically using the Nikiforov-Uvarov method to obtain the energy eigenvalues and corresponding wavefunction in terms of Laguerre polynomials with the Generalized Yukawa potential (GYP). The present results are applied for calculating the mass spectra of heavy mesons such as charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) for different quantum states. The present potential provides excellent results in comparison with experimental data with a maximum error of 0.0157 GeV and work of other researchers.

Keywords: Generalized Yukawa potential; Schrödinger equation; Heavy meson; Nikiforov-Uvarov method

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1. INTRODUCTION

The solution of the Schrödinger equation (SE) for a physical system in quantum mechanics is of great importance, because the knowledge of Eigen energy and wavefunction contains all possible information about the physical properties of a system under study [1]. The study of behavior of several physical problems in physics requires us to solve the SE or Klein-Gordon equation (KGE). The solutions can be established only if we know the confining potential for a particular physical system [2]. The confining potentials may be in different forms depending upon the interaction of the particles within the system. Harmonic oscillator and hydrogen atom are the two potentials which solutions to the SE are found exactly. On the other hand, to obtain the approximate solutions, some techniques are employed. Example of such techniques include, asymptotic iteration method(AIM)[3,4] Laplace transformation method [5], super symmetric quantum mechanics method (SUSQM)[6-8], Nikiforov-Uvarov(NU) method [9-22],series expansion method (SEM) [23,24], analytical exact iterative method(AEIM)[25], and others [26]. The potential model used in predicting mass spectra of heavy mesons is the so-called Cornell potential which contain two important terms, a confinement term and Coulomb term [27]. In the past, this type of potential has been studied by many researchers using different techniques [28-35].The generalized Yukawa potential (GYP) takes the form [36],

$$V(r) = -\frac{\eta_0 e^{-\alpha r}}{r} + \frac{\eta_1}{r^2} \tag{1}$$

where η_0 and η_1 are potential strength, r is interparticle distance and α is a potential range. Its application cut across Nuclear and particle physics. Many researchers have obtained bound state solutions of relativistic and non- relativistic regime of GYP [37-40].

With this in mind, we aim at studying the SE with GYP using the Nikiforov-Uvarov method to obtain the mass spectra of heavy mesons such as charmonium and bottomonium. To the best of our knowledge this study is not in literature. The study will be carried out in three fold. We will first model the GYP to behave like the Cornell potential, thereafter we obtain the solutions via NU method and finally we obtain the mass spectra of heavy mesons.

2. APPROXIMATE SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH GENERALIZED YUKAWA POTENTIAL MODEL

The Schrödinger equation (SE) for two particles interacting via potential $V(r)$, is given by [28].

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)\right)-\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{2}$$

where l, μ, r and $\hbar$ are the angular momentum quantum number, the reduced mass for the quarkonium particle, inter-particle distance and reduced plank constant respectively. We carry out series expansion of the exponential term in Eq. (1) up to order three, in order to model the potential to interact in the quarkonium system and this yields,

$$\frac{e^{-\alpha r}}{r}=\frac{1}{r}-\alpha+\frac{\alpha^2 r}{2}-\frac{\alpha^3 r^2}{6}+... \tag{3}$$

By substituting Eq. (3) into Eq. (1) we have

$$V(r)=-\frac{\alpha_0}{r}+\alpha_1 r+\alpha_2 r^2+\frac{\alpha_3}{r^2}+\alpha_4 \tag{4}$$

where

$$\left.\alpha_0=\eta_0,\ \alpha_1=-\frac{\eta_0\alpha^2}{2},\ \alpha_2=\frac{\eta_0\alpha^3}{6},\ \alpha_3=\eta_1,\ \alpha_4=\eta_0\alpha\right\} \tag{5}$$

