DOI: https://doi.org/10.4038/sljas.v27i1.8212

Sri Lankan Journal of Applied Statistics, Vol (27-1)

A Weighted Cosine Exponential Distribution with Application

I. Zaheer[1], T. Hussain[2], M. Shakil[3,*], M. Ahsanullah[4], and B. M. G. Kibria[5]

[1]Department of Mathematics, Dar-e-Arqam College, Mirpur 10250, Pakistan

[2]Department of Statistics, Mirpur University of Science and Technology (MUST), Mirpur 10250, Pakistan

[3]Department of Mathematics, Miami Dade College, Hialeah, FL USA

[4]Rider University, New Jersey, USA

[5]Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, USA

*Corresponding author: mshakil@mdc.edu

Received: 05th August, 2025/ Revised: 21st January, 2026/ Accepted: 22nd January, 2026/ Published: 31st March, 2026

ABSTRACT

The Weighted Cosine Exponential Distribution ($\mathcal{WCED}$) is proposed as a novel lifetime model, incorporating a trigonometric weighting mechanism into the exponential distribution to enhance flexibility and capture complex hazard behaviors, particularly the inverted bathtub shape common in reliability and survival analysis. Fundamental statistical properties are derived. A simulation study evaluates five estimation methods (MLE, MME, OLS, WLS, CVM), showing improved accuracy with larger samples. Applied to three cancer survival datasets, $\mathcal{WCED}$ demonstrates superior fit over existing weighted distributions, evidenced by lower information criteria (AIC, AICC, HQIC) and higher statistical significance. The model offers a parsimonious and adaptable tool for analyzing lifetime data with nonmonotonic failure patterns, proving valuable for reliability engineering and biomedical applications.

Keywords: Hazard function; Bathtub shape; Estimation; Reliability; Application; Weighted distribution.

1 Introduction

The motivation of this paper can be better understood in light of a key observation made by Patil and Rao (1977) and Rao (1985). They noted that although weighted distributions arise frequently across a wide range of applications, their role as a fundamental stochastic framework has not been sufficiently acknowledged. This observation provides an important basis for the resolution of many modern statistical challenges; however, research in this area remains relatively limited. In many observational studies, researchers cannot select sampling units with equal probabilities or rely on well-defined sampling frames for populations such as humans, wildlife, insects, plants, or fish. As a result, observed data are often subject to selection bias and do not reflect the underlying population distributions unless each individual has an equal probability of being observed.

The widespread occurrence of biased data across scientific disciplines has prompted researchers to develop interim methods for bias correction. Within this context, Rao (1985) and Patil and Rao (1977) introduced a comprehensive framework based on weighted distributions to model biased data in situations where complete sampling frames are unavailable and sampling units are selected with unequal probabilities. Such conditions commonly arise in non-experimental, non-replicated, and non-random study designs, often as a result of unobserved events, data contamination, or the implementation of unequal probability sampling schemes. A formal definition of a weighted distribution is given by: theoremDefinitionLet (Ω, ζ, p) be a probability space and $X : \Omega \to S$ be a continuous random variable where $S \in (a, b)$ where $a \ \ and \ \ b \geq 0$. Suppose $f(x)$ be the probability density function (PDF) of distribution function $F(x)$ and $w(x)$ be the non-negative weight function which should be greater than one. The weighted distribution of weight function w(x) with the pdf f(x) is defined as

$$f_{w_{x;\varphi}}(X) = \frac{w(x;\varphi)f(x)}{\mathbf{E}(w(X;\varphi))}, \quad x > 0. \tag{1}$$

where $\mathbb{E}(w(X;\phi)) = \int_x w(X;\phi)f(x)dx$ and φ depends upon the parameter of weight function.

The exponential distribution is a cornerstone of lifetime and reliability analysis due to its analytical simplicity and practical effectiveness in modeling survival phenomena. Its tractable probability density function (PDF) and cumulative distribution function (CDF) facilitate the derivation of closed-form results, making it a natural baseline model. Consequently, a broad range of extensions and generalizations have been developed to enhance its flexibility. Notable examples include the weighted exponential (WE), generalized exponential, gamma, and Weibull distributions, each representing a structured departure from the classical exponential model (Chesneau et al., 2022).

To address biased sampling and heterogeneity in lifetime data, the weighted exponential distribution (WED) and weighted inverted exponential distribution (WIED) were first introduced by Gupta and Kundu (2009). Subsequent developments by Kharazmi and Jabbari (2017) and Rao (1985) extended the WED through the introduction of additional shape parameters, leading to the generalized weighted exponential (GWED) and exponential weighted exponential (EWE) distributions, respectively. Moreover, Kharazmi and Jabbari (2017) proposed the new weighted exponential (NWED) distribution, offering increased flexibility within the class of weighted lifetime models. Further contributions have substantially enriched the theory of weighted exponential distributions, yielding several important extensions such as the wrapped weighted exponential (WWE), weighted inverted exponential (WIE), generalized weighted exponential (GWE), size-biased double weighted exponential (SBWE), extended exponential (ExtED), two-parameter inverted weighted exponential (TIWE), and logarithmic weighted exponential (LWED) distributions (Chesneau et al., 2022). These models have demonstrated improved adaptability in capturing diverse failure mechanisms.

Recent advances have also emphasized trigonometric-based generators as powerful tools for enhancing distributional flexibility. Sine- and cosine-based extensions—including the Sine Weibull (Almalki, 2023), Sine Modified Lindley (Mondal et al., 2022), Cosine–Weibull (Dey et al., 2020), and Cosine Exponential (Khan et al., 2022) distributions—have shown strong performance in modeling asymmetric, multimodal, and non-monotone hazard structures. More recently, the Weighted Sine–Cosine Exponential distribution (Iqbal et al., 2023) and the Sine Cosine-G family (Al-Babtain et al., 2023) have incorporated trigonometric weight functions within general generator frameworks. These developments motivate the present study, which introduces the Weighted Cosine Exponential Distribution ($\mathcal{WCED}$), a parsimonious yet flexible model capable of capturing periodic and inverted bathtub-shaped hazard rate behaviors frequently observed in biomedical, engineering, and environmental applications.

In 2021, Christophe Chesneau and Hassan S. Bakouch proposed a novel two-parameter weighted discrete distribution, known as the cosine geometric distribution, for modeling count data (Chesneau et al., 2021). The probability mass function (PMF) of this two-parameter cosine geometric distribution is given by:

$$P(X = k) = C_{p,\theta}\, p^k\, [\cos(k\theta)]^2, \qquad k \in \mathbb{N}, \tag{2}$$

where

$$C_{p,\theta} = \frac{2(1-p)(1-2p\cos(2\theta)+p^2)}{2+p\,[(p-3)\cos(2\theta)+p-1]}.$$

theoremDefinitionIf X be the non- negative random variable and $f(x)$ be the probability density function(PDF).Then the new weighted exponential distribution is obtained by introducing new weight function known as trigonometric function $w(x, \beta) = cos(\beta x)$ and substitute it into equation (1), we get

$$f(x_w; \theta, \beta) = \frac{4\beta^2\theta + \theta^3}{2\beta^2 + \theta^2}(cos(\beta x))^2 e^{-\theta x}, \quad x > 0, \quad \theta > 0, \quad \beta > 0. \tag{3}$$

where θ and β are the scale and shape parameters and $f(x)$ belongs to exponential distribution. The above established model is named as Weighted Cosine Exponential Distribution ($\mathcal{WCED}$).

Now, we represent the genesis of the model $\mathcal{WCED}$ using the following motivations
Motivation I. The new investigated model is periodic in nature due to their weight functions.
Motivation II. The proposed model gives the multimodes at their different values of random variable with their respective parametric values.
Other motivations of proposed models are discussed in the following ways. Both models contained two parameters namely, scale parameter θ and shape parameter β and under these parameters models are valuable and summarized in the following notes. Figure 1 presents values of the PDF for the$\mathcal{WCED}$ across various parameter pairs (θ, β). Each column corresponds to a specific parameter set, showing how the PDF $f(x)$ changes with the variable x. For example, when $(\theta, \beta) = (1.1325, 0.2125)$, the PDF decreases from approximately 2.00 at $x = 0$ to about 0.70 at $x = 0.5$, indicating a right-skewed, decreasing density typical of lifetime distributions. These computed values illustrate the flexibility and shape variability of the $\mathcal{WCED}$ under different parametric configurations.
Further, failure, survival and hazard rate function of model $\mathcal{WCED}$ can be expressed as

$$F(x) = 1 - \frac{e^{-\theta x}\,(4\beta^2 + \theta^2 + \theta^2 cos(2\beta x) - 2\beta\theta sin(2\beta x))}{2(2\beta^2 + \theta^2)}. \tag{4}$$

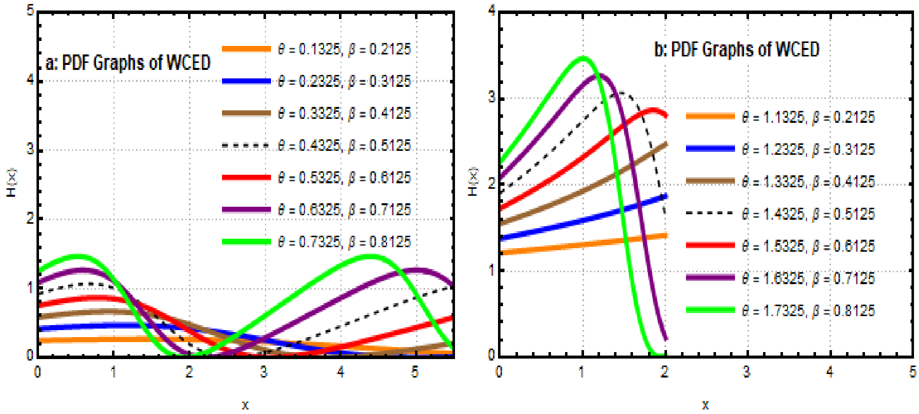

Fig. 1: PDF Graphs of $\mathcal{WCED}$ by varying the scale and shape parameters.

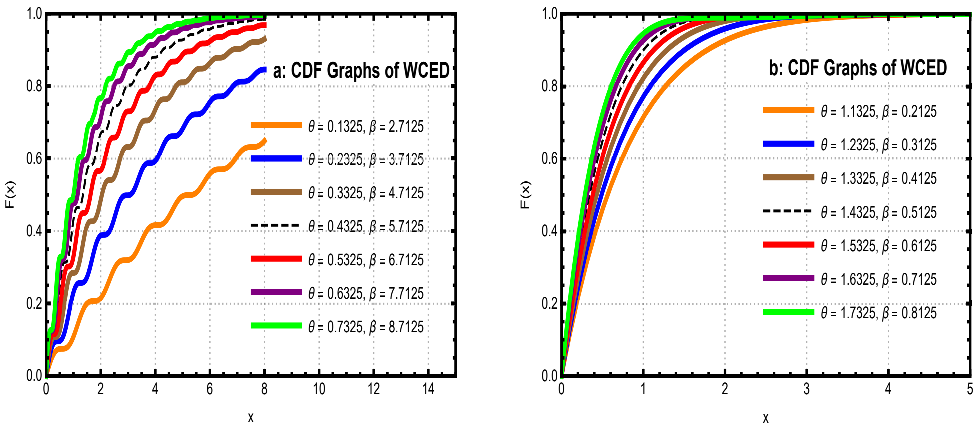

Fig. 2: CDF Graphs of $\mathcal{WCED}$ by varying the scale and shape parameters.

$$S(x) = \frac{e^{-\theta x}\left(4\beta^2 + \theta^2 + \theta^2 cos(2\beta x) - 2\beta\theta sin(2\beta x)\right)}{2(2\beta^2 + \theta^2)}. \tag{5}$$

Let X be a non-negative continuous random variable representing the lifetime of a component or individual. Let $f(x)$ and $F(x)$ denote its probability density function (PDF) and cumulative distribution function (CDF), respectively. The hazard function, also known as the failure rate or instantaneous risk function, is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x + \Delta x \mid X \geq x)}{\Delta x} = \frac{f(x)}{1 - F(x)}, \quad x \geq 0.$$

The hazard function represents the instantaneous rate at which events occur at time t, given that the individual or system has survived up to time x. Unlike the PDF, which describes the overall distribution of lifetimes, the hazard function characterizes the local risk of failure at a specific time point. The hazard function plays a crucial role in survival analysis and reliability theory due to the following reasons:

- It provides a direct measure of the instantaneous risk of failure, offering meaningful interpretation in medical, biological, and engineering applications.
- Different shapes of the hazard function (increasing, decreasing, constant, or bathtub-shaped) capture diverse aging and failure mechanisms.
- It allows for the comparison of risk patterns across populations or treatment groups, even when their survival distributions differ.
- The hazard function is central to many advanced survival models, including proportional hazards models and accelerated failure time models.
- In reliability engineering, it helps identify early-life failures, wear-out periods, and useful life phases of systems.

