DOI: https://doi.org/10.4038/sljas.v26i2.8163

Sri Lankan Journal of Applied Statistics, Vol (26-2)

Neutrosophic Quantile Regression for Robust Median Estimation: Application to Stock Price Fluctuations

M. Zohaib[1,2,*]

[1]Department of Statistics, Government College University, Faisalabad, Pakistan.

[2]Islamabad Model College for Boys Sihala Islamabad, Federal Directorate of Education, Ministry of Federal Education and Professional Training Islamabad, Pakistan

*Corresponding author: zohaibsarwar919@gmail.com

Received: 14th January, 2025/ Revised: 01st June, 2025/ Published: 25th July, 2025

ABSTRACT

In this study, we introduced a novel generalized class of ratio-type estimators for median estimation by employing quantile regression within the framework of neutrosophic statistics is designed to enhance the accuracy and reliability of estimates in the presence of data uncertainty, providing a robust alternative to traditional point estimators. The proposed methodology yields interval-based estimates for the population median, capturing a range of possible values rather than a point estimate, which allows modeling uncertainty and partial truth inherent in complex or imprecise data. This is achieved within the neutrosophic statistical framework, a generalization of classical statistics that explicitly accounts for indeterminacy and inconsistent information, making it well-suited for handling ambiguous data in interdisciplinary applications. We validate the performance of our estimators using real-life stock price data from Samsung Electronics Co. Ltd. (SMSN.IL), sourced from Yahoo Finance (2022), and further substantiate the results through a comprehensive simulation study. Comparisons between traditional and proposed estimators, based on mean squared error, percentage decrease in mean squared error, and a simulation study, demonstrate the superior precision and reliability of our approach.

Keywords: Neutrosophic Statistics; Median Estimation; Quantile Regression; Percentage Decrease Mean Square Error.

1 Introduction

This study proposes a novel estimation framework that integrates quantile regression with neutrosophic statistics to improve the accuracy of population median estimation. By explicitly modeling indeterminacy and data ambiguity, the proposed method offers a robust alternative to traditional estimators, demonstrating enhanced reliability in the presence of extreme values, measurement errors, and incomplete information. This contribution not only strengthens the robustness of median estimation, but also extends its applicability to real-world scenarios fraught with data uncertainty. Neutrosophic quantile regression has several practical applications in finance, it aids in modeling stock prices and predicting median returns under uncertain market conditions. In healthcare, it supports the estimation of median survival times in clinical trials with missing or imprecise data. In the social sciences, it enables analysis of median income levels within populations facing incomplete or uncertain responses. In engineering, it is useful for predicting median failure times in systems exposed to variability and limited information, and in environmental science, it helps to assess median pollutant levels in ecosystems characterized by uncertain or fluctuating data. Additional fields that can benefit from this methodology include economics, marketing, and political science, where data ambiguity is a frequent concern. Smarandache (1998) advocated for the application of neutrosophic statistics in uncertain systems. Irfan et al. (2021) proposed robust median estimators by incorporating auxiliary variables to address the limitations of traditional methods, particularly in the presence of outliers and extreme values. Their work highlighted the importance of robust techniques in improving the reliability of the median estimation under complex data conditions. Masood et al. (2024) extended the methodology by introducing a neutrosophic framework, integrating robust measures to handle uncertainty and imprecision more effectively. Their proposed estimators demonstrate superior performance through simulation studies and practical applications, offering a significant advancement in the field of median estimation. To validate the effectiveness of the proposed estimators, extensive simulation studies are conducted by drawing multiple samples of varying sizes from real-world stock price data. This approach ensures a rigorous evaluation of performance and robustness under realistic data conditions. The present study boasts several novel points, including:

(i) This study pioneers the use of neutrosophic quantile regression to improve the estimation of the population median.

(ii) An extensive family of novel median estimators has been developed to improve the precision and efficiency of estimating the population median.

Table 1: Neutrosophic Functions:

Notation	Description
$M_x^\eta \in \{M_{xl}^\eta, M_{xu}^\eta\}$	Median of supplementary variable for Neutrosophic Population
$M_y^\eta \in \left\{M_{yl}^\eta, M_{yu}^\eta\right\}$	Median of variable of interest for Neutrosophic Population
$\widehat{M}_x^\eta \in \left\{\widehat{M}_{xl}^\eta, \widehat{M}_{xu}^\eta\right\}$	Median of supplementary variable for Neutrosophic Sample
$\widehat{M}_y^\eta \in \left\{\widehat{M}_{yl}^\eta, \widehat{M}_{yu}^\eta\right\}$	Median of variable of interest for Neutrosophic Sample
$t_m^\eta \in \{t_{ml}^\eta, t_{mu}^\eta\}$	Neutrosophic function for Tri-mean
$d_m^\eta \in \{d_{ml}^\eta, d_{mu}^\eta\}$	Deciles-mean for neutrosophic function
$h_l^\eta \in \{h_{ll}^\eta, h_{lu}^\eta\}$	Neutrosophic function for Hodges-Lehman
$m_r^\eta \in \{m_{rl}^\eta, m_{ru}^\eta\}$	Neutrosophic function for Mid-range
$q_r^\eta \in \{q_{rl}^\eta, q_{ru}^\eta\}$	Inter-quartile range for neutrosophic function
$q_d^\eta \in \{q_{dl}^\eta, q_{du}^\eta\}$	Neutrosophic function for Quartile-deviation
$q_a^\eta \in \{q_{al}^\eta, q_{au}^\eta\}$	Quartile-average for neutrosophic function
$\rho_{yx}^\eta \in \left\{\rho_{yxl}^\eta, \rho_{yxu}^\eta\right\}$	Neutrosophic correlation Coefficient
$f_y^\eta(M_y^\eta) \in \{f_y^\eta(M_y^\eta)_l\ , f_y^\eta(M_y^\eta)_u\}$	Probability density function of M_y^η
$f_x^\eta(M_x^\eta) \in \{f_x^\eta(M_x^\eta)_l\ , f_x^\eta(M_x^\eta)_u\}$	Probability density function of M_x^η
$M_{Sy}^\eta \in \left\{M_{Syl}^\eta\ , M_{Syu}^\eta\right\}$	Standard deviation of sample median of variable of interest M_y^η
$M_{Sx}^\eta \in \{M_{Sxl}^\eta\ , M_{Sxu}^\eta\}$	Standard deviation of sample median of supplementary variable M_x^η
$C_{My}^\eta \in \left\{C_{Myl}^\eta\ , C_{Myu}^\eta\right\}$	Coefficient of variation of M_y^η
$C_{Mx}^\eta \in \{C_{Mxl}^\eta\ , C_{Mxu}^\eta\}$	Coefficient of variation of M_x^η
$\zeta^\eta \in \{\zeta_l^\eta\ , \zeta_u^\eta\}$	Finite population correction factor