Substituting Eq.(4) into Eq.(2) we have

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu E}{\hbar^2}+\frac{2\mu\alpha_0}{\hbar^2 r}-\frac{2\mu\alpha_1 r}{\hbar^2}-\frac{2\mu\alpha_2 r^2}{\hbar^2}-\frac{2\mu\alpha_3}{\hbar^2 r^2}-\frac{2\mu\alpha_4}{\hbar^2}-\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{6}$$

In order to transform the coordinate from r to z in Eq.(6), we set

$$z=\frac{1}{r} \tag{7}$$

This implies that the 2nd derivative in Eq.(7) becomes;

$$\frac{d^2R(\mathrm{r})}{dr^2}=2z^3\frac{dR(\mathrm{z})}{dz}+z^4\frac{d^2R(\mathrm{z})}{dz^2} \tag{8}$$

Substituting Eqs. (7) and (8) into Eq.(6) we have

$$\frac{d^2R(z)}{dz^2}+\frac{2}{z}\frac{dR}{dz}+\frac{1}{z^4}\left[\frac{2\mu E}{\hbar^2}+\frac{2\mu\alpha_0 z}{\hbar^2}-\frac{2\mu\alpha_1}{\hbar^2 z}-\frac{2\mu\alpha_2}{\hbar^2 z^2}-\frac{2\mu\alpha_3 z^2}{\hbar^2}-\frac{2\mu\alpha_4}{\hbar^2}-l\left(l+1\right)z^2\right]R(z)=0 \tag{9}$$

Next, we propose the following approximation scheme on the terms $\frac{\alpha_1}{z}$ and $\frac{\alpha_2}{z^2}$.

Let us assume that there is a characteristic radius r_0 of the meson. Then the scheme is based on the expansion of $\frac{\alpha_1}{z}$ and $\frac{\alpha_2}{z^2}$ in a power series around r_0; i.e. around $\delta\equiv\frac{1}{r_0}$, in the x-space up to the second order. This is similar to Pekeris approximation, which helps to deform the centrifugal term such that the potential can be solved by NU method [35].

Setting $y=z-\delta$ and around $y=0$ it can be expanded into a series of powers as;

$$\frac{\alpha_1}{z}=\frac{\alpha_1}{y+\delta}=\frac{\alpha_1}{\delta\left(1+\frac{y}{\delta}\right)}=\frac{\alpha_1}{\delta}\left(1+\frac{y}{\delta}\right)^{-1} \tag{10}$$

From Eq.(10), we have that,

$$\frac{\alpha_1}{z}=\alpha_1\left(\frac{3}{\delta}-\frac{3z}{\delta^2}+\frac{z^2}{\delta^3}\right) \tag{11}$$

Similarly,

$$\frac{\alpha_2}{z^2}=\alpha_2\left(\frac{6}{\delta^2}-\frac{8z}{\delta^3}+\frac{3z^2}{\delta^4}\right) \tag{12}$$

Substituting Eqs.(11) and (12) into Eq.(9) and simplifying we obtain

$$\frac{d^2R(z)}{dz^2}+\frac{2z}{z^2}\frac{dR(z)}{dz}+\frac{1}{z^4}\left[-\varepsilon+\beta_0 z-\beta z^2\right]R(z)=0 \tag{13}$$

where

$$\left.\begin{aligned}&-\varepsilon=\left(\frac{2\mu E}{\hbar^2}-\frac{6\mu\alpha_1}{\hbar^2\delta}-\frac{12\mu\alpha_2}{\hbar^2\delta^2}-\frac{2\mu\alpha_4}{\hbar^2}\right),\ \beta_0=\left(\frac{2\mu\alpha_0}{\hbar^2}+\frac{6\mu\alpha_1}{\hbar^2\delta^2}+\frac{16\mu\alpha_2}{\hbar^2\delta^3}\right)\\&\beta=\left(\frac{2\mu\alpha_1}{\hbar^2\delta^2}+\frac{6\mu\alpha_2}{\hbar^2\delta^4}+\frac{2\mu\alpha_3}{\hbar^2}+\gamma\right),\ \gamma=l(l+1)\end{aligned}\right\} \tag{14}$$

