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Characterization of Probability Distributions through Conditional Expectation of Order Statistics

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ABSTRACT

In this paper, we have given four characterization results of two families of continuous probability distribution. The conditional expectation of function of order statistics is used to establish the required results when the conditioned order statistics may not be adjacent one. Further, its important deduction is also discussed.

Keywords: Characterization, Continuous distribution, Conditional expectation, Order statistics

1 Introduction

Characterization is a condition involving certain properties of a random variable $X = (X_1, X_2, \ldots, X_n)$ which identifies the associated distribution function $F(x)$. The property that uniquely determines $F(x)$ may be based on a function of random variables whose joint distribution is related to that of $X = (X_1, X_2, \ldots, X_n)$. The only method of finding distribution function $F(x)$ exactly, which avoids the subjective choice, is a characterization theorem. A theorem is on a characterization of a distribution function if it concludes that a set of conditions is satisfied by $F(x)$ and only by $F(x)$.
Conditional expectation of order statistics is extensively used in characterizing probability distributions. Khan and Abu-Salih (1989) characterized some

general forms of distributions through conditional expectation of order statistics fixing adjacent order statistics. Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the distributions when the conditioning is not adjacent. Oyang (1995) characterized family of distribution through conditional expectation of a function of order statistics. Wu and Lee (1998) characterized discrete mixture of exponential distribution through conditional expectation of difference of order statistics. Franco and Ruiz (1995, 1997), Wu and Ouyang (1996), Khan *et al*. (2001), Gupta and Ahsanullah (2004), Khan and Athar (2004), Nassar (2006), Bairamov and Ozkal (2007), Khan *et al*. (2009), Lee *et al*. (2011), Khan *et al*. (2012), Khan *et al*. (2013) Athar and Noor (2013), Athar and Akhter (2015), Ahsanullah *et al*. (2015), Khan *et al*. (2016), Athar *et al*. (2021), Bakar *et al*. (2024) amongst other to characterize any particular probability distribution or general class of distributions through conditional expectation of order statistics or function of order statistics.

Let $X = (X_1, X_2, \ldots, X_n)$ be n random variables. If these are arranged in ascending order such that $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, then $X_{r:n}$ is called the r^{th} order statistic, with $X_{1:n}$ and $X_{n:n}$ being the smallest and largest order statistics, respectively (David and Nagaraja, 2003).

The probability density function (pdf) and the distribution function (df) of $X_{r:n}$, the r^{th} order statistic from a sample of size n, is given by (Arnold *et al*.,1992; David and Nagaraja, 2003):

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \tag{1}$$

$$F_{X_{r:n}}(x) = \sum_{j=r}^{n}\binom{n}{j}[F(x)]^{j}[1-F(x)]^{n-j} \tag{2}$$

The conditional pdf of $X_{r:n}$ given $X_{s:n} = y$, $1 \leq r < s \leq n$, is (David and Nagaraja, 2003):

$$\begin{aligned} f_{X_{r:n}|X_{s:n}=y}(x|y) = {} & \frac{(s-1)!}{(r-1)!(s-r-1)!} \\ & \times \frac{[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}}{[F(y)]^{s-1}}, x \leq y \end{aligned} \tag{3}$$

The conditional *pdf* of $X_{s:n}$ given $X_{r:n} = x, 1 \leq r < s \leq n$ is (David and Nagaraja, 2003):

$$f_{X_{s:n}|X_{r:n}=x}(y|x) = \frac{(n-r)!}{(r-1)!(s-r-1)!(n-s)!} \times \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{(n-r)}} f(y), x \leq y \tag{4}$$

In this paper, two general forms of exponentiated distributions are considered:

$$F(x) = [a + be^{-ch(x)}]^d, \quad x \in (\alpha, \beta) \tag{5}$$

$$F(x) = 1 - [a + be^{-ch(x)}]^d, \quad x \in (\alpha, \beta) \tag{6}$$

where $h(x)$ is a monotonic and differentiable function; a, b, c, d are constants; α, β are real numbers with $\alpha < \beta$, $x \in (\alpha, \beta)$; such that $F(x) = [a + be^{(-ch(x))}]^d$ and $F(x) = 1 - [a + be^{(-ch(x))}]^d$ are valid distribution functions. This paper aims to find characterization results of two families of continuous probability distributions defined in (5) and (6) through conditional expectation of a function of order statistics when the conditioning is any order statistics not necessarily the adjacent one. Various well-known distributions arise from above families of distributions by the proper choices of function $h(x)$ and constants a, b, c and d.