(iii) The practical applicability of the proposed estimators is demonstrated by analyzing data from the fluctuation of real-world stock prices.

1.1 Notations under Neutrosophic functions:

Suppose a neutrosophic random sample size:$n^\eta \in \{n_l^\eta, n_u^\eta\}$, is selected from a finite population $P = \{P_1, P_2, P_3, \ldots, P_N\}$ of N recognizable units. Assume $y_N(i)$ and $x_N(i), i = (1, 2, .., n)$ represent the unit values ith sampled from the neutrosophic data of the study variable $y_N(i) \in \{y_l, y_u\}$ and the supplementary variable $x_N(i) \in \{x_l, x_u\}$, respectably. Suppose that the neutrosophic range of data is $S^\eta = S_l^\eta + S_u^\eta L^\eta$with $L^\eta \in \{L_l^\eta, L_l^\eta\}$, the neutrosophic variable S^η shows that the neutrosophic samples are selected from populations that have ambiguous and imprecise measurements. We used the notation $S^\eta \in \{a, b\}$, where a shows the lower value and b shows the upper value of neutrosophic data, respectively. Table 1 presents the neutrosophic functions used to derive the properties of the estimators.

The following relative error terms are used to find the mathematical properties of the neutrosophic median estimator. $e_0^\eta = \frac{\widehat{M}_y^\eta - M_y^\eta}{M_y^\eta}, e_1^\eta = \frac{\widehat{M}_x^\eta - M_x^\eta}{M_x^\eta}$ such that $E\left(e_0^\eta\right) = E\left(e_1^\eta\right) = 0$,

$E\left(e_0^{\eta^2}\right) = \zeta^\eta C_{My}^{\eta^2},\ E\left(e_1^{\eta^2}\right) = \zeta^\eta C_{Mx}^{\eta^2}\ E\left(e_0^\eta e_1^\eta\right) = \zeta^\eta C_{Myx}^\eta$,

$C_{My}^\eta = \frac{1}{M_y^\eta f_y^\eta(M_y^\eta)}\ , C_{Mx}^\eta = \frac{1}{M_x^\eta f_x^\eta(M_x^\eta)}\ , C_{Myx}^\eta = \rho_{yx}^\eta C_{My}^\eta C_{Mx}^\eta,\ \zeta^\eta = \frac{1}{4}\left(\frac{1}{n^\eta} - \frac{1}{N^\eta}\right)$,

$M_{Sy}^{\eta} = \sqrt{(1-f^{\eta})(4n^{\eta})^{-1}\left\{f_y^{\eta}(M_y^{\eta})\right\}^{-2}}$, $M_{Sxy}^{\eta} = \rho_{yx}^{\eta} M_{Sy}^{\eta} M_{Sx}^{\eta}$, $M_{Sx}^{\eta} = \sqrt{(1-f^{\eta})(4n^{\eta})^{-1}\left\{f_x^{\eta}(M_x^{\eta})\right\}^{-2}}$.

The structure of the article is organized as follows. Section 2 provides an in-depth review of conventional studies that utilize the neutrosophic approach. Section 3 presents the proposed framework for neutrosophic estimators within the context of quantile regression, where the minimum mean squared error (MSE) for novel family of estimator is derived, using first-degree approximations. Section 4 focuses on the practical application, showcasing stock price fluctuation data.

2 Methodology

This section presents a comprehensive review of traditional studies and methods analyzed through the lens of the neutrosophic statistical approach, highlighting key developments and their relevance to our work.