A constant hazard function corresponds to the exponential distribution, indicating a memoryless property, while non-constant hazard functions arise in more flexible lifetime models. Now the hazard rate function (HRF) of $\mathcal{WCED}$ is expressed as

$$h(x) = \frac{f(x)}{S(x)} = \frac{2(4\beta^2\theta + \theta^3 cos(\beta x)^2)}{4\beta^2 + \theta^2 + \theta^2 cos(2\beta x) - 2\beta\theta sin(2\beta x)}. \tag{6}$$

Figure 2 depicts the CDF of the $\mathcal{WCED}$, with curves generated by varying its scale parameter θ and shape parameter β. Each curve illustrates the probability $F(x) = P(X \leq x)$ as x increases, showing how the distribution's shape and rate of accumulation change under different parameter settings. This visualization highlights the flexibility of the $\mathcal{WCED}$ in modeling various types of positively skewed lifetime data. Figure 3 portays the survival function $S(x)$ of $\mathcal{WCED}$ with different scale (θ) and shape (β) parameters. Each value corresponds to a specific parameter set, showing how $S(x)$—the probability that a unit survives beyond time x—decreases as x increases. For example, when $(\theta, \beta) = (1.1325, 0.2125)$, the survival probability drops from 1.00

at $x = 0$ to approximately 0.00 by $x = 4$, illustrating the model's ability to describe various failure-time behaviors through its parameters. Figure 4 portrays the HRF $h(x)$ for $\mathcal{WCED}$ across various parame-

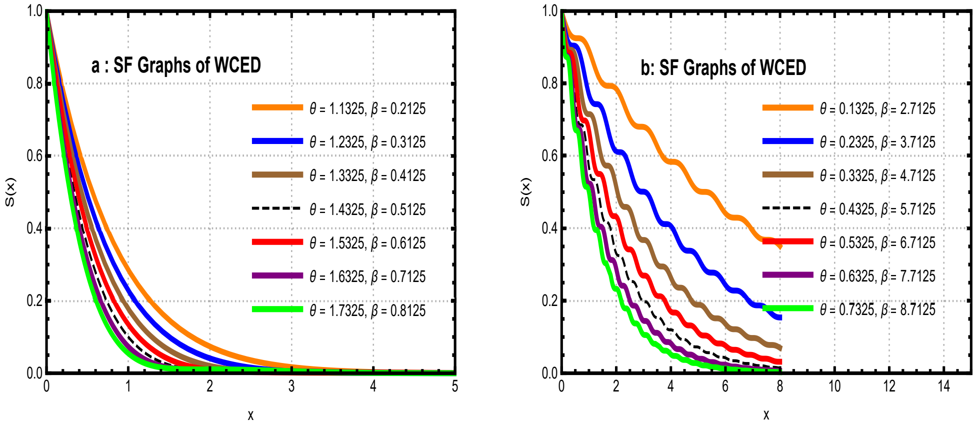

Fig. 3: Survival behavior of $\mathcal{WCED}$ on variation of parameters.

ter pairs (θ, β). Each column corresponds to a different combination of the scale and shape parameters, showing how the instantaneous failure rate evolves over time x. The HRF exhibits non-monotonic behavior for certain parameter sets, with values initially increasing to a peak and then decreasing or stabilizing, characteristic of an inverted bathtub-shaped hazard—a key feature of the $\mathcal{WCED}$ model's flexibility. The subsequent sections of the paper are structured as follows:

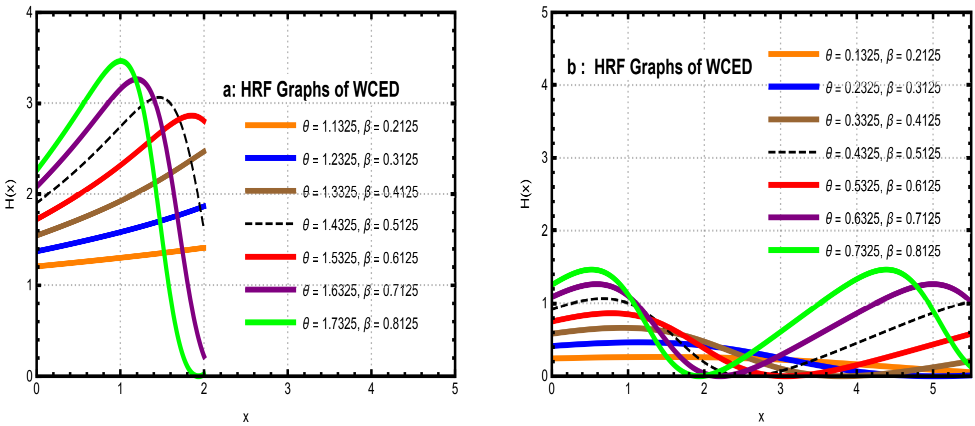

Fig. 4: HRF behavior of $\mathcal{WCED}$ on variation of parameters.

Section 2 explores various statistical and mathematical characteristics of the new model, such as moment generating functions, moments, mean deviation around the mean and median, entropy, and the stress-strength parameter. Parameter estimation for the $\mathcal{WCED}$ using maximum likelihood and the method of moments, along with simulation studies, is covered in Section 3. Section 4 involves fitting the models to suitable real-world datasets, and Section 5 provides the conclusion.

2 Some Statistical and Mathematical Properties of $\mathcal{WCED}$

2.1 Moment Generating Function

Moment generating function (MGF) plays an important role in describing the characteristics of distribution like shape, skewness and kurtosis of the probability function . It is defined as ($\mathbf{E}(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$) for $\mathcal{WCED}$ can be expressed as

$$M_X(t) = \mathbf{E}\left(e^{tX}\right) = \int\limits_0^\infty e^{tx}\frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}(cos(\beta x))^2 e^{-\theta x}dx,$$

which on simplification yields

$$M_X(t) = \mathbf{E}\left(e^{tX}\right) = \frac{\theta\left(4\beta^2+\theta^2\right)\left(2\beta^2+(t-\theta)^2\right)}{\left(2\beta^2+\theta^2\right)(\theta-t)\left(4\beta^2+(t-\theta)^2\right)}. \tag{7}$$

By differentiating the above equation (7) w.r.t t and then putting $t=0$ we can find out the required moments of new model. The first four moments are;

$$\mu_1 = M_1'(0) = \frac{8\beta^4+2\beta^2\theta^2+\theta^4}{\theta\left(2\beta^2+\theta^2\right)\left(4\beta^2+\theta^2\right)}.$$

$$\mu_2 = M_2''(0) = \frac{2\left(32\beta^6+24\beta^4\theta^2+\theta^6\right)}{\theta^2\left(2\beta^2+\theta^2\right)\left(4\beta^2+\theta^2\right)^2}.$$

$$\mu_3 = M_3'''(0) = \frac{6\left(128\beta^8+128\beta^6\theta^2+56\beta^4\theta^4-4\beta^2\theta^6+\theta^8\right)}{\theta^3\left(2\beta^2+\theta^2\right)\left(4\beta^2+\theta^2\right)^3}.$$

$$\mu_4 = M_4''''(0) = \frac{24\left(512\beta^{10}+640\beta^8\theta^2+320\beta^6\theta^4+120\beta^4\theta^6-10\beta^2\theta^8+\theta^{10}\right)}{\theta^4\left(2\beta^2+\theta^2\right)\left(4\beta^2+\theta^2\right)^4}.$$

2.2 Skweness and Kurtosis Of $\mathcal{WCED}$

Skewness and kurtosis gave the dual concept to define shape or peaknes of probability function in statistics. Skewness tells about the symmetry of distribution whereas kurtosis notify about the peakedness or height of the distribution . Skewness and kurtosis of $\mathcal{WCED}$ are illustrated in the below figure. Skewness and kurtosis of $\mathcal{WCED}$ distribution are expressed as

$$\begin{aligned}
\mathbb{S} &= \frac{2\left(512\beta^{12}+1152\beta^{10}\theta^{2}+1632\beta^{8}\theta^{4}+680\beta^{6}\theta^{6}+60\beta^{4}\theta^{8}-6\beta^{2}\theta^{10}+\theta^{12}\right)}{\left(8\beta^{4}\theta+6\beta^{2}\theta^{3}+\theta^{5}\right)^{3}}.\\
\mathbb{K} &= \frac{3\left(12288\beta^{16}+40960\beta^{14}\theta^{2}+48640\beta^{12}\theta^{4}+38912\beta^{10}\theta^{6}\right)}{\left(8\beta^{4}\theta+6\beta^{2}\theta^{3}+\theta^{5}\right)^{4}}\\
&\quad+\frac{3\left(19440\beta^{8}\theta^{8}+4800\beta^{6}\theta^{10}+312\beta^{4}\theta^{12}-32\beta^{2}\theta^{14}+3\theta^{16}\right)}{\left(8\beta^{4}\theta+6\beta^{2}\theta^{3}+\theta^{5}\right)^{4}}.
\end{aligned}$$

Figure 5 consists of two 3D surface plots visualizing the skewness

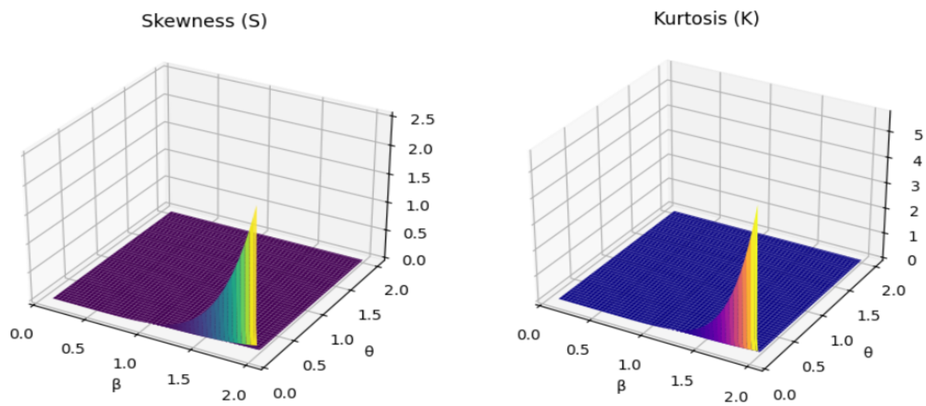

Fig. 5: Skewness and Kurtosis 3D Plots of $\mathcal{WCED}$

and kurtosis of the $\mathcal{WCED}$ as functions of its scale θ and shape β parameters. Skewness (S) illustrates how the asymmetry of the distribution changes across the parameter space, while kurtosis (K) shows the corresponding variation in the tailedness (peak sharpness) of the distribution. These plots confirm that the $\mathcal{WCED}$ can exhibit a wide range of distributional shapes—from symmetric to highly skewed and from platykurtic to leptokurtic—depending on the parameter choices, highlighting its modeling flexibility.