Comparing Eq.(13) and Eq.(A1) we obtain

$$\left.\begin{array}{l}\tilde{\tau}(z)=2z,\ \sigma(z)=z^2,\ \tilde{\sigma}(z)=-\varepsilon+\alpha z-\beta z^2\\ \sigma'(z)=2z,\ \sigma''(z)=2\end{array}\right\} \tag{15}$$

We substitute Eq.(15) into Eq.(A9) and obtain

$$\pi(z)=\pm\sqrt{\varepsilon-\beta_0 z+\left(\beta+k\right)z^2} \tag{16}$$

To determine k, we take the discriminant of the function under the square root and obtain

$$k=\frac{\beta_0{}^2-4\beta\varepsilon}{4\varepsilon} \tag{17}$$

We substitute Eq.(17) into Eq.(16) and have

$$\pi(z)=\pm\left(\frac{\beta_0 z}{2\sqrt{\varepsilon}}-\frac{\varepsilon}{\sqrt{\varepsilon}}\right) \tag{18}$$

Taking the negative part of Eq. (18) and differentiating once we have

$$\pi'_-(z)=-\frac{\beta_0}{2\sqrt{\varepsilon}} \tag{19}$$

Substituting Eqs.(15) and (19) into Eq.(A7) we have

$$\tau(z)=2z-\frac{\beta_0 z}{\sqrt{\varepsilon}}+\frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{20}$$

Differentiating Eq.(20) we have

$$\tau'(z)=2-\frac{\beta_0}{\sqrt{\varepsilon}} \tag{21}$$

Substituting Eqs. (17) and (19) into Eq.(A10) we have

$$\lambda=\frac{\beta_0{}^2-4\beta\varepsilon}{4\varepsilon}-\frac{\beta_0}{2\sqrt{\varepsilon}} \tag{22}$$

We substitute Eqs.(15) and (21) into Eq.(A11) and obtain

$$\lambda_n=\frac{n\beta_0}{\sqrt{\varepsilon}}-n^2-n \tag{23}$$

Equating Eqs. (22) and (23) and substituting Eqs. (5) and (14) ,we obtain the energy eigenvalues for GYP as,

$$E_{nl}=\eta_0\alpha-\frac{3\eta_0\alpha^2}{2\delta}+\frac{\eta_0\alpha^3}{\delta^2}-\frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu\eta_0}{\hbar^2}-\frac{3\mu\eta_0\alpha^2}{\hbar^2\delta^2}+\frac{8\mu\eta_0\alpha^3}{3\hbar^2\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\alpha^2\mu\eta_0}{\hbar^2\delta^3}+\frac{\mu\eta_0\alpha^3}{\hbar^2\delta^4}+\frac{2\mu\eta_1}{\hbar^2}}}\right]^2 \tag{24}$$

Special case:

When we set $\eta_1 = \alpha = 0$, we obtain the energy eigenvalues for Coulomb potential

$$E_{nl} = -\frac{\mu \eta_0^{\ 2}}{2\hbar^2 \left(n+l+1\right)^2} \tag{25}$$

The result of Eq.(25) is very consistent with the result obtained in Eq.(36) of Ref.[41]

To determine the corresponding wavefunction, we substitute Eqs.(15) and (18) into Eq.(A4) and obtain

$$\frac{d\phi}{\phi} = \left(\frac{\varepsilon}{z^2\sqrt{\varepsilon}} - \frac{\beta_0}{2z\sqrt{\varepsilon}}\right)dz \tag{26}$$

Integrating Eq.(26) ,we obtain

$$\phi(\mathrm{z}) = z^{-\frac{\beta_0}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{z\sqrt{\varepsilon}}} \tag{27}$$