2.1 Traditional Method 1: S.T. Gross (1980)

Gross (1980) introduced the foundational methodology for estimating the variance of the usual neutrosophic median estimator:

$$\widehat{M}_0^{\eta} = \widehat{M}_y^{\eta} \tag{1}$$

variance of the usual neutrosophic median estimator is expressed as

$$\text{Var}\left(\widehat{M}_0^{\eta}\right) = \zeta^{\eta}\left(M_y^{\eta}\right)^2\left(C_{My}^{\eta}\right)^2 \tag{2}$$

2.2 Traditional Method 2: A.Y.C. Kuk & T. K Mak (1989)

Kuk and Mak (1989) proposed a ratio-type neutrosophic estimator assuming that the median of the auxiliary variable X is known.

$$\widehat{M}_R^{\eta} = \widehat{M}_y^{\eta}\left(\frac{M_x^{\eta}}{\widehat{M}_x^{\eta}}\right) \tag{3}$$

The mean square error expression of $\widehat{M}_R^{\eta}$ the estimator is provided as follows.

$$\text{MSE}\left(\widehat{M}_R^{\eta}\right) \cong \zeta^{\eta}\left(M_y^{\eta}\right)^2\left(C_{My}^{\eta 2} + C_{Mx}^{\eta 2} - 2C_{Myx}^{\eta}\right) \tag{4}$$

The ratio type neutrosophic estimator $\left(\widehat{M}_R^{\eta}\right)$ exhibits improved performance over $\left(\widehat{M}_0^{\eta}\right)$ if $\rho_{yx}^{\eta} > 0.5\frac{C_{Mx}^{\eta}}{C_{My}^{\eta}}$

2.3 Traditional Method 3: S. Bahl & R.K. Tuteja (1991)

Bahl and Tuteja (1991) introduced an exponential ratio-type neutrosophic estimator for estimating the median.

$$\widehat{M}^{\eta}_{ex} = \widehat{M}^{\eta}_{y} \exp\left(\frac{M^{\eta}_{x} - \widehat{M}^{\eta}_{x}}{M^{\eta}_{x} + \widehat{M}^{\eta}_{x}}\right) \tag{5}$$

$$\text{MSE}\left(\widehat{M}^{\eta}_{ex}\right) \cong \zeta^{\eta}\left(M^{\eta}_{y}\right)^{2}\left(C^{\eta 2}_{My} + \frac{1}{4}C^{\eta 2}_{Mx} - C^{\eta}_{Myx}\right) \tag{6}$$

Exponential ratio-type neutrosophic estimator $\widehat{M}^{\eta}_{ex}$ is comparatively efficient than $\widehat{M}^{\eta}_{R}$ and $\widehat{M}^{\eta}_{0}$ if $\rho^{\eta}_{yx} > 0.25\frac{C^{\eta}_{Mx}}{C^{\eta}_{My}}$ and $\rho^{\eta}_{yx} < 0.75\frac{C^{\eta}_{Mx}}{C^{\eta}_{My}}$, respectively.

2.4 Traditional Method 4: Singh (2003)

Singh (2003) developed an unbiased difference-type estimator, which can be reinterpreted in the context of neutrosophic logic.

$$\widehat{M}^{\eta}_{d0} = \widehat{M}^{\eta}_{y} + D^{\eta}_{do}\left(M^{\eta}_{x} - \widehat{M}^{\eta}_{x}\right) \tag{7}$$

Given the optimal value of D^{η}_{do}as $D^{\eta}_{do(opt)} = \frac{M^{\eta}_{y}\rho^{\eta}_{yx}C^{\eta}_{My}}{M^{\eta}_{x}C^{\eta}_{Mx}}$

$$\left(\text{MSE}\left(\widehat{M}^{\eta}_{d0}\right)\right)_{\text{min}} \cong \zeta^{\eta}\left(M^{\eta}_{y}\right)^{2} C^{\eta 2}_{My}\left(1 - \rho^{\eta}_{yx}\right) \tag{8}$$

The MSE of $\widehat{M}^{\eta}_{d0}$ is consistently lower than the MSE of $\widehat{M}^{\eta}_{R}$, $\widehat{M}^{\eta}_{0}$, and $\widehat{M}^{\eta}_{ex}$.

2.5 Traditional Method 5: S. Gupta, J. Shabbir and S. Ahmad (2008)

Gupta et al. (2008) proposed three improved difference-type estimators for estimating the median, which can also be extended within the framework of neutrosophic logic.

$$\widehat{M}^{\eta}_{d1} = \widehat{M}^{\eta}_{y}D^{\eta}_{d1} + D^{\eta}_{d2}\left(M^{\eta}_{x} - \widehat{M}^{\eta}_{x}\right) \tag{9}$$

$$\widehat{M}^{\eta}_{d2} = \left\{\widehat{M}^{\eta}_{y}D^{\eta}_{d3} + D^{\eta}_{d4}\left(M^{\eta}_{x} - \widehat{M}^{\eta}_{x}\right)\right\}\left(\frac{M^{\eta}_{x}}{\widehat{M}^{\eta}_{x}}\right) \tag{10}$$

$$\widehat{M}^{\eta}_{d3} = \left\{\widehat{M}^{\eta}_{y}D^{\eta}_{d5} + D^{\eta}_{d6}\left(M^{\eta}_{x} - \widehat{M}^{\eta}_{x}\right)\right\}\exp\left(\frac{M^{\eta}_{x} - \widehat{M}^{\eta}_{x}}{M^{\eta}_{x} + \widehat{M}^{\eta}_{x}}\right) \tag{11}$$

The terms $D^{\eta}_{d1(opt)}, D^{\eta}_{d2(opt)}, D^{\eta}_{d3(opt)}, D^{\eta}_{d4(opt)}, D^{\eta}_{d5(opt)}$, and $D^{\eta}_{d6(opt)}$ represent unspecified constants.