2.3 Mean Deviation about Mean and Median

Mean deviation about mean and median is usually used to measure the spread of population. Mean deviation about mean is an absolute difference between the random variable and their mean μ. Mathematically, $E \mid x-\mu \mid = 2\mu G(\mu)-2\mu+2\int_{\mu}^{\infty} xg(x)dx$. Mean devia-

tion about median offers a direct measure of a random variable 'y' around their median($\tilde{\mu}$). Mathematically it takes the form, $E \mid x - \tilde{\mu} \mid = \tilde{\mu} - 2\int_0^{\tilde{\mu}} xg(x)dx.$
For $\mathcal{WCED}$, mean deviation about mean can expressed as;

$$\begin{aligned} E|x-\mu| &= -\frac{\mu e^{-\theta\mu}\left(4\beta^2+\theta^2+\theta^2 cos(2\beta\mu)-2\beta\theta sin(2\beta\mu)\right)}{(2\beta^2+\theta^2)}+ \\ & \left(\frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}\right)\sum_{n=0}^{\infty}\frac{(-1^n)(2\beta)^{2n}}{2n!}\left(\theta^{-(2n+2)}\Gamma(2n+2,-\theta\mu)\right). \end{aligned}$$

For $\mathcal{WCED}$, mean deviation about median can expressed as;

$$E \mid X-\mu \mid = \mu - \frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}\sum_{n=0}^{\infty}\frac{(-1^n)(2\beta)^{2n}}{2n!}\left(\theta^{-(2n+2)}\Gamma(2n+2,-\theta\mu)\right).$$

2.4 Concavity and Convexity of ($\mathcal{WCED}$)

In order to determine the concavity and convexity of our new model we apply the sufficient condition for concavity up(convexity down) and convexity up(concavity down) which can express as

Consider a function $f(x)$ whose first derivative exist in closed interval $[c, d]$ and their second derivative occurs in an open interval (a, b). Then the following sufficient conditions are acceptable for concavity and convexity of a function:

• A function is said to be concave up/convex down at their open interval if $f''(x) > 0$.
• A function is said to be concave down/convex up at their open interval if $f''(x) < 0$.

Now we apply this test for our model to find out their convexity and concavity .
Therefore, the function ($\mathcal{WCED}$) shows the concavity up/convexity down at point

$$x < -\frac{1}{\beta}\tan^{-1}\left(\frac{\theta \pm \sqrt{2\beta^2-\theta^2}}{2\beta}\right), \ \ \theta > 0 \ \ \beta > 0.$$

Similarly, the PDF of ($\mathcal{WCED}$) shows the concavity down/convexity up at point

$$x > -\frac{1}{\beta}\tan^{-1}\left(\frac{\theta \pm \sqrt{2\beta^2-\theta^2}}{2\beta}\right), \ \ \theta > 0 \ \ \beta > 0.$$

By taking the different values of parameters we can detect the curve where function is concave or convex.

2.5 Stress-Strength Parameter

When we discuss the analysis of reliability, the stress and strength parameter explain the life of an item or device. Suppose X and Y be the two independent random variables of an item. When strength (Y) being exposed under the stress(X) then reliability parameter gives the performance of an item. Furthermore, reliability describes the failure of a device when stress was exceeded the strength than the item will not be able to perform well or have no more chances to live more.i.e, $(R = P\,(X \geq Y))$.In mathematically, it can be express as,

$$R = P\,(X \geq Y) = \int_{0}^{\infty} \left(f(x) \int_{y}^{\infty} F(y)dy \right) dx. \tag{8}$$

Where $f(x)$ is a pdf of new investigated models and 2nd integral represent the survival function of a new proposed weighted distribution. Now, for new proposed model $(\mathcal{WCED})$ sress- strength parameter can be evaluated by substitution of CDF and PDF of $(\mathcal{WCED})$ in equation(8), then we get,

$$R = \frac{e^{-\theta_2 y}\left(4\beta_2^2 + \theta_2^2 + \theta_2^2 cos(2\beta_2 y) - 2\beta_2\theta_2 sin(2\beta_2 y)\right)}{2(2\beta_2^2 + \theta_2^2)}.$$

2.6 Entropy

Entropy measures the disorderness or unpredictable state of a system which is associated to the random variable $'X'$. To designating and identifying the behavior of engineering and biological systems the concept of information coding and transmissions play their important rule. In information theory it provides a baseline idea which is associated to statistical mechanics and thermodynamics. Here, the three different and most popular entropies are obtained to measure the uncertainty of a random variable. The entropies are Shannon, Renyi and residual Entropy.

2.6.1 Shannon Entropy:

In 1948, Shannon introduced the concept of entropy in communication theory. The Shannon entropy is a classical entropy and is defined as the negative expected value of the logarithm of the PDF (Shannon, 1951). In mathematically,

$$H(x) = -E[\log f(x)] = -\int_{0}^{\infty} f(x) log f(x) dx. \tag{9}$$

By solving the equation(9), Shannon Entropy for new model $\mathcal{WCED}$ can be formulated as;

$$H(x) = -\log\left(\frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}\right) + \left(\frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}\right)\left[1 - \frac{4\beta^2\theta^2}{8\beta^4+6\beta^2\theta^2+\theta^4} + J\right].$$

Where J is

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+m} 2^{2(n+m)} (\beta x)^{2n+2m} (2^{2m}-1)|B_{2m}|}{2m(2n!)(2m!)} \frac{2(n+m)}{\theta^{2(n+m)+1}} \Gamma(2(n+m)).$$

2.6.2 Renyi Entropy:

If X be a random variable with CDF $F(x)$ and PDF $f(x)$. The Renyi Entropy of X can be defined as

$$H_\gamma(X) = \frac{1}{1-\gamma} \log \int_{0}^{\infty} f^\gamma(x)\, dx. \tag{10}$$

The Rényi entropy is a sub-case of Shannon entropy (Rényi, 1961). Now, Renyi entropy for $\mathcal{WCED}$ can be evaluated by solving the above integral and finally it takes the following form;

$$H_\gamma(x) = \frac{1}{1-\gamma} \log\left(\left(\frac{4\beta^2+\theta^3}{2\beta^2+\theta^2}\right)^\gamma \sum_{n=0}^{\infty} \left(\frac{(-1)^n (2\beta)^{2n}}{2(2n)!}\right)^\gamma \frac{1}{(\alpha\theta)^{2n\alpha+1}} \Gamma(2n\gamma+1)\right).$$

2.6.3 Residual Entropy:

Suppose X is a non-negative random variable. The residual entropy explains the lifetime of a system as a dynamic form of improbability of X at time t (Belzunce et al., 2004). Mathematically, it can be defined as

$$R.E = H(x:t) = -\int_{t}^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)}. \tag{11}$$

Where $\bar{F(t)}$ is the survival function of investigated models.
By solving the equation(11), Residual entropy of $\mathcal{WCED}$ can be expressed as;

$$H(x;t) = \frac{2\left(4\beta^2\theta+\theta^3\right)}{e^{-t\theta}\left(4\beta^2+\theta^2+\theta^2 cos(2\beta t)-2\beta\theta sin(2\beta t)\right)}$$
$$\times \log\left(\frac{2(4\beta^2\theta+\theta^3)}{e^{-t\theta}\left(4\beta^2+\theta^2+\theta^2\cos(2\beta t)-2\beta\theta\sin(2\beta t)\right)}\right)$$
$$\left(-\sum_{n=0}^{\infty}\frac{(-1)^n(2\beta)^{2n}}{2n!}\left(\theta^{-(2n+1)}\Gamma(2n+2,\theta t)\right)\right)$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}2^{2(n+m)}(\beta x)^{2n+2m}(2^{2m}-1)|B_{2m}|}{m(2n!)(2m!)}\left(\theta^{-2(m+n)+1}\Gamma(2n+2m+1,\theta t)\right).$$

3 Parameter Estimation and Simulation Study

The parameters of our proposed model are estimated using a selection of five widely recognized methods. These include maximum-likelihood (ML), ordinary least-squares (OLS), weighted least-squares (WLS), maximum product of spacing (MPS), and Cramér-von Mises (CVM).

3.1 Simulation Study

To investigate the performance and behavior of the estimators discussed in the previous section, we conduct a Monte Carlo simulation study. In this process, the sample size n is increased from 15, 25, 50, 150, 350 to 500 by varying the parameters of the models under investigation, i.e., $(\theta,\beta) = (0.3596, 0.1529)$, $(\theta,\beta) = (1.2965, 0.0593)$, and $(\theta,\beta) = (1.0329, 2.3054)$. For each sample size n, we generate 1000 independent samples and estimate the parameters using various methods. The bias, mean squared error (MSE), root mean squared error (RMSE), and convergence rate (Con.Rate) for each estimator $\hat{\Theta} = (\hat{\theta},\hat{\beta})$ are evaluated using the following formulas.

$$Bias(\hat{\Theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\Theta}_i-\Theta),\;\; MSE(\hat{\Theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\Theta}_i-\Theta)^2$$

,

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{\Theta}_i-\Theta)^2}\ \text{Con.Rate} = \frac{\left|\hat{\Theta}_{n+1}-\Theta\right|}{\left|\hat{\Theta}_n-\Theta\right|}$$

. Now we shall study different estimation methods to estimate the parameters of both new proposed models.

3.2 Maximum Likelihood Method

The Maximum Likelihood Estimation (MLE) approach is a widely adopted and structured technique for parameter estimation. It minimizes bias and maximizes the likelihood of observing the given data under the assumed probability distribution. MLE's popularity stems from its desirable characteristics, including asymptotic efficiency and consistency. Let's consider n observations,$x_1, x_2, x_3, \ldots, x_n$drawn from a continuous random variable, $\mathcal{WCED}$. For this random sample $x_1, x_2, \ldots, x_n$, the likelihood function is:

$$L(\theta,\beta) = \prod_{i=1}^{n} f(x_i;\theta,\beta) = \left(\frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}\right)^n \prod_{i=1}^{n} \cos^2(\beta x_i) e^{-\theta\sum_{i=1}^{n} x_i}.$$

The log-likelihood function for the $\mathcal{WCED}$ distribution is then defined as:

$$\ell(\theta,\beta) = n\ln\left(\frac{4\beta^2\theta+\theta^3}{2\beta^2+\theta^2}\right) + 2\sum_{i=1}^{n} \ln\cos(\beta x_i) - \theta\sum_{i=1}^{n} x_i. \tag{12}$$

Taking the partially derivative of above equation w.r.t θ and β and equating them equal to zero to find out the MLEs of θ and β respectively as,

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} &= n\cdot\frac{2\beta^2+\theta^2}{4\beta^2\theta+\theta^3}\cdot\frac{8\beta^4+2\beta^2\theta^2+\theta^4}{(2\beta^2+\theta^2)^2} - \sum_{i=1}^{n} x_i \\ &= n\cdot\frac{8\beta^4+2\beta^2\theta^2+\theta^4}{(4\beta^2\theta+\theta^3)(2\beta^2+\theta^2)} - n\bar{x}\end{aligned}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. and

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} &= n\cdot\frac{2\beta^2+\theta^2}{4\beta^2\theta+\theta^3}\cdot\frac{4\beta\theta^3}{(2\beta^2+\theta^2)^2} - 2\sum_{i=1}^{n} x_i\tan(\beta x_i) \\ &= \frac{4n\beta\theta^3}{(4\beta^2\theta+\theta^3)(2\beta^2+\theta^2)} - 2\sum_{i=1}^{n} x_i\tan(\beta x_i).\end{aligned}$$

So we have

$$\frac{8\beta^4 + 2\beta^2\theta^2 + \theta^4}{(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2)} = \bar{x}. \tag{13}$$

$$\frac{4n\beta\theta^3}{(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2)} = 2\sum_{i=1}^{n} x_i \tan(\beta x_i). \tag{14}$$

The system of Equations (13) and (14) cannot be solved analytically. The MLEs $(\hat{\theta}, \hat{\beta})$ are obtained by solving:

$$(\hat{\theta}, \hat{\beta}) = \arg \max_{\theta>0, \beta>0} \ell(\theta, \beta) \tag{15}$$

via Mathematica and MatLab where $\ell(\theta, \beta)$ is given in Equation (12). For parameters estimation we have used the following algorithm.