2 Characterization Theorems

Theorem 1: Let X be absolutely continuous with *df* $F(x)$ and *pdf* $f(x)$. Suppose $0 < F(x) < 1$ for all $x \in (\alpha, \beta)$. Then for $1 \leq r < s \leq n$,

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = a_{r|s}e^{-ch(y)} + b_{r|s} \tag{7}$$

if and only if

$$F(x) = [a + be^{-ch(x)}]^d, \qquad x \in (\alpha, \beta)$$

where $a_{r|s} = \frac{\Gamma(s)\Gamma(r+\frac{1}{d})}{\Gamma(r)\Gamma(s+\frac{1}{d})}$, $b_{r|s} = \frac{a}{b}\left[a_{r|s} - 1\right]$, and Γ is the gamma function, $\Gamma n = (n-1)!$.

Proof: To prove the necessary part, in view of (3), we have

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = \frac{(s-1)!}{(r-1)!(s-r-1)!} \times \int_{\alpha}^{y} e^{-ch(x)} \left(\frac{F(x)}{F(y)}\right)^{r-1} \left(1 - \frac{F(x)}{F(y)}\right)^{s-r-1} \frac{f(x)}{F(y)} dx \tag{8}$$

Substituting $u = \frac{F(x)}{F(y)}$ and putting $F(x)$ and $F(y)$ from equation (5) in equation (8), we have

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = \frac{(s-1)!}{(r-1)!(s-r-1)!}$$
$$\times \int_0^1 \Big\{ \frac{[a + be^{-ch(y)}]u^{\frac{1}{d}} - a}{b} \Big\} u^{r-1}(1-u)^{s-r-1} du$$

or

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = \frac{(s-1)![a + be^{-ch(y)}]}{b(r-1)!(s-r-1)!} \int_0^1 u^{r+\frac{1}{d}-1}(1-u)^{s-r-1} du$$
$$- \frac{a(s-1)!}{b(r-1)!(s-r-1)!} \int_0^1 u^{r-1}(1-u)^{s-r-1} du \tag{9}$$

After solving equation (9) and rearranging the terms, we have

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = \frac{1}{b}\Big\{ \frac{\Gamma s \Gamma(r + \frac{1}{d})}{\Gamma r \Gamma(s + \frac{1}{d})}[a + be^{-ch(y)}] - a \Big\}.$$

which can be written as

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = \frac{\Gamma s \Gamma(r + \frac{1}{d})}{\Gamma r \Gamma(s + \frac{1}{d})} e^{-ch(y)} - \frac{a}{b}\Big[\frac{\Gamma s \Gamma(r + \frac{1}{d})}{\Gamma r \Gamma(s + \frac{1}{d})} - 1 \Big].$$

or

$$E\big[e^{-ch(X_{r:n})} \mid X_{s:n} = y\big] = a_{r|s} e^{-ch(y)} + b_{r|s}$$

where $a_{r|s} = \frac{\Gamma(s)\Gamma(r+\frac{1}{d})}{\Gamma(r)\Gamma(s+\frac{1}{d})}$, $b_{r|s} = \frac{a}{b}\left[a_{r|s} - 1\right]$
hence the necessary part.

To prove sufficiency part, Let $E[e^{-ch(X_{r:n})} | X_{s:n} = y] = a_{r|s} e^{-ch(y)} + b_{r|s}$ where $a_{r|s}$ and $b_{r|s}$ are given as above.