The minimum mean squared error $(\text{MSE})_{\text{min}}$ of $\widehat{M}^{\eta}_{d1}$ at optimum values

$$D^{\eta}_{d1(opt)} = \frac{1}{1+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)} \quad \text{and} \quad D^{\eta}_{d2(opt)} = \frac{M^{\eta}_{y}}{M^{\eta}_{x}}\left(\frac{\frac{\rho^{\eta}_{yx}C^{\eta}_{My}}{C^{\eta}_{Mx}}}{1+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}\right)$$

is expressed as

$$\text{MSE}\left(\widehat{M}^{\eta}_{d1}\right)_{\text{min}} \cong \frac{\zeta^{\eta}M^{\eta 2}_{y}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}{1+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)} \tag{12}$$

The $(MSE)_{\text{min}}$of $\widehat{M}^{\eta}_{d2}$ at optimum values of $D^{\eta}_{d3(opt)} = \frac{1}{1-\zeta^{\eta}C^{\eta 2}_{Mx}+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}$ $and\ D^{\eta}_{d4(opt)} = \frac{M^{\eta}_{y}}{M^{\eta}_{x}}\left(1+D^{\eta}_{d3(opt)}\frac{\rho^{\eta 2}_{yx}C^{\eta}_{My}}{C^{\eta}_{My}}-2\right)$ is

$$\text{MSE}\left(\widehat{M}^{\eta}_{d2}\right)_{\text{min}} \cong M^{\eta 2}_{y}\left[\frac{\left(1-\zeta^{\eta}C^{\eta 2}_{Mx}\right)\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}{\left(1-\zeta^{\eta}C^{\eta 2}_{Mx}\right)+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}\right] \tag{13}$$

The $(MSE)_{\text{min}}$ of $\widehat{M}^{\eta}_{d3}$ at optimum values of $D^{\eta}_{d5(opt)} = \frac{1-\left(\frac{\zeta^{\eta}C^{\eta 2}_{Mx}}{8}\right)}{1+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}$ $and\ D^{\eta}_{d6(opt)} = \frac{M^{\eta}_{y}}{M^{\eta}_{x}}\left\{\frac{1}{2}+D^{\eta}_{d5(opt)}\left(\frac{\rho^{\eta}_{yx}C^{\eta}_{My}}{C^{\eta}_{My}}-1\right)\right\}$ is given by

$$\text{MSE}\left(\widehat{M}^{\eta}_{d3}\right)_{\text{min}} \cong M^{\eta 2}_{y}\left[\frac{\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)-\frac{1}{64}\zeta^{2\eta}C^{\eta 4}_{Mx}-\frac{1}{4}\zeta^{2\eta}C^{\eta 2}_{My}C^{\eta 2}_{Mx}\left(1-\rho^{\eta 2}_{yx}\right)}{\left(1-\zeta^{\eta}C^{\eta 2}_{Mx}\right)+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}\right] \tag{14}$$

2.6 Traditional Method 6: J. Shabbir & S. Gupta (2017)

Shabbir & Gupta (2017) proposed a generalized difference-type estimator for median estimation, incorporating a neutrosophic approach to enhance its applicability in dealing with uncertainty and indeterminacy.

$$\widehat{M}^{\eta}_{d4} = \left[D^{\eta}_{d7}M^{\eta}_{y}+D^{\eta}_{d8}\left(M^{\eta}_{x}-\widehat{M}^{\eta}_{x}\right)\right]\left[\exp\left(\frac{M^{\eta}_{x}}{\widehat{M}^{\eta}_{x}}-1\right)\right] \tag{15}$$

Table 2: Irfan et al. (2021) Suggested families of estimators

$\widehat{\mathbf{M}}^{\eta}_{\mathbf{Ti}}$	Families of Estimators under Median Estimation Using Neutrosophic Approach	$\square^{\eta}_{\mathbf{a}}$	$\square^{\eta}_{\mathbf{b}}$
1	$\widehat{\mathbf{M}}^{\eta\circleddash}_{\mathbf{Ti}} = M^{\eta}_{y}\left[\left\{D^{\eta}_{d9}\left(\frac{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)\exp\left(\frac{M^{\eta}_{x}-\widehat{M}^{\eta}_{x}}{M^{\eta}_{x}+\widehat{M}^{\eta}_{x}}\right)\right\}+\left\{D^{\eta}_{d10}\left(\frac{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)^{2}\right\}\right]$	1	2
2	$\widehat{\mathbf{M}}^{\eta\oplus}_{\mathbf{Ti}} = M^{\eta}_{y}\left[\left\{D^{\eta}_{d9}\left(\frac{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)\exp\left(\frac{M^{\eta}_{x}-\widehat{M}^{\eta}_{x}}{M^{\eta}_{x}+\widehat{M}^{\eta}_{x}}\right)\right\}+\left\{D^{\eta}_{d10}\left(\frac{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)\right\}\right]$	-1	-1
3	$\widehat{\mathbf{M}}^{\eta\otimes}_{\mathbf{Ti}} = M^{\eta}_{y}\left[\left\{D^{\eta}_{d9}\left(\frac{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)\exp\left(\frac{M^{\eta}_{x}-\widehat{M}^{\eta}_{x}}{M^{\eta}_{x}+\widehat{M}^{\eta}_{x}}\right)\right\}+\left\{D^{\eta}_{d10}\left(\frac{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}M^{\eta}_{x}+H^{\eta}_{l}}\right)^{2}\right\}\right]$	-1	-2
4	$\widehat{\mathbf{M}}^{\eta\circledast}_{\mathbf{Ti}} = M^{\eta}_{y}\left[\left\{D^{\eta}_{d9}\left(\frac{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)^{2}\exp\left(\frac{M^{\eta}_{x}-\widehat{M}^{\eta}_{x}}{M^{\eta}_{x}+\widehat{M}^{\eta}_{x}}\right)\right\}+\left\{D^{\eta}_{d10}\left(\frac{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)^{2}\right\}\right]$	2	2
5	$\widehat{\mathbf{M}}^{\eta\circledcirc}_{\mathbf{Ti}} = M^{\eta}_{y}\left[\left\{D^{\eta}_{d9}\left(\frac{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)^{2}\exp\left(\frac{M^{\eta}_{x}-\widehat{M}^{\eta}_{x}}{M^{\eta}_{x}+\widehat{M}^{\eta}_{x}}\right)\right\}+\left\{D^{\eta}_{d10}\left(\frac{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)\right\}\right]$	-2	-1