Algorithm 1: Maximum Likelihood Estimation via Numerical Optimization

- **Input:** Sample $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$, initial guesses $\theta_0 > 0$, $\beta_0 > 0$, tolerance $\epsilon > 0$, max iterations K.
- **Output:** MLEs $\hat{\theta}$, $\hat{\beta}$.
- **Steps:**
  1. Compute $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.
  2. Define log-likelihood function:

$$\ell(\theta, \beta) = n \ln\left(\frac{4\beta^2\theta + \theta^3}{2\beta^2 + \theta^2}\right) + 2\sum_{i=1}^{n} \ln\cos(\beta x_i) - \theta \sum_{i=1}^{n} x_i.$$

  3. Initialize: $(\theta^{(0)}, \beta^{(0)}) = (\theta_0, \beta_0)$, $k = 0$.
  4. While $k < K$ and $\|\nabla \ell(\theta^{(k)}, \beta^{(k)})\| > \epsilon$:
     (a) Compute gradient:

$$g_\theta = n \cdot \frac{8\beta^4 + 2\beta^2\theta^2 + \theta^4}{(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2)} - n\bar{x}$$

$$g_\beta = \frac{4n\beta\theta^3}{(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2)} - 2\sum_{i=1}^{n} x_i \tan(\beta x_i)$$

(b) Compute Hessian $H^{(k)}$ (or approximate via BFGS update).

(c) Update parameters: $(\theta^{(k+1)}, \beta^{(k+1)}) = (\theta^{(k)}, \beta^{(k)}) - [H^{(k)}]^{-1}\nabla\ell^{(k)}$.

(d) Project onto feasible set: $\theta^{(k+1)} \leftarrow \max(\theta^{(k+1)}, 10^{-6})$, $\beta^{(k+1)} \leftarrow \max(\beta^{(k+1)}, 10^{-6})$.

(e) $k \leftarrow k+1$.

5. Return $(\hat{\theta}, \hat{\beta}) = (\theta^{(k)}, \beta^{(k)})$.

The Hessian matrix elements are derived as:

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \theta^2} &= n\left[\frac{8\beta^4 + 2\beta^2\theta^2 + \theta^4}{(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2)}\right]^2 \\
&\quad - n\left[\frac{(8\beta^2\theta + 3\theta^2)(2\beta^2 + \theta^2) - 2\theta(8\beta^4 + 2\beta^2\theta^2 + \theta^4)}{(4\beta^2\theta + \theta^3)^2(2\beta^2 + \theta^2)}\right] \\
\frac{\partial^2 \ell}{\partial \beta^2} &= n\left[\frac{4\beta\theta^3}{(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2)}\right]^2 \\
&\quad - n\left[\frac{4\theta^3(2\beta^2 + \theta^2)^2 - 8\beta^2\theta^3(2\beta^2 + \theta^2)}{(4\beta^2\theta + \theta^3)^2(2\beta^2 + \theta^2)^2}\right] - 2\sum_{i=1}^{n} x_i^2 \sec^2(\beta x_i) \\
\frac{\partial^2 \ell}{\partial \theta \partial \beta} &= \frac{\partial^2 \ell}{\partial \beta \partial \theta} = n\frac{1}{(4\beta^2\theta + \theta^3)^2(2\beta^2 + \theta^2)^2} \\
&\quad \times \{(8\beta^3 + 4\beta\theta^2)(4\beta^2\theta + \theta^3)(2\beta^2 + \theta^2) \\
&\quad - (8\beta^4 + 2\beta^2\theta^2 + \theta^4)(8\beta\theta(2\beta^2 + \theta^2) + 4\beta\theta^3)\}
\end{aligned}$$

The Fisher information matrix for a single observation is:

$$I(\theta, \beta) = -E\begin{bmatrix} \frac{\partial^2 \ln f}{\partial \theta^2} & \frac{\partial^2 \ln f}{\partial \theta \partial \beta} \\ \frac{\partial^2 \ln f}{\partial \beta \partial \theta} & \frac{\partial^2 \ln f}{\partial \beta^2} \end{bmatrix} \tag{17}$$

For large samples, the MLEs have asymptotic distribution:

$$\sqrt{n}\begin{pmatrix} \hat{\theta} - \theta \\ \hat{\beta} - \beta \end{pmatrix} \xrightarrow{d} N\left(\mathbf{0}, I^{-1}(\theta, \beta)\right)$$

3.3 Ordinary and Weighted Least-Square Estimators

Suppose that $X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(n)}$ are the order statistics of a random sample from any probability distribution. The i^{th} order statistic has the mean and the variance as follows:

$$\mathbf{E}(F(X_{(i)})) = \frac{i}{n+1}, \mathbf{var}(F(X_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \forall i \in 1, 2, 3, ..., n.$$

Ordinary Least-Squares (OLS) and Weighted Least-Squares (WLS) estimators were proposed in 1988 by Swain et al. (1988). We can get OLS estimates for the parameters by minimising the following function with respect to the parameters, as follows:

$$\mathbb{R} = \sum_{i=1}^{n} \left(F(x_i) - \mathcal{F}(i)\right)^2,$$

where $F(x_{(i)})$ represents the theoretical CDF of the observation $x_{(i)}$ of the distribution under study and $\mathcal{F}(i)$ represents the empirical CDF which is usually estimated by $\mathcal{F}(i) = \frac{i}{n+1}$, then we have

$$\mathbb{R} = \sum_{i=1}^{n} \left(F(x_i) - \frac{i}{n+1}\right)^2,$$

where $F(x_i)$ is defined in Equation (4). Similarly we can get WLS estimates for the parameters by minimising the following function with respect to the parameters, as follows:

$$\mathbb{R} = \sum_{i=1}^{n} w_i \left(F(x_i) - \frac{n+1-i}{n+1}\right)^2,$$

where $w_i = \frac{(n+1)^2(n+2)}{i(n+1-i)den}$.

3.4 Method of Moments

Method of Moments(MME) is a slow method comparatively to MLE. MME of $\mathcal{WCED}$ can be obtained by equating the sample moments to the parent or new weighted distributions moments. Equating the first two moments of $\mathcal{WCED}$ corresponding to their sample moments we get,

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{8\beta^4 + 2\beta^2\theta^2 + \theta^4}{\theta\left(2\beta^2 + \theta^2\right)\left(4\beta^2 + \theta^2\right)},$$
$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{2\left(32\beta^6 + 24\beta^4\theta^2 + \theta^6\right)}{\theta^2\left(2\beta^2 + \theta^2\right)\left(4\beta^2 + \theta^2\right)^2}.$$

Since closed form solution are not possible so we have estimated its parameters by minimizing the squared residual w.r.t β and θ of the form

$$\mathbb{R} = \left(\frac{8\beta^4 + 2\beta^2\theta^2 + \theta^4}{\theta\left(2\beta^2 + \theta^2\right)\left(4\beta^2 + \theta^2\right)} - \frac{\sum_{i=1}^{n} x_i}{n}\right)^2 + \left(\frac{2\left(32\beta^6 + 24\beta^4\theta^2 + \theta^6\right)}{\theta^2\left(2\beta^2 + \theta^2\right)\left(4\beta^2 + \theta^2\right)^2} - \frac{\sum_{i=1}^{n} x_i^2}{n}\right)^2.$$

3.5 Cramér-von Mises Estimators

Cramér–von Mises estimators (CVM) were proposed by MacDonald in 1971 (Macdonald, 1971). The idea of this method is to min- imise the following function:

$$\mathcal{CVM} = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_i) - \frac{2i-1}{2n}\right)^2, \tag{16}$$

where $F(x_i)$ is defined in Equation (4). We can determine the $\mathcal{CVM}$ estimates by maximising (16) via computationl package Mathematica 11.0. Since, in all above discussed methods the parameters θ and β are not in closed form so we use numerical approach via Mathematica 11.0. to estimate the parameters.

Algorithm 2: Monte Carlo Simulation Study for Estimator Comparison

- **Input:** True parameters $\theta_{\text{true}}, \beta_{\text{true}}$; sample sizes $\mathcal{S} = \{n_1, n_2, \ldots, n_m\}$; number of replications R; estimation methods $\mathcal{M} = \{\text{MLE}, \text{MME}, \text{LSE}, \text{WLSE}, \text{CVM}\}$.
- **Output:** Bias, MSE, and variance tables for each estimator.
- **Steps:**
  1. For each $n \in \mathcal{S}$:
     (a) Initialize arrays: $\text{Bias}_\theta[\mathcal{M}], \text{MSE}_\theta[\mathcal{M}], \text{Var}_\theta[\mathcal{M}]$, similarly for β.
     (b) For $r = 1$ to R:
        i. Generate sample: $x_1, \ldots, x_n \sim \text{WCED}(\theta_{\text{true}}, \beta_{\text{true}})$.
        ii. For each method $M \in \mathcal{M}$:

A. Obtain estimates $(\hat{\theta}_r, \hat{\beta}_r)$ using method M.
B. Accumulate: $\text{Sum}_\theta[M] \leftarrow \text{Sum}_\theta[M] + \hat{\theta}_r$.
C. Accumulate squares: $\text{SumSq}_\theta[M] \leftarrow \text{SumSq}_\theta[M] + \hat{\theta}_r^2$.
D. Similarly for β.

(c) For each method $M \in \mathcal{M}$:

i. Compute:

$$\text{Bias}_\theta[M] = \frac{\text{Sum}_\theta[M]}{R} - \theta_{\text{true}}$$

$$\text{MSE}_\theta[M] = \frac{\text{SumSq}_\theta[M]}{R} - \left(\frac{\text{Sum}_\theta[M]}{R}\right)^2$$

$$\text{Var}_\theta[M] = \text{MSE}_\theta[M] - (\text{Bias}_\theta[M])^2$$

ii. Similarly for β.

2. Compile results into tables (Tables 1–6 in manuscript).

Interpretation of Simulation Results for Weighted Cosine Exponential Distribution The simulation study evaluates five estimation methods for the $\mathcal{WCED}$ parameters (θ, β) across three different parameter configurations namely

- Configuration-I: $\theta = 0.3596$, $\beta = 0.1529$.
- Configuration-III: $\theta = 1.2965$, $\beta = 0.0593$.
- Configuration-IV: $\theta = 1.0329$, $\beta = 2.3054$.

and varying sample sizes $n = \{15, 25, 50, 150, 350, 500\}$. The estimators compared are via (i) Maximum Likelihood Estimation, (ii) Method of Moments Estimation, (iii) Ordinary Least Squares Estimation, (iv) Weighted Least Squares Estimation, and (v) Cramér–von Mises Estimation. The performance metrics are Bias and Mean Squared Error (MSE). The simulation results presented in Tables 1–3 provide a comprehensive evaluation of five estimation methods for the Weighted Cosine Exponential Distribution ($\mathcal{WCED}$) across different parameter configurations and sample sizes. For Scenario 1 as portrayed in Table-1 and Figures-6 and -9 ($\theta = 0.3596$, $\beta = 0.1529$), the Cramér–von Mises (CVM) estimator demonstrates the most consistent performance for the scale parameter θ, achieving the lowest MSE across all sample sizes, particularly for $n = 15$ (MSE = 0.005044) and $n = 500$ (MSE

= 0.002177). However, for the shape parameter β, estimation proves considerably more challenging, with all methods exhibiting substantial bias and MSE that fails to decrease meaningfully with increasing sample size. This indicates that β is inherently difficult to estimate accurately, even with large samples. In Scenario 2 as portrayed in

Table 1: Summary Tables for Different Sample Sizes ($\theta = 0.3596$, $\beta = 0.1529$)