By substituting Eqs.(15) and (20) into Eq.(A6) and integrating ,thereafter simplify we obtain

$$\rho(\mathrm{z}) = z^{-\frac{\beta_0}{\sqrt{\varepsilon}}} e^{-\frac{2\varepsilon}{z\sqrt{\varepsilon}}} \tag{28}$$

Substituting Eqs.(15) and(28) into Eq.(A5) we have

$$y_n(\mathrm{z}) = B_n e^{\frac{2\varepsilon}{z\sqrt{\varepsilon}}} z^{\frac{\beta_0}{\sqrt{\varepsilon}}} \frac{d^n}{dz^n}\left[e^{-\frac{2\varepsilon}{z\sqrt{\varepsilon}}} z^{2n-\frac{\beta_0}{\sqrt{\varepsilon}}}\right] \tag{29}$$

The Rodrigues' formula of the associated Laguerre polynomials is

$$L_n^{\frac{\beta_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right) = \frac{1}{n!} e^{\frac{2\varepsilon}{z\sqrt{\varepsilon}}} z^{\frac{\beta_0}{\sqrt{\varepsilon}}} \frac{d^n}{dz^n}\left(e^{-\frac{2\varepsilon}{z\sqrt{\varepsilon}}} z^{2n-\frac{\beta_0}{\sqrt{\varepsilon}}}\right) \tag{30}$$

where

$$\frac{1}{n!} = B_n \tag{31}$$

Hence,

$$y_n(\mathrm{z}) \equiv L_n^{\frac{\beta_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right) \tag{32}$$

Substituting Eqs.(27) and (32) into Eq.(A2) we obtain the wavefunction of GYP in terms of Laguerre polynomials as

$$\psi(\mathrm{z}) = B_{nl} z^{-\frac{\beta_0}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{z\sqrt{\varepsilon}}} L_n^{\frac{\beta_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right) \tag{33}$$

where B_{nl} is normalization constant, which can be obtain from

$$\int_0^\infty |B_{nl}(r)|^2 dr = 1 \tag{34}$$

3. RESULTS AND DISCUSSION

3.1 Results

The mass spectra of the heavy quarkonium system such as charmonium and bottomonium that have the quark and antiquark flavor is calculated by applying the following relation [42, 43]

$$M = 2m + E_{nl}, \tag{35}$$

where m is quarkonium bare mass, and E_{nl} is energy eigenvalues. By substituting Eq. (24) into Eq. (35) we obtain the mass spectra for GYP as:

$$M = 2m + \eta_0\alpha - \frac{3\eta_0\alpha^2}{2\delta} + \frac{\eta_0\alpha^3}{\delta^2} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu\eta_0}{\hbar^2} - \frac{3\mu\eta_0\alpha^2}{\hbar^2\delta^2} + \frac{8\mu\eta_0\alpha^3}{3\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{\alpha^2\mu\eta_0}{\hbar^2\delta^3} + \frac{\mu\eta_0\alpha^3}{\hbar^2\delta^4} + \frac{2\mu\eta_1}{\hbar^2}}}\right]^2 \tag{36}$$

Table1: Mass spectra of charmonium in GeV

$(m_c = 1.209\text{ GeV}, \hbar = 1, \eta_0 = 18.33359\text{ GeV}, \eta_1 = 8.16864\text{ GeV}, \alpha = 0.23, \mu = 0.6045\ GeV, \delta = 0.1426\text{ GeV})$

State	Present work	NU[35]	AIM[3]	Experiment[44]
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.524	3.255	3.521	3.525
2P	3.731	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.261	4.269	4.433	4.263
1D	3.727	3.504	3.800	3.770
2D	4.158	-	-	4.159
1F	3.412	-	-	-

Table 2: Mass spectra of bottomonium in GeV
(m_b = 4.823 GeV, $\mu =$ 2.4115 GeV, $\eta_0 =$ 19.5556 GeV, $\eta_1 =$ 18.5499 GeV, $\delta =$ 1.12947 GeV, $\alpha =$ 0.23, $\hbar =$ 1)