The $(MSE)_{\min}$ of $\widehat{M}^{\eta}_{d4}$ at optimum values of $D^{\eta}_{d7(opt)} = \frac{1-\frac{1}{2}\zeta^{\eta 2}C^{\eta 2}_{Mx}}{1+\zeta^{\eta 2}C^{\eta 2}_{My}(1-\rho^{\eta 2}_{yx})}$ $and\ D^{\eta}_{d8(opt)} = \frac{M^{\eta}_{y}}{M^{\eta}_{x}}\left[1+D^{\eta}_{d7(opt)}\left\{\frac{\rho^{\eta}_{yx}C^{\eta}_{My}}{C^{\eta}_{My}}-2\right\}\right]$ is

$$\text{MSE}\left(\widehat{M}^{\eta}_{d4}\right)_{\min} \cong M^{\eta 2}_{y}\left[\frac{\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)-\frac{1}{4}\zeta^{2\eta}C^{\eta 4}_{Mx}-\zeta^{2\eta}C^{\eta 2}_{My}C^{\eta 2}_{Mx}\left(1-\rho^{\eta 2}_{yx}\right)}{1+\zeta^{\eta}C^{\eta 2}_{My}\left(1-\rho^{\eta 2}_{yx}\right)}\right] \tag{16}$$

2.7 Traditional Method 7: Irfan et al. (2021) and Masood et al. (2024)

Irfan et al. (2021) developed generalized estimator for median estimation based on robust measures of an auxiliary variable. Masood et al. (2024) extended this concept by converting the estimator into a neutrosophic approach, enabling better handling of uncertainty and imprecision in data. A generalized estimator for estimating the population median using neutrosophic approach can be defined as a linear combination of various robust measures associated with the auxiliary variable and the study variable.

$$\begin{aligned}\widehat{M}^{\eta}_{Ti} = M^{\eta}_{y}\Bigg[\, & D^{\eta}_{d9}\left(\frac{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)^{\Theta^{\eta}_{a}}\exp\left(\frac{M^{\eta}_{x}-\widehat{M}^{\eta}_{x}}{M^{\eta}_{x}+\widehat{M}^{\eta}_{x}}\right) \\ & + D^{\eta}_{d10}\left(\frac{\theta^{\eta}_{i}M^{\eta}_{x}+\vartheta^{\eta}_{i}}{\theta^{\eta}_{i}\widehat{M}^{\eta}_{x}+\vartheta^{\eta}_{i}}\right)^{\Theta^{\eta}_{b}}\,\Bigg]\end{aligned} \tag{17}$$

Given that $D^{\eta}_{d9}\ and\ D^{\eta}_{d10}$ are constants, and $\Theta^{\eta}_{a}\ and\ \Theta^{\eta}_{a}$ can takes on values such as 1, -1, 2, -2 these elements are used in the development of new estimators. By substituting different values of $\Theta^{\eta}_{a}\ and\ \Theta^{\eta}_{b}$ in Eq. (17), the resulting families of estimators are presented in Table.2. Additionally, θ^{η}_{i} and ϑ^{η}_{i} could be constant values or functions that depend on known non-conventional / robust measures associated with the variable X.

By substituting D_{d9}^{η} *and* D_{d10}^{η} for optimal values, we obtain $(MSE)_{Min}$ as follows.

$$\text{MSE}\left(\widehat{M}_{Ti}^{\eta}\right)_{\min} \cong M_y^{\eta 2}\left[1-\frac{\varphi_b^{\eta}\varphi_d^{\eta 2}+\varphi_a^{\eta}\varphi_e^{\eta 2}-2\varphi_c^{\eta}\varphi_d^{\eta}\varphi_e^{\eta}}{\varphi_a^{\eta}\varphi_b^{\eta}-\left(\varphi_c^{\eta}\right)^2}\right] \tag{18}$$