Method	θ Bias	β Bias	θ MSE	β MSE	θ RMSE	β RMSE	Conv. Rate
Sample Size $n = 15$							
MLE	0.099699	0.339057	0.020296	0.117059	0.142463	0.342139	1.0
MME	0.114018	7.572614	0.027461	73.178977	0.165714	8.554471	1.0
LSE	-0.018320	0.827114	0.008155	1.256030	0.090304	1.120727	1.0
WLSE	-0.037976	0.653315	0.008275	1.124258	0.090967	1.060310	1.0
CVM	-0.035678	0.643845	0.005044	0.792219	0.071020	0.890067	1.0
Sample Size $n = 25$							
MLE	0.098532	0.347100	0.020632	0.120478	0.143640	0.347100	1.0
MME	0.111477	8.457200	0.021705	82.262425	0.147325	9.069864	1.0
LSE	-0.031799	1.066377	0.004002	2.203755	0.063262	1.484505	1.0
WLSE	-0.038575	0.673628	0.005007	0.906992	0.070759	0.952361	1.0
CVM	-0.021786	1.059822	0.004710	2.491709	0.068631	1.578515	1.0
Sample Size $n = 50$							
MLE	0.086113	0.347100	0.011842	0.120478	0.108820	0.347100	1.0
MME	0.081785	6.780354	0.012983	63.712852	0.113941	7.982033	1.0
LSE	-0.007234	1.554487	0.003255	4.092142	0.057049	2.022904	1.0
WLSE	-0.020298	1.322984	0.003804	4.578537	0.061679	2.139752	1.0
CVM	-0.007939	1.685710	0.002977	4.633923	0.054563	2.152655	1.0
Sample Size $n = 150$							
MLE	0.068322	0.347100	0.005977	0.120478	0.077310	0.347100	1.0
MME	0.082333	7.427209	0.009048	69.762024	0.095122	8.352366	1.0
LSE	0.034993	4.593741	0.002279	25.316062	0.047736	5.031507	1.0
WLSE	0.016079	2.813290	0.002316	12.848793	0.048123	3.584521	1.0
CVM	0.038484	4.096537	0.002415	20.243352	0.049144	4.499261	1.0
Sample Size $n = 350$							
MLE	0.068568	0.347100	0.005213	0.120478	0.072199	0.347100	1.0
MME	0.080288	7.494597	0.007236	69.095380	0.085062	8.312363	1.0
LSE	0.045032	5.745554	0.002558	36.238989	0.050573	6.019883	1.0
WLSE	0.044845	6.313446	0.002980	49.375751	0.054590	7.026788	1.0
CVM	0.042225	6.040183	0.002297	40.643038	0.047927	6.375189	1.0
Sample Size $n = 500$							
MLE	0.070179	0.347100	0.005208	0.120478	0.072164	0.347100	1.0
MME	0.073477	8.188957	0.006031	77.416467	0.077657	8.798663	1.0
LSE	0.045444	6.132741	0.002454	41.236003	0.049539	6.421527	1.0
WLSE	0.050787	7.227967	0.003149	59.307995	0.056112	7.701168	1.0
CVM	0.043181	6.784402	0.002177	50.628996	0.046657	7.115406	1.0

Table-2,Figures-7 and -10 ($\theta = 1.2965$, $\beta = 0.0593$), a distinct pattern emerges: Maximum Likelihood Estimation (MLE) shows excellent behavior for θ, with bias decreasing from -0.1727 at $n = 15$ to near-zero (0.0130) at $n = 500$, and corresponding MSE reduction from 0.1490 to 0.0038. For β estimation, the Weighted Least Squares (WLSE) method performs best at larger sample sizes ($n = 500$: bias = -0.0082, MSE = 0.0109), demonstrating superior convergence properties. Notably, the Method of Moments (MME) exhibits persistent negative bias for θ across all sample sizes, suggesting systematic underestimation in this parameter regime. Scenario 3 ($\theta = 1.0329$, $\beta = 2.3054$) reveals particularly challenging estimation conditions, especially for the shape parameter β as portrayed in Table-3 and Figures-8 and -11. All methods show substantial negative bias for β, ranging from -0.4474 (MME at $n = 15$) to -2.3054 (WLSE at $n = 150$), indicating systematic underestimation that does not improve with increased sample size. For θ estimation, MLE performs most consistently with MSE decreasing

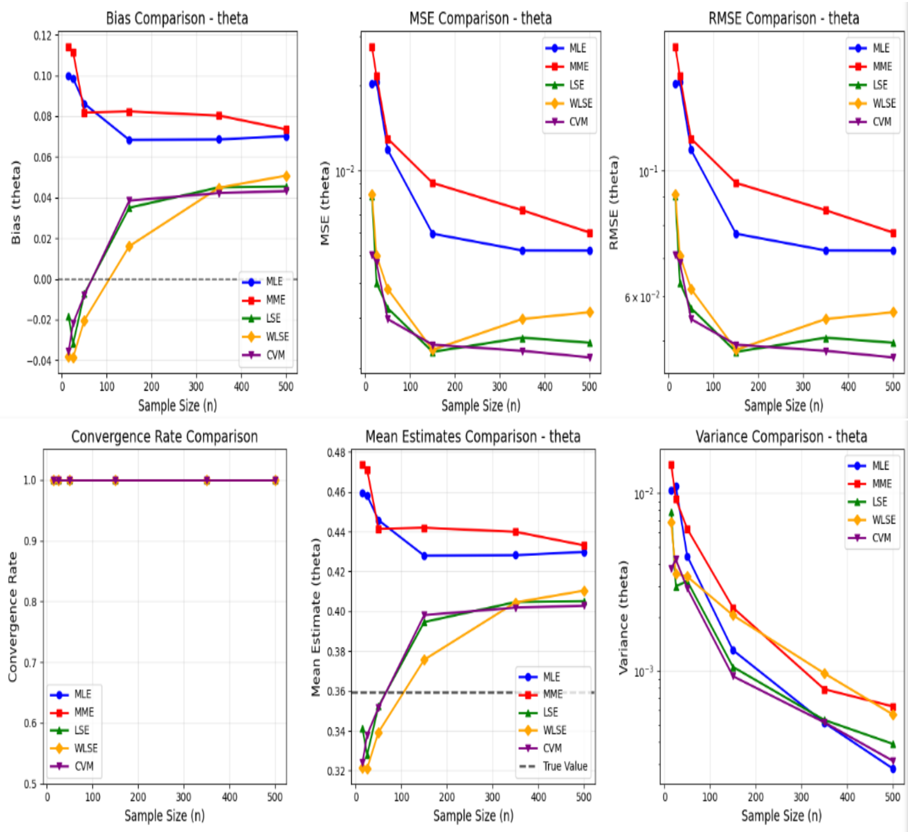

Fig. 6: Bias, MSE, RMSE and Conv.Rate of θ for $\theta = 0.3596$, $\beta = 0.1529$ under different methods of $\mathcal{WCED}$

from 0.2635 ($n = 15$) to 0.0128 ($n = 500$). The convergence rates remain at 100% across all methods and scenarios, confirming numerical stability of the optimization procedures.

Across all scenarios, several general trends are evident: (1) θ estimation consistently improves with sample size for most methods, particularly MLE and CVM; (2) β estimation remains problematic even with large samples, suggesting parameter identifiability challenges; (3) MLE generally provides the best overall performance for θ, while WLSE often excels for β; (4) MME frequently exhibits the highest MSE values, especially for β; and (5) sample sizes of $n \geq 150$ are necessary for reasonably stable parameter estimates. These findings provide valuable guidance for practitioners selecting estimation methods based on their specific parameter regimes and sample size constraints.

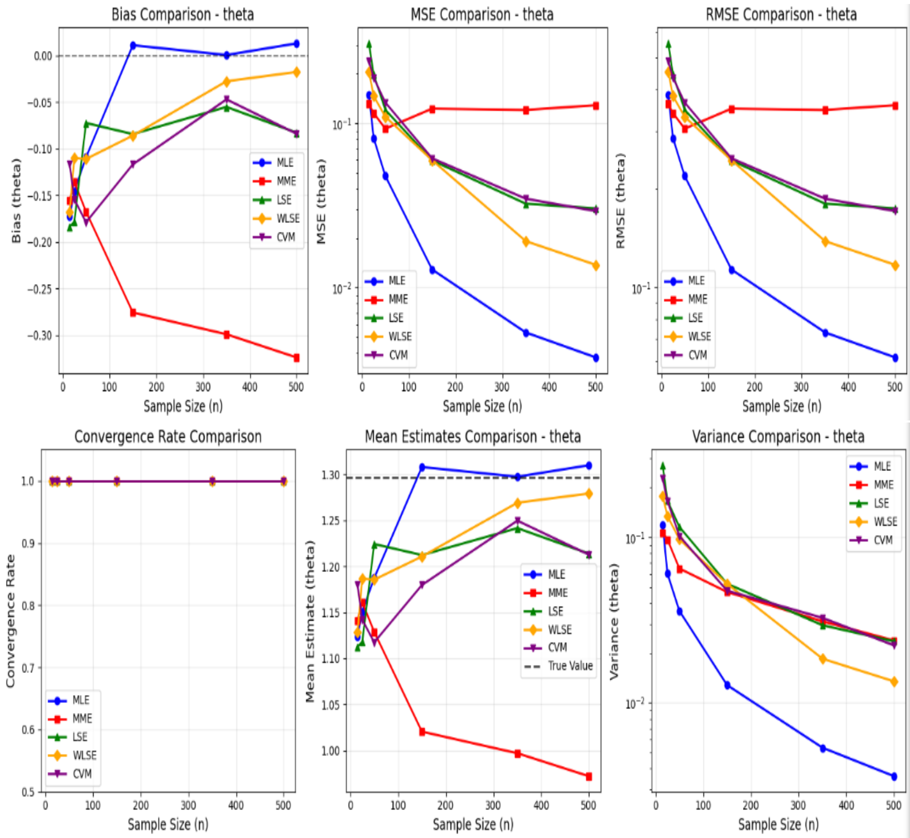

Fig. 7: Bias, MSE, RMSE and Conv.Rate of θ for $\theta = 1.2965$, $\beta = 0.0593$ under different methods of $\mathcal{WCED}$

4 Evaluation Statistics and Data Applications

4.1 Evaluation Tests

This subsection will provides us to apply our new model on the real time data sets and for this purpose we will contrast our models with another models. The unknown parameters of $\mathcal{WCED}$ are estimated by MLE method and log-likelihood of new investigated models are compared with other models. Further we will find out the information criteria of proposed model and compare it with others models in order to exhibit that our model is best life-time distribution. Here we will use the different information criteria methods in which includes AIC, AICC and HQIC. These information criteria are defined as;

$$AIC = 2k - 2(\mathfrak{L}), \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad HQIC = -2\mathfrak{L} + 2kln(ln(n)).$$