Now,

$$E\left[e^{-ch(X_{r:n})} \mid X_{s:n}=y\right]=\frac{(s-1)!}{(r-1)!(s-r-1)!}$$
$$\times\int\limits_{\alpha}^{y} e^{-ch(x)}\left(\frac{F(x)}{F(y)}\right)^{r-1}\left(1-\frac{F(x)}{F(y)}\right)^{s-r-1}\frac{f(x)}{F(y)}dx=a_{r|s}e^{-ch(y)}+b_{r|s} \tag{10}$$

which implies

$$\frac{(s-1)!}{(r-1)!(s-r-1)!}\int\limits_{\alpha}^{y} e^{-ch(x)}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}f(x)dx$$
$$=a_{r|s}e^{-ch(y)}[F(y)]^{s-1}+b_{r|s}[F(y)]^{s-1} \tag{11}$$

Differentiating both sides of Differentiating both sides of equation (11) *w.r.t.* y and rearranging the terms, we have

$$\frac{(s-1)!}{(r-1)!(s-r-1)!}f(y)\int\limits_{\alpha}^{y} e^{-ch(x)}\frac{[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}}{[F(y)]^{s-2}}f(x)dx$$
$$=(s-1)a_{r|s}e^{-ch(y)}f(y)+(s-1)b_{r|s}f(y)-ch'(y)e^{-ch(y)}F(y) \tag{12}$$

Now from equation (10) and (12), we have

$$E\left[e^{-ch(X_{r:n})} \mid X_{s-1:n}=y\right]=a_{r|s}e^{-ch(y)}+b_{r|s}-\frac{a_{r|s}}{(s-1)}ch'(y)e^{-ch(y)}\frac{F(y)}{f(y)}. \tag{13}$$

which implies

$$a_{r|s-1}e^{-ch(y)}+b_{r|s-1}=a_{r|s}e^{-ch(y)}+b_{r|s}-\frac{a_{r|s}}{(s-1)}ch'(y)e^{-ch(y)}\frac{F(y)}{f(y)}. \tag{14}$$

After simplifying equation (14), we have

$$\frac{f(y)}{F(y)}=\frac{bcdh'(y)e^{-ch(y)}}{[a+be^{-ch(y)}]}$$

which gives

$$F(y)=[a+be^{-ch(y)}]^{d}$$

and hence the theorem.

Remark 2.1: Putting $a = 0,\ b = 1,\ d = 1$, we get the characterization result for $F(y) = e^{-ch(y)}$ as

$$E[e^{-ch(X_{r:n})} \mid X_{s:n} = y] = \frac{r}{s} e^{-ch(y)}$$

Remark 2.2: Putting $a = 1,\ b = -1,\ d = 1$, we get the characterization result for $F(y) = 1 - e^{-ch(y)}$ as

$$E[e^{-ch(X_{r:n})} \mid X_{s:n} = y] = \frac{r}{s} e^{-ch(y)} + \frac{s-r}{s}$$

Theorem 2: Under the condition given in Theorem 1 and for $1 \leq r < s \leq n$,

$$E[e^{-ch(X_{m:n})} \mid X_{s:n} = y] = a_{m|r} E[e^{-ch(X_{r:n})} \mid X_{s:n} = y] + b_{m|r} \tag{15}$$

if and only if equation (5) holds, where $a_{m|r}$ and $b_{m|r}$ are defined as in Theorem 1.
Proof: In view of Theorem 1:

$$E[e^{-ch(X_{m:n})} \mid X_{s:n} = y] = a_{m|s} e^{-ch(y)} + b_{m|s}$$
$$E[e^{-ch(X_{m:n})} \mid X_{r:n} = y] = a_{m|r} e^{-ch(y)} + b_{m|r}$$

Now,

$$\begin{aligned} a_{m|s} &= \frac{\Gamma(s)\Gamma(m+\frac{1}{d})}{\Gamma(m)\Gamma(s+\frac{1}{d})} = \frac{\Gamma(s)\Gamma(r+\frac{1}{d})}{\Gamma(r)\Gamma(s+\frac{1}{d})} \cdot \frac{\Gamma(r)\Gamma(m+\frac{1}{d})}{\Gamma(m)\Gamma(r+\frac{1}{d})} \\ &= a_{r|s} a_{m|r} \end{aligned} \tag{16}$$

and

$$a_{m|r} b_{r|s} + b_{m|r} = a_{m|r} \frac{a}{b}(a_{r|s} - 1) + \frac{a}{b}(a_{m|r} - 1) = \frac{a}{b}(a_{m|s} - 1) = b_{m|s} \tag{17}$$