Where $\xi^{\eta}=\frac{\theta_i^{\eta}M_x^{\eta}}{(\theta_i^{\eta}M_x^{\eta}+\vartheta_i^{\eta})}$,

$$\varphi_a^{\eta}=\Bigg[1+\zeta^{\eta}\Big\{C_{My}^{\eta 2}+C_{Mx}^{\eta 2}\big(1+2\Theta_a^{\eta 2}\xi^{\eta 2}-\Theta_a^{\eta}\xi^{\eta 2}-2\Theta_a^{\eta}\xi^{\eta}\big)$$
$$-2\rho_{yx}^{\eta}C_{My}^{\eta}C_{Mx}^{\eta}\left(1-2\Theta_a^{\eta}\xi^{\eta}\right)\Big\}\Bigg]$$

$$\varphi_b^{\eta}=\left[1+\zeta^{\eta}\left\{C_{My}^{\eta 2}+\xi^{\eta 2}C_{Mx}^{\eta 2}\left(2\Theta_b^{\eta 2}+\Theta_b^{\eta}\right)-4\Theta_b^{\eta}\xi^{\eta}\rho_{yx}^{\eta}C_{Mx}^{\eta}C_{My}^{\eta}\right\}\right]$$

$$\varphi_c^{\eta}=\Bigg[1+\zeta^{\eta}\Big\{C_{My}^{\eta 2}-\rho_{yx}^{\eta}C_{My}^{\eta}C_{Mx}^{\eta}\left(2\xi^{\eta}(\Theta_b^{\eta}-\Theta_a^{\eta})+1\right)$$
$$-C_{Mx}^{\eta 2}\Big(\Theta_a^{\eta}\Theta_b^{\eta}\xi^{\eta 2}-\frac{3}{8}+\frac{\Theta_a^{\eta}\xi^{\eta}}{2}-\frac{\Theta_b^{\eta}\xi^{\eta}}{2}$$
$$-\frac{\Theta_b^{\eta}(\Theta_b^{\eta}+1)\xi^{\eta 2}}{2}-\frac{1}{2}\Theta_a^{\eta}(\Theta_a^{\eta}-1)\xi^{\eta 2}\Big)\Big\}\Bigg]$$

$$\varphi_d^{\eta}=\left\{1+\zeta^{\eta}\rho_{yx}^{\eta}C_{My}^{\eta}C_{Mx}^{\eta}\left(\Theta_a^{\eta}\xi^{\eta}-\tfrac{1}{2}\right)+\zeta^{\eta}\tfrac{C_{Mx}^{\eta 2}}{2}\left(\tfrac{3}{4}-\Theta_a^{\eta}\xi^{\eta}+\Theta_a^{\eta}\left(\Theta_a^{\eta}-1\right)\xi^{\eta 2}\right)\right\}$$
$$\varphi_e^{\eta}=\left\{1+\tfrac{1}{2}\Theta_b^{\eta}\left(\Theta_b^{\eta}+1\right)\xi^{\eta 2}\zeta^{\eta}C_{Mx}^{\eta 2}-\Theta_b^{\eta}\xi^{\eta}\zeta^{\eta}\rho_{yx}^{\eta}C_{My}^{\eta}C_{Mx}^{\eta}\right\}$$

Partially differentiating with respect to D_{d9}^{η} *and* D_{d10}^{η} and equating them to zero, we obtain the optimal values of D_{d9}^{η} *and* D_{d10}^{η} as follows.

$$D_{d9(opt)}^{\eta}=\frac{\left(\varphi_b^{\eta}\varphi_d^{\eta}-\varphi_c^{\eta}\varphi_e^{\eta}\right)}{\left(\varphi_a^{\eta}\varphi_b^{\eta}-\varphi_c^{\eta 2}\right)}\qquad ,D_{d10(opt)}^{\eta}=\frac{\left(\varphi_a^{\eta}\varphi_e^{\eta}-\varphi_c^{\eta}\varphi_d^{\eta}\right)}{\left(\varphi_a^{\eta}\varphi_b^{\eta}-\varphi_c^{\eta 2}\right)}$$

Remark 2.1.
By substituting appropriate constants or known non-conventional population parameters of the auxiliary variable for θ_i^{η} and ϑ_i^{η} in Eq. (17), various optimal estimators can be derived. Table 2 presents various combinations of conventional and non-conventional measures for the class of $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\ominus}$. Placing the same values of θ_i^{η} and ϑ_i^{η} in $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\oplus}$, $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\otimes}$, $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\circledast}$, and $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\circledcirc}$, we obtain a number of

Table 3: Family Members of $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\ominus}$