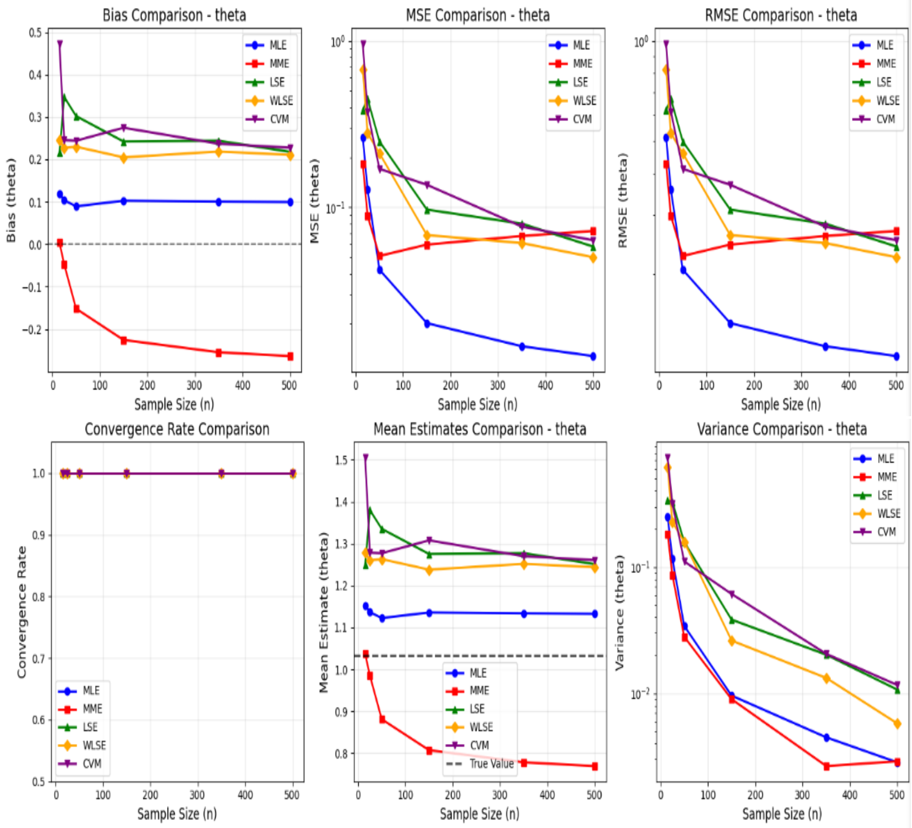

Fig. 8: Bias, MSE, RMSE and Conv.Rate of θ for $\theta = 1.0329$, $\beta = 2.3054$ under different methods of $\mathcal{WCED}$

Algorithm 3: Model Comparison via Information Criteria

- **Input:** Dataset $\mathcal{D} = \{t_1, t_2, \ldots, t_N\}$; candidate models $\mathcal{G} = \{\text{WCED}, \text{NWED}, \text{WWD}, \text{EIWD}, \text{WEIWD}\}$.
- **Output:** Best-fitting model based on information criteria.
- **Steps:**
  1. For each model $G \in \mathcal{G}$:
     (a) Estimate parameters $\hat{\boldsymbol{\eta}}_G$ via MLE (Algorithm 1 for WCED).
     (b) Compute log-likelihood: $\ell_G = \sum_{i=1}^{N} \ln f_G(t_i; \hat{\boldsymbol{\eta}}_G)$.

Table 2: Summary Tables for Different Sample Sizes ($\theta = 1.2965$, $\beta = 0.0593$)

Method	θ Bias	β Bias	θ MSE	β MSE	θ RMSE	β RMSE	Conv. Rate
Sample Size $n = 15$							
MLE	-0.172744	0.307694	0.149008	0.125122	0.386015	0.353727	1.0
MME	-0.155339	0.611030	0.130885	2.530814	0.361780	1.590853	1.0
LSE	-0.183659	0.173645	0.306135	0.118296	0.553295	0.343942	1.0
WLSE	-0.168318	0.160443	0.206230	0.105971	0.454125	0.325531	1.0
CVM	-0.116547	0.280410	0.239216	0.286894	0.489098	0.535625	1.0
Sample Size $n = 25$							
MLE	-0.145582	0.294485	0.081496	0.111806	0.285475	0.334374	1.0
MME	-0.135601	0.289577	0.115355	0.174646	0.339640	0.417907	1.0
LSE	-0.178600	0.153060	0.198092	0.083267	0.445076	0.288560	1.0
WLSE	-0.110066	0.120461	0.146552	0.086775	0.382821	0.294576	1.0
CVM	-0.154276	0.172976	0.189511	0.088357	0.435329	0.297250	1.0
Sample Size $n = 50$							
MLE	-0.109324	0.356301	0.047831	0.141325	0.218703	0.375933	1.0
MME	-0.167564	0.223929	0.092808	0.067688	0.304644	0.260169	1.0
LSE	-0.072395	0.087376	0.120741	0.052507	0.347478	0.229145	1.0
WLSE	-0.111064	0.095002	0.109623	0.043806	0.331093	0.209299	1.0
CVM	-0.179036	0.145717	0.133818	0.069104	0.365811	0.262876	1.0
Sample Size $n = 150$							
MLE	0.011043	0.416183	0.012908	0.179267	0.113613	0.423399	1.0
MME	-0.275788	0.228789	0.122777	0.069436	0.350395	0.263508	1.0
LSE	-0.084345	0.077773	0.059472	0.036353	0.243868	0.190664	1.0
WLSE	-0.085947	0.071178	0.059656	0.036238	0.244246	0.190362	1.0
CVM	-0.116865	0.108908	0.061201	0.046366	0.247389	0.215327	1.0
Sample Size $n = 350$							
MLE	0.000551	0.436847	0.005349	0.191566	0.073135	0.437683	1.0
MME	-0.299235	0.253968	0.120565	0.077063	0.347225	0.277603	1.0
LSE	-0.055191	0.053061	0.032448	0.025788	0.180134	0.160585	1.0
WLSE	-0.027627	0.002409	0.019210	0.013768	0.138599	0.117336	1.0
CVM	-0.047092	0.060952	0.034801	0.026746	0.186550	0.163542	1.0
Sample Size $n = 500$							
MLE	0.013024	0.440700	0.003768	0.194216	0.061385	0.440700	1.0
MME	-0.324164	0.259268	0.128855	0.078309	0.358963	0.279838	1.0
LSE	-0.082880	0.068386	0.030371	0.026090	0.174273	0.161524	1.0
WLSE	-0.017639	-0.008232	0.013770	0.010921	0.117344	0.104506	1.0
CVM	-0.084005	0.083538	0.029246	0.027069	0.171015	0.164526	1.0

(c) Compute information criteria:

$$\text{AIC}_G = 2k_G - 2\ell_G, \quad \text{AICC}_G = \text{AIC}_G + \frac{2k_G(k_G+1)}{N-k_G-1}, \quad \text{HQIC}_G = -2\ell_G + 2k_G \ln\ln N$$

where k_G = number of parameters in model G.

(d) Compute goodness-of-fit statistics:

$$\text{KS}_G = \max_i \left| F_G(t_{(i)}) - \frac{i}{N+1} \right|,$$

$$\text{AD}_G = -N - \sum_{i=1}^{N} \frac{2i-1}{N} [\ln F_G(t_{(i)}) + \ln(1 - F_G(t_{(N+1-i)}))]$$

(e) Compute p-value via appropriate distribution for KS_G.

2. Identify best model: $G^* = \arg\min_{G\in\mathcal{G}} \text{AIC}_G$ (or AICC_G, HQIC_G).

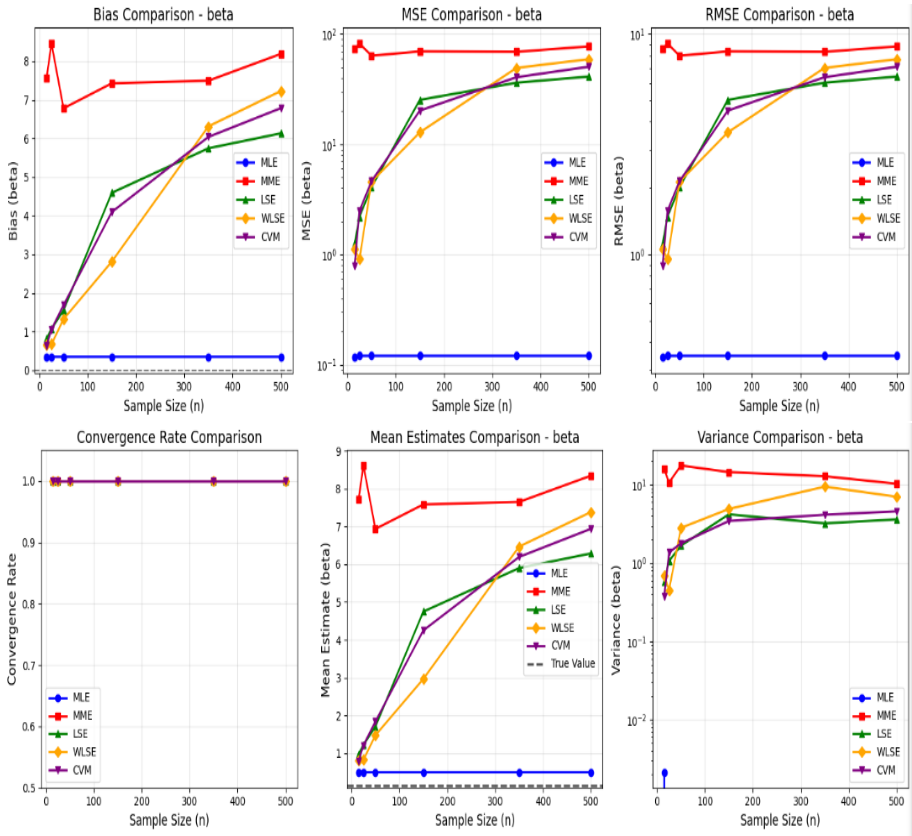

Fig. 9: Bias, MSE, RMSE and Conv.Rate of β for $\theta = 0.3596$, $\beta = 0.1529$ under different methods of $\mathcal{WCED}$

3. Validate with TTT-plot:

$$\text{TTT}_i = \frac{\sum_{j=1}^{i} t_{(j)} + (N-i)t_{(i)}}{\sum_{j=1}^{N} t_{(j)}}, \quad i = 1, \dots, N$$

4. Compare TTT-plot shape with hazard shapes of candidate models.

5. Return G^*, information criteria table, goodness-of-fit statistics.

4.2 Statistical Analysis of Real Life Time Data Sets

Before undertaking formal model comparisons, we first establish the appropriateness of the $\mathcal{WCED}$ for the empirical datasets through graphical alignment of data patterns and model properties. The Total

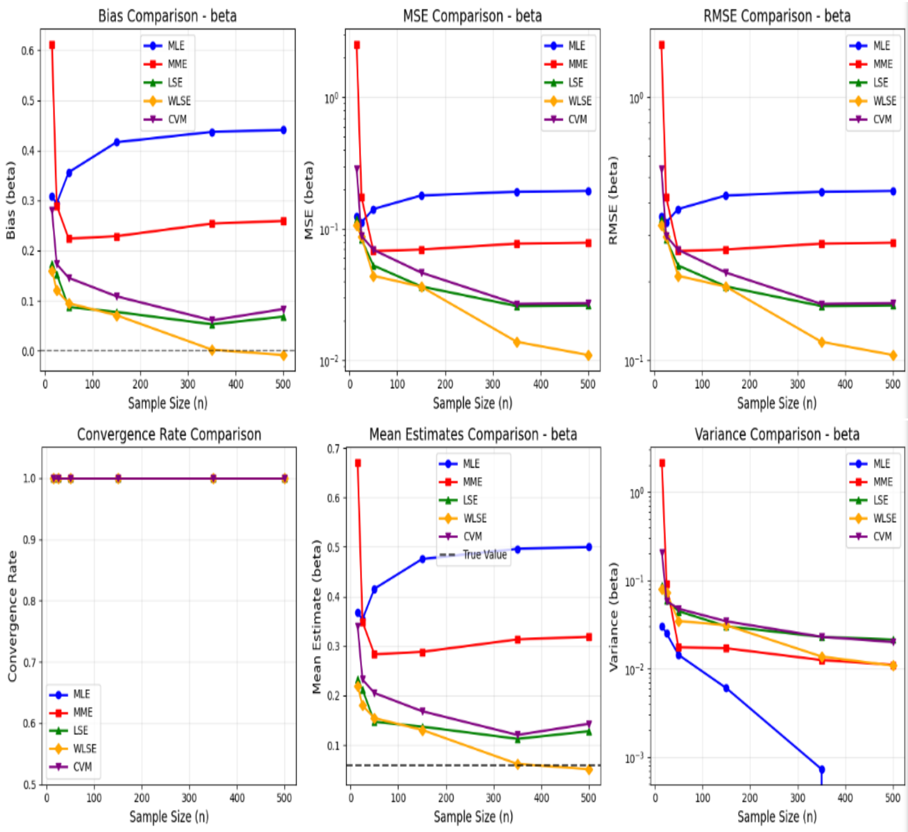

Fig. 10: Bias, MSE, RMSE and Conv.Rate of β for $\theta = 1.2965$, $\beta = 0.0593$ under different methods of $\mathcal{WCED}$

Time on Test (TTT) plot (Aarset, 2009) is employed to diagnose the empirical hazard shape, while the theoretical hazard function of the $\mathcal{WCED}$ (derived in Section 2 and visualized in Figure 4) exhibits flexible non-monotonic behavior, including inverted bathtub forms.