Now from Theorem 1, equation (16) and (17), we have

$$\begin{aligned} E\left[e^{-ch(X_{m:n})} \mid X_{s:n} = y\right] &= a_{m|s} e^{-ch(y)} + b_{m|s} \\ &= a_{m|r} a_{r|s} e^{-ch(y)} + a_{m|r} b_{r|s} + b_{m|r} \end{aligned}$$

which may be written as

$$E\left[e^{-ch(X_{m:n})} \mid X_{s:n}=y\right]=a_{m|r}\left[a_{r|s}e^{-ch(y)}+b_{r|s}\right]+b_{m|r}$$

In view of Theorem 1, we have

$$E\left[e^{-ch(X_{m:n})} \mid X_{s:n}=y\right]=a_{m|r}E\left[e^{-ch(X_{r:n})} \mid X_{s:n}=y\right]+b_{m|r}$$

and hence the necessary part.

For the sufficiency part,

$$E\left[e^{-ch(X_{m:n})} \mid X_{s:n}=y\right]=a_{m|r}E\left[e^{-ch(X_{r:n})} \mid X_{s:n}=y\right]+b_{m|r}$$

From equation (3), we have

$$\frac{(s-1)!}{(m-1)!(s-m-1)!}\int_{\alpha}^{y} e^{-ch(x)}\left(\frac{F(x)}{F(y)}\right)^{m-1}\left[1-\frac{F(x)}{F(y)}\right]^{s-m-1}\frac{f(x)}{F(y)}dx$$
$$=a_{m|r}\frac{(s-1)!}{(r-1)!(s-r-1)!}\int_{\alpha}^{y} e^{-ch(x)}\left(\frac{F(x)}{F(y)}\right)^{r-1}\left[1-\frac{F(x)}{F(y)}\right]^{s-r-1}\frac{f(x)}{F(y)}dx$$
$$+b_{m|r}$$

or

$$\frac{(s-1)!}{(m-1)!(s-m-1)!}\int_{\alpha}^{y} e^{-ch(x)}[F(x)]^{m-1}[F(y)-F(x)]^{s-m-1}f(x)dx$$
$$=a_{m|r}\frac{(s-1)!}{(r-1)!(s-r-1)!}v\int_{\alpha}^{y} e^{-ch(x)}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}f(x)dx$$
$$+b_{m|r}[F(x)]^{s-1} \tag{18}$$

Differentiating $(s-r)$ times both sides of equation (18) and rearranging the terms, we have

$$\frac{(r-1)!}{(m-1)!(r-m-1)!}\int_{\alpha}^{y} e^{-ch(x)}\left(\frac{F(x)}{F(y)}\right)^{m-1}\left(1-\frac{F(x)}{F(y)}\right)^{r-m-1}\frac{f(x)}{F(x)}dx$$
$$=a_{m|r}e^{-ch(x)}+b_{m|r} \tag{19}$$

Proceeding on the lines of Theorem 1, we get the required result.

Remark 2.3: Putting $h(x) = \frac{1}{c} \ln h(x)$ we get characterization result for family of $F(x) = [a + bh(x)]^d$ as obtained by Khan *et al*. (2013).

Remark 2.4: Putting $a = 0, b = 1, d = 1$,we get the characterization results for $F(y) = 1 - e^{-ch(y)}$.

Remark 2.5: Putting $a = 1, b = -1, d = 1$,we get the characterization results for $F(y) = e^{-ch(y)}$.

Theorem 3: Let X be an absolutely continuous random variable with *df* $F(x)$ and *pdf* $f(x)$. Suppose $0 < F(x) < 1$ for all $x \in (\alpha, \beta)$. Then for $1 \leq r < s \leq n$,

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = a_{s|r} e^{-ch(x)} + b_{s|r} \tag{20}$$

if and only if

$$F(x) = 1 - [a + be^{-ch(x)}]^d$$

where $a_{s|r} = \frac{\Gamma(n-r+1)\Gamma\left(n-s+\frac{1}{d}+1\right)}{\Gamma(n-s+1)\Gamma\left(n-r+\frac{1}{d}+1\right)}$ and $b_{s|r} = \frac{a}{b}(a_{s|r} - 1)$.
Proof First, we shall prove the necessary part. In view of (4), we have