$\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\ominus}$	Estimator	θ_i^η	ϑ_i^η
1	$\widehat{\mathbf{M}}_{\mathbf{T1}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{q_a^\eta\widehat{M}_x^\eta+t_m^\eta}{q_a^\eta M_x^\eta+t_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{q_a^\eta M_x^\eta+t_m^\eta}{q_a^\eta\widehat{M}_x^\eta+t_m^\eta}\right)^2\right\}\right]$	q_a^η	t_m^η
2	$\widehat{\mathbf{M}}_{\mathbf{T2}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{m_r^\eta\widehat{M}_x^\eta+t_m^\eta}{m_r^\eta M_x^\eta+t_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{m_r^\eta M_x^\eta+t_m^\eta}{m_r^\eta\widehat{M}_x^\eta+t_m^\eta}\right)^2\right\}\right]$	m_r^η	t_m^η
3	$\widehat{\mathbf{M}}_{\mathbf{T3}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{h_l^\eta\widehat{M}_x^\eta+t_m^\eta}{h_l^\eta M_x^\eta+t_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{h_l^\eta M_x^\eta+t_m^\eta}{h_l^\eta\widehat{M}_x^\eta+t_m^\eta}\right)^2\right\}\right]$	h_l^η	t_m^η
4	$\widehat{\mathbf{M}}_{\mathbf{T4}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{t_m^\eta\widehat{M}_x^\eta+h_l^\eta}{t_m^\eta M_x^\eta+h_l^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{t_m^\eta M_x^\eta+h_l^\eta}{t_m^\eta\widehat{M}_x^\eta+h_l^\eta}\right)^2\right\}\right]$	t_m^η	h_l^η
5	$\widehat{\mathbf{M}}_{\mathbf{T5}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{q_a^\eta\widehat{M}_x^\eta+h_l^\eta}{q_a^\eta M_x^\eta+h_l^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{q_a^\eta M_x^\eta+h_l^\eta}{q_a^\eta\widehat{M}_x^\eta+h_l^\eta}\right)^2\right\}\right]$	q_a^η	h_l^η
6	$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{h_l^\eta\widehat{M}_x^\eta+q_r^\eta}{h_l^\eta M_x^\eta+q_r^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{h_l^\eta M_x^\eta+q_r^\eta}{h_l^\eta\widehat{M}_x^\eta+q_r^\eta}\right)^2\right\}\right]$	h_l^η	q_r^η
7	$\widehat{\mathbf{M}}_{\mathbf{T7}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{m_r^\eta\widehat{M}_x^\eta+h_l^\eta}{m_r^\eta M_x^\eta+h_l^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{m_r^\eta M_x^\eta+h_l^\eta}{m_r^\eta\widehat{M}_x^\eta+h_l^\eta}\right)^2\right\}\right]$	m_r^η	h_l^η
8	$\widehat{\mathbf{M}}_{\mathbf{T8}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{h_l^\eta\widehat{M}_x^\eta+d_m^\eta}{h_l^\eta M_x^\eta+d_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{h_l^\eta M_x^\eta+d_m^\eta}{h_l^\eta\widehat{M}_x^\eta+d_m^\eta}\right)^2\right\}\right]$	h_l^η	d_m^η
9	$\widehat{\mathbf{M}}_{\mathbf{T9}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{d_m^\eta\widehat{M}_x^\eta+t_m^\eta}{d_m^\eta M_x^\eta+t_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{d_m^\eta M_x^\eta+t_m^\eta}{d_m^\eta\widehat{M}_x^\eta+t_m^\eta}\right)^2\right\}\right]$	d_m^η	t_m^η
10	$\widehat{\mathbf{M}}_{\mathbf{T10}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{m_r^\eta\widehat{M}_x^\eta+d_m^\eta}{m_r^\eta M_x^\eta+d_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{m_r^\eta M_x^\eta+d_m^\eta}{m_r^\eta\widehat{M}_x^\eta+d_m^\eta}\right)^2\right\}\right]$	m_r^η	d_m^η
11	$\widehat{\mathbf{M}}_{\mathbf{T11}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{q_d^\eta\widehat{M}_x^\eta+d_m^\eta}{q_d^\eta M_x^\eta+d_m^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{q_d^\eta M_x^\eta+d_m^\eta}{q_d^\eta\widehat{M}_x^\eta+d_m^\eta}\right)^2\right\}\right]$	q_d^η	d_m^η
12	$\widehat{\mathbf{M}}_{\mathbf{T12}}^{\eta\ominus}=M_y^\eta\left[\left\{D_{d9}^\eta\left(\frac{d_m^\eta\widehat{M}_x^\eta+q_a^\eta}{d_m^\eta M_x^\eta+q_a^\eta}\right)\exp\left(\frac{M_x^\eta-\widehat{M}_x^\eta}{M_x^\eta+\widehat{M}_x^\eta}\right)\right\}+\left\{D_{d10}^\eta\left(\frac{d_m^\eta M_x^\eta+q_a^\eta}{d_m^\eta\widehat{M}_x^\eta+q_a^\eta}\right)^2\right\}\right]$	d_m^η	q_a^η

optimal estimators.

All suggested estimators $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\ominus}$, $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\oplus}$,$\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\otimes}$, $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\circledast}$and $\widehat{\mathbf{M}}_{\mathbf{Ti}}^{\eta\circledcirc}$ having minimum MSE compared to all reviewing estimators and a deep insight of family members $\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\circledcirc}$ Performs efficiently compared to all family members.

3 A Proposed Framework of Neutrosophic Estimators under Quantile Regression:

This research presents an innovative solution by combining neutrosophic statistics with quantile regression to create a new general family of estimators for median estimation. The integration of neutrosophic logic with quantile regression is theoretically inspired by the need to address both uncertainty in data and variation across its distribution. Neutrosophic statistics effectively models imprecision and indeterminacy, while quantile regression offers a comprehensive perspective by analyzing data at multiple points along its distribution. By leveraging distinctive metrics like quartile deviation, quartile average, midrange and interquartile range, these estimators improve data distribution anal-