Figure 12 as portrayed in Appendix, presents the TTT plots for the three survival datasets, each displaying a characteristic convex–concave trajectory that mirrors the increasing-then-decreasing hazard pattern producible by the $\mathcal{WCED}$. This visual congruence confirms that the datasets indeed reflect the type of failure process that the $\mathcal{WCED}$ is designed to model, thereby validating its use in the subsequent comparative analysis.

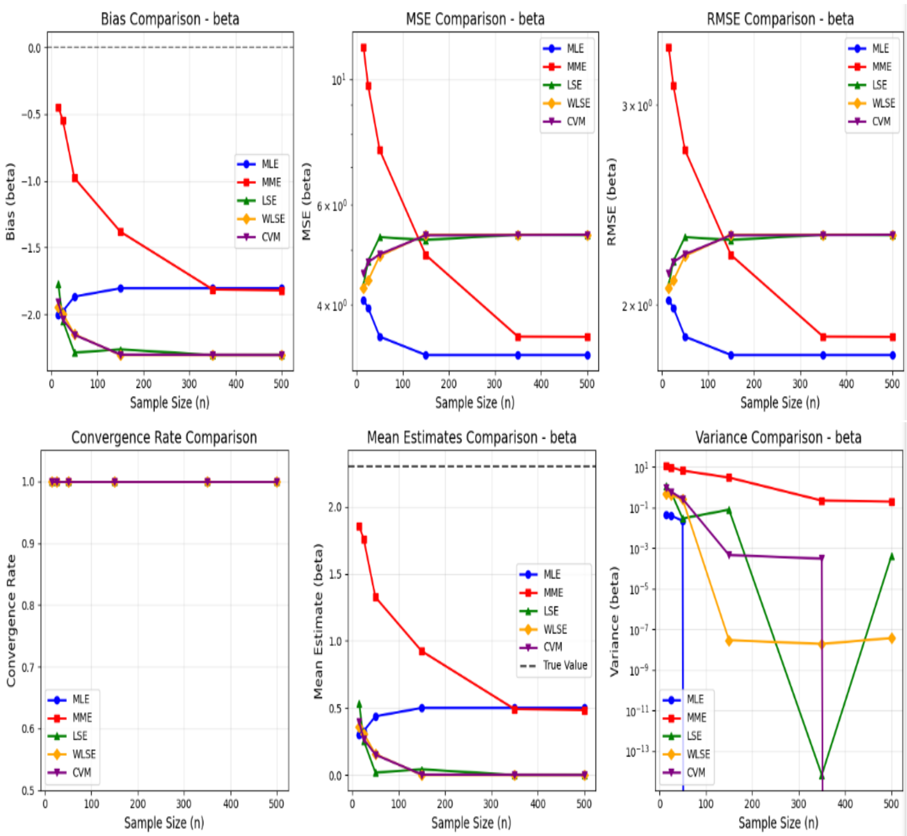

Fig. 11: Bias, MSE, RMSE and Conv.Rate of β for $\theta = 1.0329$, $\beta = 2.3054$ under different methods of $\mathcal{WCED}$

4.2.1 Rationale for Selection of Competing Models

To fairly assess the performance of the proposed $\mathcal{WCED}$, we have selected the following established weighted and lifetime distributions for comparison:

- **New Weighted Exponential Distribution (NWED)**: Selected as a direct competitor because it represents a recent and flexible generalization within the broader family of weighted exponential distributions. It was proposed by Oguntunde (2015). Comparing $\mathcal{WCED}$ to NWED allows us to evaluate whether the introduction of the trigonometric weight function offers a substantive improvement over another modern, parameterized weighted exponential model.

Table 3: Summary Tables for Different Sample Sizes ($\theta = 1.0329$, $\beta = 2.3054$)

Method	θ Bias	β Bias	θ MSE	β MSE	θ RMSE	β RMSE	Conv. Rate
Sample Size $n = 15$							
MLE	0.119333	-2.008838	0.263547	4.077126	0.513369	2.019189	1.0
MME	0.003999	-0.447426	0.181747	11.377114	0.426318	3.372998	1.0
LSE	0.216154	-1.771849	0.384024	4.342358	0.619696	2.083832	1.0
WLSE	0.245292	-1.950256	0.673810	4.281719	0.820859	2.069232	1.0
CVM	0.471972	-1.909754	0.958746	4.544993	0.979156	2.131899	1.0
Sample Size $n = 25$							
MLE	0.103648	-1.976700	0.127649	3.947243	0.357280	1.986767	1.0
MME	-0.047546	-0.549065	0.089059	9.733397	0.298427	3.119839	1.0
LSE	0.346859	-2.051519	0.449629	4.772775	0.670544	2.184668	1.0
WLSE	0.227081	-1.996944	0.279026	4.412085	0.528229	2.100496	1.0
CVM	0.245420	-2.039326	0.377309	4.751266	0.614255	2.179740	1.0
Sample Size $n = 50$							
MLE	0.089518	-1.868093	0.042252	3.511756	0.205552	1.873968	1.0
MME	-0.151651	-0.979411	0.051090	7.497459	0.226031	2.738149	1.0
LSE	0.302058	-2.287538	0.249101	5.261670	0.499100	2.293833	1.0
WLSE	0.229902	-2.150922	0.211416	4.875985	0.459800	2.208163	1.0
CVM	0.244018	-2.154430	0.170456	4.907199	0.412863	2.215220	1.0
Sample Size $n = 150$							
MLE	0.102740	-1.805400	0.020165	3.259469	0.142005	1.805400	1.0
MME	-0.225339	-1.382408	0.059780	4.892599	0.244499	2.211922	1.0
LSE	0.242141	-2.263641	0.096991	5.201735	0.311433	2.280731	1.0
WLSE	0.204863	-2.305325	0.068175	5.314523	0.261104	2.305325	1.0
CVM	0.274666	-2.302462	0.136705	5.301783	0.369736	2.302560	1.0
Sample Size $n = 350$							
MLE	0.100787	-1.805400	0.014648	3.259469	0.121027	1.805400	1.0
MME	-0.254297	-1.814401	0.067327	3.511710	0.259474	1.873956	1.0
LSE	0.244058	-2.305399	0.079813	5.314865	0.282512	2.305399	1.0
WLSE	0.218664	-2.305346	0.061066	5.314620	0.247115	2.305346	1.0
CVM	0.236889	-2.303651	0.076665	5.307108	0.276885	2.303716	1.0
Sample Size $n = 500$							
MLE	0.099744	-1.805400	0.012794	3.259469	0.113111	1.805400	1.0
MME	-0.263234	-1.820463	0.072194	3.507697	0.268689	1.872885	1.0
LSE	0.217891	-2.303377	0.058186	5.305950	0.241218	2.303465	1.0
WLSE	0.210777	-2.305325	0.050209	5.314524	0.224073	2.305325	1.0
CVM	0.227687	-2.305399	0.063487	5.314865	0.251966	2.305399	1.0

- **Weighted Weibull Distribution (WWD)**: The Weibull distribution is a cornerstone of reliability and survival analysis due to its flexible hazard rate. The WWD incorporates a weighting mechanism, making it a natural and strong benchmark. It was examined by Nasiru (2015). This comparison tests if the specific structure of $\mathcal{WCED}$ (with its periodic weight) can outperform a weighted version of one of the most widely used lifetime models.

- **Exponentiated Inverted Weibull Distribution (EIWD) and Weighted Exponentiated Inverted Weibull Distribution (WEIWD)**: These models were developed by Flaih et al. (2012) and Saghir et al. (2017), respectively, and were chosen for two primary reasons. First, they are capable of modeling inverted bathtub-shaped hazard rates, which is a key characteristic we have identified for the $\mathcal{WCED}$ (see Figure 4). This ensures we are comparing against models suited for the same failure pattern. Second, they represent a different philosophical extension—using exponentiation and inversion of a base distribution—providing a contrast to the weighted approach of the $\mathcal{WCED}$. This tests the $\mathcal{WCED}$'s performance against alternative parametric strategies for achieving similar hazard shapes.

The selection is therefore not arbitrary; it is designed to benchmark the $\mathcal{WCED}$ against:

1. Contemporary peers within its own weighted class (NWED),
2. A weighted version of a classical standard (WWD), and
3. Established alternatives known to exhibit the same inverted bathtub hazard shape (EIWD, WEIWD).

This multi-faceted comparison provides a comprehensive evaluation of the $\mathcal{WCED}$'s efficacy across different lineages of lifetime distributions and for the specific hazard profile it aims to model.
Analysis of real data I. The first dataset consists of survival times for 121 breast cancer patients, measured in months, and is reported in Lawless (2011). The proposed $\mathcal{WCED}$ demonstrates superior performance on breast cancer survival data (Dataset I) across all statistical criteria. In Table 1(a), $\mathcal{WCED}$ achieves the lowest AIC (1174.94), AICC (1175.04), and HQIC (1177.10) values, with an AIC difference of 11.02 over the second-best model (NWED), indicating very strong empirical support.

Table-1(a) Log Likelihood ($-\mathfrak{L}$)and Information Criterion for data set I

Distribution	$-\mathfrak{L}$	AIC	AICC	HQIC
$\mathcal{WCED}$	585.47	1174.94	1175.04	1177.10
WWD	791.42	1586.84	1586.94	1589.09
WEIWD	837.91	1679.82	1679.92	1682.09
NWED	590.98	1185.96	1186.06	1188.21
EIWD	635.89	1275.88	1275.88	1277.95

Table 1(b) shows $\mathcal{WCED}$'s exceptional goodness-of-fit with a Kolmogorov-Smirnov p-value of 0.9089, while all competing models yield $p < 0.001$, statistically rejecting their adequacy. The $\mathcal{WCED}$ also produces the most plausible parameter estimates ($\hat{\theta} = 0.0129$, $\hat{\beta} = 0.0057$) and the lowest distance statistics ($A^2 = 0.0442$, $W^2 = 0.0587$), confirming its optimal balance of parsimony and fit quality for modeling survival data with complex hazard patterns.

Table-1(b) Maximum Likelihood Estimators and Goodness Fitness Statistics of real data set I

Distribution	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	A_0^*	W_0^*	KS	$p-value$
$\mathcal{WCED}$	0.0129	0.0057		0.4289	0.0442	0.0587	0.9089
WWD	0.1202	0.3250	0.9828	114.30	20.699	0.8182	0.0000
NWED	0.0898	0.0173		36.064	5.2610	0.3443	0.000
EIWD	6.7445	0.6534		36.333	7.0387	0.4336	0.000
WEIWD	0.5758	0.081	0.0387	35.843	7.7672	0.5652	0.000

Analysis of real data II: The second dataset comprises survival times of bile duct cancer patients following therapy, as documented in Lawless (2011), and is also presented in the Appendix. The analysis of Dataset II (bile duct cancer survival after therapy, $n = 25$) reveals that the $\mathcal{WCED}$ provides the optimal statistical fit among competing models. $\mathcal{WCED}$ achieves the lowest information criteria values (AIC = 347.76, AICC = 348.30, HQIC = 348.43) and yields a statistically acceptable Kolmogorov-Smirnov p-value (0.0322), while all alternative models show poor fit ($p < 0.001$). The parameter estimates ($\hat{\theta} = 0.0014$, $\hat{\beta} = 0.0009$) suggest a distribution with gradual hazard changes appropriate for post-treatment survival patterns see Table-2(a) and Table-2(b). These results establish $\mathcal{WCED}$ as the most suitable model for characterizing bile duct cancer survival data following therapeutic intervention.