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = \frac{(n-r)!}{(s-r-1)!(n-s)!}$$
$$\times \int_x^\beta e^{-ch(y)} \left(1 - \frac{\overline{F}(y)}{\overline{F}(x)}\right)^{s-r-1} \left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{n-s} \frac{f(y)}{\overline{F}(x)} dy, \tag{21}$$

where $\bar{F}(x) = 1 - F(x)$.
Substituting $u = \frac{\bar{F}(y)}{\bar{F}(x)}$, we get

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = \frac{(n-r)!}{(s-r-1)!(n-s)!}$$
$$\times \int_0^1 \left\{ \frac{[a + be^{-ch(x)}]u^{\frac{1}{d}} - a}{b} \right\} u^{n-s}(1-u)^{s-r-1} du.$$

which can be rewritten as

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = \frac{(n-r)![a+be^{-ch(x)}]}{b(s-r-1)!(n-s)!}\int_0^1 u^{n-s+\frac{1}{d}}(1-u)^{s-r-1}du$$
$$-\frac{a(n-r)!}{b(s-r-1)!(n-s)!}\int_0^1 u^{n-s}(1-u)^{s-r-1}du.$$

After solving the above equation and rearranging the terms, we have

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = \frac{1}{b}\left\{\frac{\Gamma(n-r+1)\Gamma(n-s+\frac{1}{d}+1)}{\Gamma(n-s+1)\Gamma(n-r+\frac{1}{d}+1)}[a+be^{-ch(x)}] - a\right\}$$

which can be written as

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = \frac{\Gamma(n-r+1)\Gamma(n-s+\frac{1}{d}+1)}{\Gamma(n-s+1)\Gamma(n-r+\frac{1}{d}+1)}e^{-ch(x)}$$
$$+\frac{a}{b}\left\{\frac{\Gamma(n-r+1)\Gamma(n-s+\frac{1}{d}+1)}{\Gamma(n-s+1)\Gamma(n-r+\frac{1}{d}+1)} - 1\right\}$$

or

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = a_{s|r}e^{-ch(x)} + b_{s|r} \tag{22}$$

where $a_{s|r} = \frac{\Gamma(n-r+1)\Gamma(n-s+\frac{1}{d}+1)}{\Gamma(n-s+1)\Gamma(n-r+\frac{1}{d}+1)}$ and $b_{s|r} = \frac{a}{b}[a_{s|r} - 1]$
hence the necessary part.

For the sufficiency part, let

$$E\left[e^{-ch(X_{s:n})} \mid X_{r:n} = x\right] = a_{s|r}e^{-ch(x)} + b_{s|r} = g_{s|r}(x)$$

Now

$$g_{s|r}(x) = a_{s|r}e^{-ch(x)} + b_{s|r} \tag{23}$$

Differentiating both sides of equation (23) *w.r.t.* x, we have

$$g'_{s|r}(x) = -ca_{s|r}h'(x)e^{-ch(x)}.$$

Substituting the values of $a_{s|r}$ and $b_{s|r}$ in above equation, we have

$$g'_{s|r}(x) = -\frac{\Gamma(n-r+1)\Gamma\left(n-s+\frac{1}{d}+1\right)}{\Gamma(n-s+1)\Gamma\left(n-r+\frac{1}{d}+1\right)} ch'(x)e^{-ch(x)}. \tag{24}$$

Furthermore,

$$g_{s|r+1}(x) - g_{s|r}(x) = a_{s|r+1}e^{-ch(x)} + b_{s|r+1} - a_{s|r}e^{-ch(x)} - b_{s|r}.$$

or

$$g_{s|r+1}(x) - g_{s|r}(x) = \frac{1}{bd}\frac{\Gamma(n-r+1)\Gamma\left(n-s+\frac{1}{d}+1\right)}{\Gamma(n-s+1)\Gamma\left(n-r+\frac{1}{d}+1\right)}[a+be^{-ch(x)}]. \tag{25}$$