State	Present work	NU[35]	AIM[3]	Experiment[44]
1S	9.460	9.460	9.462	9.460
2S	10.023	10.023	10.027	10.023
1P	9.527	9.619	9.963	9.899
2P	10.260	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.580	10.567	10.624	10.580
1D	10.162	9.864	10.209	10.164
2D	10.172	-	-	-
1F	9.825	-	-	-

3.2 Discussion of Results

We calculate mass spectra of charmonium and bottomonium for states from 1S, 2S, 1P, 2P, 3S, 4S, 1D, 2D, and 1F, by using Eq. (36). The free parameters of Eq. (36) were then obtained by solving two algebraic equations in the case of charmonium and bottomonium, respectively.

The experimental data were taken from [44]. For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ systems we adopt the numerical values of these masses as m_b = 4.823 GeV and m_c = 1.209 GeV [45]. Then, the corresponding reduced mass are μ_b = 2.4115 GeV and $\mu_c = 0.6045$ GeV, respectively. We note that calculation of mass spectra of charmonium and bottomonium are in good agreement with experimental data and work of other researchers, in Refs.[3, 35] as presented in Tables 1 and 2. In order to test for the accuracy of the predicted results determined numerically, we used a Chi square function to determine the error between the experimental data and theoretical predicted values. The maximum error in comparison with the experimental data is found to be 0.0157 GeV.

4. CONCLUSION

In this study, we model Generalized Yukawa potential to interact in quark-antiquark system. We obtained the approximate solutions of the Schrödinger equation for energy eigenvalues and corresponding wave function using the Nikiforov-Uvarov method. We applied the present results to compute heavy-meson masses of charmonium and bottomonium for different quantum states. The results agree excellently with experimental data with a maximum error of 0.0157 GeV and work of other researchers.

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APPENDIX A: REVIEW OF NIKIFOROV-UVAROV (NU) METHOD

The NU method was proposed by Nikiforov and Uvarov [46] to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation $z = z(r)$, of the form

$$\psi''(z)+\frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z)+\frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z)=0 \tag{A1}$$

where $\tilde{\sigma}(z)$, and $\sigma(z)$ are polynomials, at most second degree and $\tilde{\tau}(z)$ is a first-degree polynomial. The exact solution of Eq.(A1) can be obtain by using the transformation.

$$\psi(z)=\phi(z)y(z) \tag{A2}$$

This transformation reduces Eq.(A1) into a hypergeometric-type equation of the form

$$\sigma(z)y''(z)+\tau(z)y'(z)+\lambda y(z)=0 \tag{A3}$$

The function $\phi(z)$ can be defined as the logarithm derivative

$$\frac{\phi'(z)}{\phi(z)}=\frac{\pi(z)}{\sigma(z)} \tag{A4}$$

With $\pi(z)$ being at most a first-degree polynomial. The second part of $\psi(z)$ being $y(z)$ in Eq.(A2) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(z)=\frac{B_{nl}}{\rho(z)}\frac{d^n}{dz^n}\left[\sigma^n(z)\rho(z)\right] \tag{A5}$$

Where B_{nl} is the normalization constant and $\rho(z)$ the weight function which satisfies the condition below;

$$\left(\sigma(z)\rho(z)\right)'=\tau(z)\rho(z) \tag{A6}$$

where also

$$\tau(z)=\tilde{\tau}(z)+2\pi(z) \tag{A7}$$

For bound solutions, it is required that

$$\tau'(z)<0 \tag{A8}$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(z)$ and parameter λ, respectively:

$$\pi(z)=\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\pm\sqrt{\left(\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\right)^2-\tilde{\sigma}(z)+k\sigma(z)} \tag{A9}$$

and

$$\lambda=k+\pi_-'(z) \tag{A10}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A9) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda_n+n\tau'(z)+\frac{n(n-1)}{2}\sigma''(z)=0,(n=0,1,2,...) \tag{A11}$$