Table-2(a) Log Likelihood ($-\mathfrak{L}$)and Information Criterion for data set II

Distribution	$-\mathfrak{L}$	AIC	AICC	HQIC
$\mathcal{WCED}$	171.882	347.76	348.30	348.43
WWD	172.286	348.56	349.08	349.23
WEIWD	195.528	395.10	395.23	395.23
NWED	220.62	349.24	349.76	349.91
EIWD	175.34	354.68	355.20	359.87

Table-2(b) Maximum Likelihood Estimators and Goodness Fitness Statistics of real data set II

Distribution	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	A_0^*	W_0^*	KS	$p-value$
$\mathcal{WCED}$	0.0014	0.0009		2.4823	0.5154	0.2932	0.0322
WWD	0.8956	0.0027	0.9546	NA	7.6186	0.91652	0.000
WEIWD	0.6469	0.081	0.0207	9.7661	2.0903	0.5550	0.000
NWED	0.1406	0.0023		5.8972	0.6845	0.3007	0.0260
EIWD	14.899	0.58		10.791	2.0951	0.4391	0.0001

Analysis of real data III: Third data set shows the the survival times of bile duct cancer patient during radiation-drug therapy see Lawless (2011) and portrayed in Appendix. For Dataset III (bile duct cancer during therapy, $n = 22$), the $\mathcal{WCED}$ demonstrates superior performance across all evaluation metrics see Table-3(a) and Table-3(b). $\mathcal{WCED}$ achieves the lowest information criteria (AIC = 290.01, AICC = 290.11, HQIC = 292.26) and the highest Kolmogorov-Smirnov p-value (0.5152), indicating excellent agreement with the empirical distribution. The competing models either show substantially higher AIC values (WEIWD: 391.76) or statistically insignificant fits ($p < 0.001$). $\mathcal{WCED}$'s parameter estimates ($\hat{\theta} = 0.0026$, $\hat{\beta} = 0.0011$) suggest a survival distribution well-suited for modeling treatment-phase failure times, confirming its utility in active therapeutic monitoring scenarios.

Table-3(a) Log Likelihood ($-\mathfrak{L}$)and Information Criterion for data set III

Distribution	$-\mathfrak{L}$	AIC	AICC	HQIC
$\mathcal{WCED}$	143.005	290.01	290.113	292.26
WWD	143.84	291.68	292.31	292.19
WEIWD	193.88	391.76	392.39	392.27
NWED	143.89	291.78	292.41	296.29
EIWD	153.47	310.94	311.57	311.45

Table-3(b) Maximum Likelihood Estimators and Goodness Fitness Statistics real data set III

Distribution	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	A_0^*	W_0^*	KS	$p-value$
$\mathcal{WCED}$	0.0026	0.0011		0.9187	0.1537	0.1784	0.5152
WWD	1.0042	0.0025	0.4866	128.74	6.4663	0.8984	3.88 $\times 10^{-15}$
WEIWD	0.6430	0.081	0.0180	8.9364	1.9177	0.5910	0.000
NWED	0.7786	0.0022		11.982	2.0969	0.4993	0.0000
EIWD	14.9	0.5890		10.170	2.1740	0.5547	4.87$\times 10^{-6}$

5 Conclusion

This study has introduced and systematically investigated the Weighted Cosine Exponential Distribution ($\mathcal{WCED}$), a novel trigonometric-weighted lifetime model derived through cosine-based transformation of the exponential distribution. The proposed distribution exhibits enhanced flexibility, particularly in its capacity to model inverted bathtub-shaped hazard rates, a characteristic of considerable importance in reliability engineering and survival analysis.

A comprehensive examination of the $\mathcal{WCED}$'s mathematical properties has been conducted, establishing its fundamental characteristics including moments, generating functions, mean deviations, entropy measures, and stress-strength reliability. The model's statistical robustness has been validated through extensive simulation studies evaluating five distinct estimation methodologies: Maximum Likelihood, Method of Moments, Ordinary Least Squares, Weighted Least Squares, and Cramér–von Mises estimation. Results demonstrate that maximum likelihood estimation generally provides the most reliable parameter recovery, particularly for the scale parameter θ, while adequate sample sizes ($n \geq 150$) are recommended for stable estimation of the shape parameter β.

The practical utility of the $\mathcal{WCED}$ has been substantiated through application to three real-world survival datasets from oncology studies. Comparative analyses against established weighted distributions—including NWED, WWD, EIWD, and WEIWD—consistently reveal the WCED's superior goodness-of-fit, as evidenced by lower information criteria values (AIC, AICC, HQIC) and higher p-values in Kolmogorov–Smirnov tests. These empirical results confirm the model's effectiveness in capturing complex survival patterns observed in biomedical data.

The $\mathcal{WCED}$ offers researchers and practitioners a parsimonious yet flexible alternative for modeling lifetime data characterized by non-monotonic hazard patterns. Its trigonometric weighting mechanism introduces periodic behavior that may prove particularly valuable in applications where failure processes exhibit cyclical or multimodal characteristics. Future research directions include extension to censored data scenarios, development of Bayesian estimation frameworks, and investigation of regression formulations incorporating covariates, which would further enhance the model's applicability in clinical trials and reliability studies.

In summary, the $\mathcal{WCED}$ represents a valuable addition to the family of weighted lifetime distributions, combining mathematical tractability with enhanced modeling capability for complex hazard patterns. Its demonstrated performance on real survival data suggests significant potential for applications in reliability engineering, biomedical statistics, and related fields where accurate characterization of failure-time distributions is essential.

References

Aarset, M. V. (2009). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1):106–108.

Al-Babtain, A. A., Elbatal, I., Elgarhy, M., and Almetwally, E. M. (2023). The sine cosine-g family of distributions: properties and applications. *AIMS Mathematics*, 8(2):2958–2984.

Almalki, S. J. (2023). Sine weibull distribution with applications to failure data. *Journal of Statistical Theory and Practice*, 17(1):1–18.

Belzunce, F., Navarro, J., Ruiz, J. M., and Aguila, Y. D. (2004). Some results on residual entropy function. *Metrika*, 59(2):147–161.

Chesneau, C., Bakouch, H. S., Hussain, T., and Para, B. A. (2021). The cosine geometric distribution with count data modeling. *Journal of Applied Statistics*, 48(1):124–137.

Chesneau, C., Kumar, V., Khetan, M., and Arshad (2022). On a modified weighted exponential distribution with applications. *Math. Comput. Appl.*, 27:17.

Dey, S., Alzaatreh, A., Zhang, C., and Kumar, D. (2020). The cosine–weibull distribution with applications to lifetime data. *Journal of Applied Statistics*, 47(13–15):2517–2538.

Flaih, A., Elsalloukh, H., Mendi, E., and Milanova, M. (2012). The exponentiated inverted weibull distribution. *Applied Mathematical Information Sciences*, 6(2):167–171.

Gupta, R. D. and Kundu, D. (2009). A new class of weighted exponential distributions. *Statistics*, 43(6):621–634.

Iqbal, M., Tahir, M. H., and Alshanbari, H. M. (2023). Weighted sine–cosine exponential distribution: properties and applications in reliability. *Reliability Engineering System Safety*, 235:109246.

Khan, M. S., Khosa, S. K., and Asghar, Z. (2022). Cosine exponential distribution: a new flexible lifetime model with applications. *Pakistan Journal of Statistics*, 38(4):473–490.

Kharazmi, O. and Jabbari, L. (2017). A new weighted exponential distribution and its application to the complete and censored data. *Gazi University Journal of Science*, 30(2):219–236.

Lawless, J. F. (2011). *Statistical models and methods for lifetime data*, volume 362. John Wiley Sons.

Macdonald, P. (1971). An estimation procedure for mixtures of distribution. *Journal of the Royal Statistical Society: Series B*, 33:326–329.

Mondal, S., Kundu, D., and Al-Mofleh, H. (2022). The sine modified lindley distribution: properties and estimation with application to reliability data. *Journal of Statistical Computation and Simulation*, 92(8):1719–1739.

Nasiru, S. (2015). Another weighted weibull distribution from azzalini's family. *European Scientific Journal*, 11(9):1857–7881.

Oguntunde, P. E. (2015). On the exponentiated weighted exponential distribution and its basic statistical properties. *Applied Science Reports*, 10(3):160–167.

Patil, G. P. and Rao, C. R. (1977). *Weighted distributions: a survey of their applications*. North Holland Publishing Company.

Rao, C. R. (1985). Weighted distributions arising out of methods of ascertainment: What population does a sample represent? In *A celebration of statistics*, pages 543–569.

Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California.

Saghir, A., Tazeema, S., and Ahmad, I. (2017). The weighted exponentiated inverted weibull distribution: properties and application. *Journal of Informatics and Mathematical Sciences*, 9(1):137–151.

Shannon, C. E. (1951). Prediction and entropy of printed english. *Bell System Technical Journal*, 30(1):50–64.

Swain, J. J., Venkatraman, S., and Wilson, J. R. (1988). Least-squares estimation of distribution functions in johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4):271–297.

Appendix

Table 4: Complete Dataset Listing with References

Dataset	**Survival Times (in months)**
Dataset I: Breast Cancer Patients	0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0 $n = 121$ *Source: Lawless (2011) (p. 564)*
Dataset II: Bile Duct Cancer Patients (After Therapy)	57, 58, 74, 79, 89, 98, 101, 104, 110, 118, 125, 132, 154, 159, 188, 203, 257, 257, 431, 461, 497, 723, 747, 1313, 2636 $n = 25$ *Source: Lawless (2011) (p. 565)*
Dataset III: Bile Duct Cancer Patients (During Radiation–Drug Therapy)	30, 67, 79, 82, 95, 148, 170, 171, 176, 193, 200, 221, 243, 261, 262, 263, 399, 414, 446, 446, 464, 777 $n = 22$ *Source: Lawless (2011), (p. 566)*

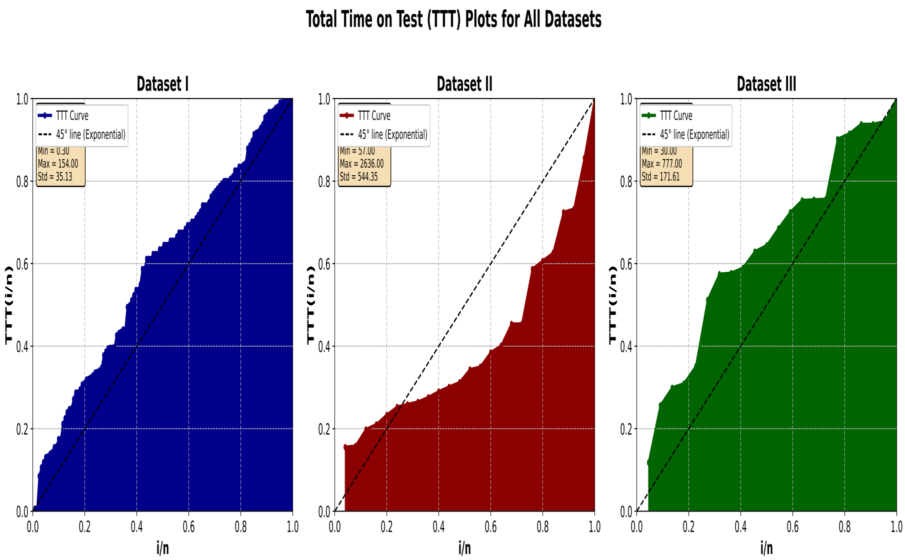

Fig. 12: TTT Plots of Three Data Sets