From equations (24) and (25), we have

$$\frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = -\frac{(n-r)bcdh'(x)e^{-ch(x)}}{[a+be^{-ch(x)}]}$$

or

$$-\frac{1}{(n-r)}\frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{bcdh'(x)e^{-ch(x)}}{[a+be^{-ch(x)}]} \tag{26}$$

Now using the result given by Khan *et al*. (2006), we have

$$\frac{f(x)}{\bar{F}(x)} = -\frac{1}{(n-r)}\frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} \tag{27}$$

From equations (26) and (27), we have

$$\frac{f(x)}{\bar{F}(x)} = \frac{bcdh'(x)e^{-ch(x)}}{[a+be^{-ch(x)}]}$$

which leads to

$$F(x) = 1 - [a+be^{-ch(x)}]^d$$

and hence the theorem.

Remark 2.6: Putting $a = 0, b = 1, d = 1$, we get the characterization results for $F(x) = 1 - e^{-ch(x)}$ as:

$$E[e^{-ch(X_{s:n})} \mid X_{r:n} = x] = \frac{n-s}{n-r}e^{-ch(x)}.$$

Remark 2.7: Putting $a = 1, b = -1, d = 1$, we get the characterization results for $F(x) = e^{-ch(x)}$

$$E[e^{-ch(X_{r:n})} \mid X_{s:n} = x] = \frac{n-s}{n-r} e^{-ch(x)} + \frac{s-r}{n-r}.$$

Theorem 4: Under the condition given in Theorem 3 and for $1 \leq r < s \leq n$

$$E[e^{-ch(X_{s:n})} \mid X_{m:n} = x] = a_{s|r} E[e^{-ch(X_{r:n})} \mid X_{m:n} = x] + b_{s|r}, \tag{28}$$

if and only if equation (6) holds, where $a_{s|r}$ and $b_{s|r}$ are defined as in Theorem 3.

Proof: In view of Theorem 3, we have

$$E[e^{-ch(X_{s:n})} \mid X_{m:n} = x] = a_{s|m} e^{-ch(x)} + b_{s|m}, \tag{29}$$

and

$$E[e^{-ch(X_{r:n})} \mid X_{m:n} = x] = a_{r|m} e^{-ch(x)} + b_{r|m}. \tag{30}$$

Now

$$\begin{aligned} a_{s|m} = \frac{\Gamma(n-m+1)\Gamma\left(n-s+\frac{1}{d}+1\right)}{\Gamma(n-s+1)\Gamma\left(n-m+\frac{1}{d}+1\right)} &= \frac{\Gamma(n-m+1)\Gamma\left(n-r+\frac{1}{d}+1\right)}{\Gamma(n-r+1)\Gamma\left(n-m+\frac{1}{d}+1\right)} \\ \times \frac{\Gamma(n-r+1)\Gamma\left(n-s+\frac{1}{d}+1\right)}{\Gamma(n-s+1)\Gamma\left(n-r+\frac{1}{d}+1\right)} &= a_{r|m} a_{s|r}. \end{aligned} \tag{31}$$

and

$$\begin{aligned} a_{s|r} b_{r|m} + b_{s|r} &= \frac{a}{b}\left[a_{r|m} - 1\right] a_{s|r} + \frac{a}{b}\left[a_{s|r} - 1\right] \\ &= \frac{a}{b}\left[a_{s|r} a_{r|m} - 1\right] = b_{s|m}. \end{aligned} \tag{32}$$

Now from Theorem 3 and equations (31) and (32), we have

$$\begin{aligned} E\left[e^{-ch(X_{s:n})} \mid X_{m:n} = x\right] &= a_{s|m} e^{-ch(x)} + b_{s|m} \\ &= a_{s|r} a_{r|m} e^{-ch(x)} + a_{s|r} b_{r|m} + b_{s|r}, \end{aligned}$$

which can be written as

$$E\left[e^{-ch(X_{s:n})} \mid X_{m:n} = x\right] = a_{s|r}\left[a_{r|m} e^{-ch(x)} + b_{r|m}\right] + b_{s|r}.$$

Now in view of Theorem 3, we have

$$E\left[e^{-ch(X_{s:n})} \mid X_{m:n}=x\right]=a_{s|r}E\left[e^{-ch(X_{r:n})} \mid X_{m:n}=x\right]+b_{s|r}$$

and hence the necessary part.

For the sufficiency part, let

$$E\left[e^{-ch(X_{s:n})} \mid X_{m:n}=x\right]=a_{s|r}E\left[e^{-ch(X_{r:n})} \mid X_{m:n}=x\right]+b_{s|r}$$

Now from equation (4), we have

$$\frac{(n-m)!}{(s-m-1)!(n-s)!}\int_x^\beta e^{-ch(x)}\left[1-\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{s-m-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{n-s}\frac{f(y)}{\overline{F}(x)}dy$$
$$=a_{s|r}\frac{(n-m)!}{(r-m-1)!(n-r)!}\int_x^\beta e^{-ch(x)}\left[1-\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{r-m-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{n-r}\frac{f(y)}{\overline{F}(x)}dy$$
$$+b_{s|r},$$

or

$$\frac{(n-m)!}{(s-m-1)!(n-s)!}\int_x^\beta e^{-ch(x)}[\overline{F}(x)-\overline{F}(y)]^{s-m-1}[\overline{F}(y)]^{n-s}f(y)dy$$
$$=a_{s|r}\frac{(n-m)!}{(r-m-1)!(n-r)!}\int_x^\beta e^{-ch(x)}[\overline{F}(x)-\overline{F}(y)]^{r-m-1}[\overline{F}(y)]^{n-r}f(y)dy$$
$$+b_{s|r}[\overline{F}(x)]^{n-m}. \tag{33}$$

Differentiating $(r-m)$ times both sides of equation (33) and rearranging the terms, we have

$$\frac{(n-r)!}{(s-r-1)!(n-s)!}\int_x^\beta e^{-ch(x)}\left[1-\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{n-s}\frac{f(y)}{\overline{F}(x)}dy$$
$$=a_{s|r}e^{-ch(x)}+b_{s|r}.$$

Now proceeding on the lines of Theorem 3, we get the required result.

Remark 2.8: Putting $h(x)=-\frac{1}{c}\ln h(x)$, we get the characterization result for the family $F(x) = 1 - [a + bh(x)]^d$, as obtained by Khan et al. (2013).

Remark 2.9: Putting $a = 0, b = 1$ and $d = 1$ we get the characterization result for $F(x) = 1 - e^{-ch(x)}$.

Remark 2.10: Putting $a = 1, b = -1$ and $d = 1$ we get the characterization result for $F(x) = e^{-ch(x)}$.

Examples:

1. For $a = 0, b = 1$ and $d = 1$, the the family of distribution $F(x) = [a+be^{-ch(x)}]^d$ reduces to $F(x) = e^{-ch(x)}$ and for $a = 1, b = -1$ and $d = 1$, the family of distribution $\overline{F}(x) = [a + be^{-ch(x)}]^d$ reduces to $F(x) = e^{-ch(x)}$ and A special distribution for this family is given in Table 4.2 in Noor *et al*. (2015).

2. For $a = 1, b = -1$ and $d = 1$, the the family of distribution $F(x) = [a + be^{-ch(x)}]^d$ reduces to $F(x) = 1 - e^{-ch(x)}$ and for $a = 0, b = 1$ and $d = 1$, the family of distribution $\overline{F}(x) = [a+be^{-ch(x)}]^d$ reduces to $F(x) = 1-e^{-ch(x)}$ and A special distribution for this family is given in Table 4.1 in Noor *et al*. (2015).

3. Some other distributions for the family $F(x) = [a + be^{-ch(x)}]^d$ are given in Table 1.

4. Some other distribution for the family of distribution $\bar{F}(x) = [a+be^{-ch(x)}]^d$ is given in Table 2.

Table 1: Example distributions for the family $F(x) = [a + be^{-ch(x)}]^d$

Distribution	$F(x)$	a	b	c	$h(x)$	d
Exponentiated Exponential	$[1-e^{-\lambda x}]^{\alpha}$	1	-1	λ	x	α
Exponentiated Weibull	$\left[1-e^{\{-(\lambda x)^{\gamma}\}}\right]^{\alpha}$	1	-1	λ^{γ}	x^{γ}	α
Exponentiated Inverse Weibul	$\exp\left[-\alpha(\frac{1}{(\lambda x)})^{\gamma}\right]$	0	1	$\frac{1}{\lambda^{\gamma}}$	$\frac{1}{x^{\gamma}}$	α
Exponentiated Log-Logistic	$\left[1-e^{-\frac{x}{\beta}}\right]^{-\alpha}$	1	1	$\frac{1}{\beta}$	x	$-\alpha$
Exponentiated Pareto	$\left[1-(\frac{\beta}{x})^{\gamma}\right]^{\alpha}$	1	-1	$-\gamma$	$\ln(\frac{\beta}{x})$	α
Exponentiated Generalized Uniform	$\left(\frac{x}{\beta}\right)^{\alpha(\gamma+1)}$	0	1	$-(\gamma$ $+1)$	$\ln(\frac{x}{\beta})$	α
Exponentiated Kumaraswamy	$\left[1-(1-x^{\gamma})^{\beta}\right]^{\alpha}$	1	-1	$-\beta$	$\ln(1-x^{\gamma})$	α
Exponentiated Lomax	$\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha}$	1	-1	θ	$\ln(1+\lambda x)$	α
Exponentiated Gamma	$\left[1-e^{-\lambda x}(\lambda x+1)\right]^{\alpha}$	1	-1	λ	$x-\frac{1}{\lambda}\ln(\lambda x+1)$	α
Exponentiated Extreme Value	$\left[1-\exp\{-e^{-\frac{(x-\theta)}{\sigma}}\}\right]^{\alpha}$	1	-1	1	$e^{-(x-\theta)/\sigma}$	α
Gompertz-Verhulst	$\left[1-\rho e^{-\lambda x}\right]^{\alpha}$, $x > \frac{1}{\lambda}\ln\rho > 0$	1	$-\rho$	λ	x	α

3 Conclusion

Conditional expectation of order statistics conditioned on non-adjacent order statistics are used to find characterization results for the families of distributions $F(x) = [a + be^{-ch(x)}]^d$ and $\overline{F}(x) = [a + be^{-ch(x)}]^d$ where $h(x)$ is a monotonic and differentiable function such that $F(x)$ is the distribution function for fixed a, b, c and d. The specific distributions for which the results are true are given by the proper choices of a, b, c, d and $h(x)$.

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Table 2: Example distributions for the family $\bar{F}(x) = [a + be^{-ch(x)}]^d$

Distribution	$F(x)$	a	b	c	$h(x)$	d
Exponentiated Inverse Exponential	$1-\left[1-e^{-(\frac{\theta}{x})}\right]^b$	1	-1	1	$\frac{\theta}{x}$	b
Exponentiated Inverse Rayleigh	$1-\left[1-e^{-(\frac{\theta}{x})^2}\right]^b$	1	-1	1	$(\frac{\theta}{x})^2$	b
Kumaraswamy Inverse Exponential	$1-\left[1-e^{-a(\frac{\theta}{x})}\right]^b$	1	-1	a	$\frac{\theta}{x}$	b
Kumaraswamy Inverse Rayleigh	$1-\left[1-e^{-(\frac{\theta}{x})^2}\right]^b$	1	-1	1	$(\frac{\theta}{x})^2$	b
Exponentiated Frechet	$1-\left[1-e^{-(\frac{\theta}{x})^\beta}\right]^b$	1	-1	1	$(\frac{\theta}{x})^\beta$	b
Kumaraswamy Frechet	$1-\left[1-e^{-a(\frac{\theta}{x})^\beta}\right]^b$	1	-1	a	$(\frac{\theta}{x})^\beta$	b
Kumaraswamy Generalized Inverse Weibull	$1-\left[1-e^{-a\left(\frac{qc^{\frac{1}{\beta}}}{x}\right)^\beta}\right]^b$	1	-1	a	$\left(\frac{qc^{\frac{1}{\beta}}}{x}\right)^\beta$	b
Exponentiated Gumbel Type-2	$1-\left[1-e^{-a\frac{p}{x^\beta}}\right]^b$	1	-1	a	$\frac{p}{x^\beta}$	b

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