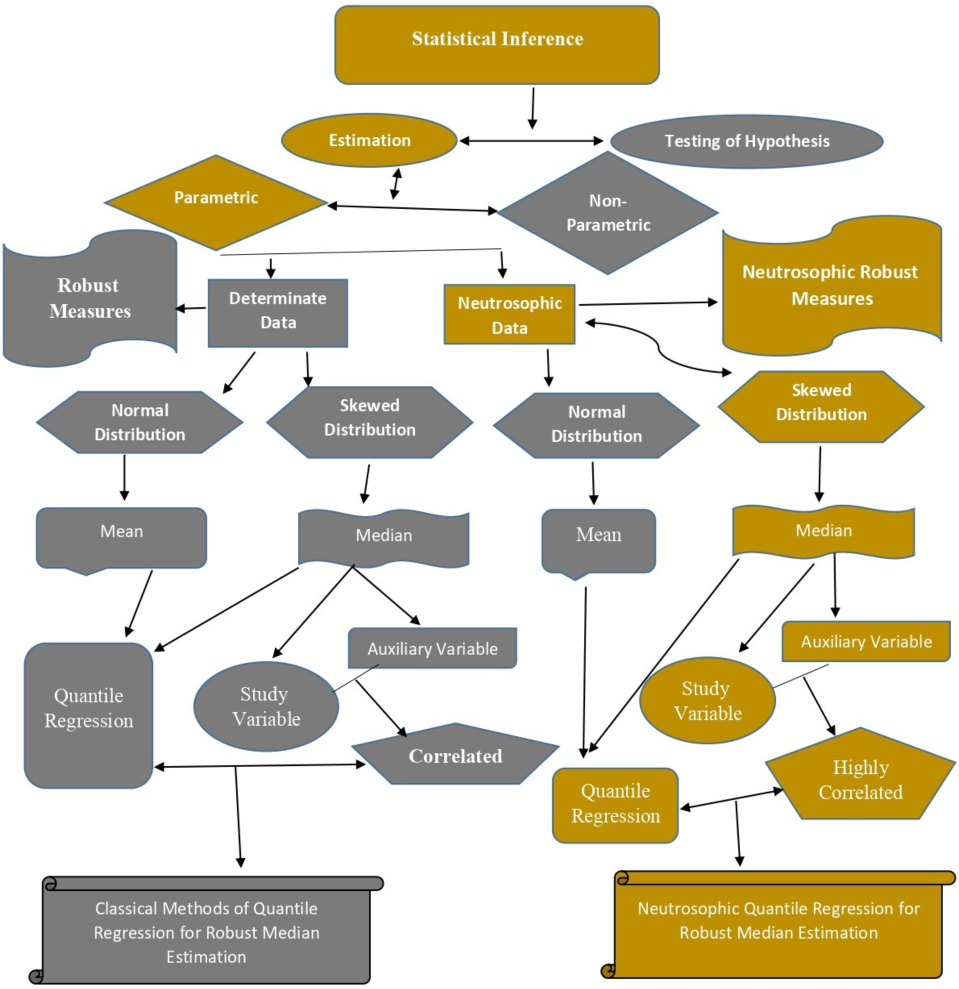

Figure 1: Flow Chart of Neutrosophic Quantile Regression for Robust Median Estimation

ysis and effectively handle outliers. Utilizing robust measures including decile means, the Hodges-Lehmann estimator, and the tri-mean these methods provide more accurate and reliable estimates.
Inspired by the works of Irfan et al. (2021), Masood et al. (2024), Hussain et al. (2024) and Zohaib et al. (2024), we introduced generalized class of ratio-type median estimators that employ quantile regression under neutrosophic statistics.

$$\widehat{T}^{\eta}_{(Q)m} = S^{\eta}_{z}\left[M^{\eta}_{y}\left(\frac{M^{\eta}_{x}\theta^{\eta}_{i}+R^{\eta}\vartheta^{\eta}_{i}}{\widehat{M}^{\eta}_{x}\theta^{\eta}_{i}+R^{\eta}\vartheta^{\eta}_{i}}\right)^{\frac{M^{\eta}_{x}\theta^{\eta}_{i}}{M^{\eta}_{x}\theta^{\eta}_{i}+R^{\eta}\vartheta^{\eta}_{i}}} + \widehat{\beta}^{\eta}_{(Q)}\left(M^{\eta}_{x}-\widehat{M}^{\eta}_{x}\right)\right] \tag{19}$$

For $m = 1, 2, \ldots, 12$

Here The components $\widehat{T}^{\eta}_{(Q)m} \in \left\{\widehat{T}^{\eta}_{(Q)ml}, \widehat{T}^{\eta}_{(Q)mu}\right\}$, $\theta^{\eta}_{i} \in \{\theta^{\eta}_{il}, \theta^{\eta}_{iu}\}$, and $\vartheta^{\eta}_{i} \in \{\vartheta^{\eta}_{il}, \vartheta^{\eta}_{iu}\}$ may represent either constant values or functions, derived from robust or non-traditional location measures, as well as conventional measures related to the auxiliary variable.

The term $S^{\eta}_{z} \in \{S^{\eta}_{zl}, S^{\eta}_{zu}\}$ denotes an unknown constant that will be specified later, while $\widehat{\beta}^{\eta}_{(Q)} \in \left\{\widehat{\beta}^{\eta}_{(Q)l}, \widehat{\beta}^{\eta}_{(Q)u}\right\}$ represents the quantile regression coefficients corresponding to $p = 2$ variables. For an in-depth discussion on quantile regression, refer to Koenker and Hallock (2001), and Zohaib et al. (2024).

The following notations define neutrosophic functions:

$$\Delta^{\eta}_{P} \in \{\Delta^{\eta}_{Pl}, \Delta^{\eta}_{Pu}\}, \quad S^{\eta}_{z} \in \{S^{\eta}_{zl}, S^{\eta}_{zu}\}, \quad \Lambda^{\eta}_{1} \in \{\Lambda^{\eta}_{1l}, \Lambda^{\eta}_{1u}\}, \quad \Lambda^{\eta}_{2} \in \{\Lambda^{\eta}_{2l}, \Lambda^{\eta}_{2u}\}$$

For further insights into conventional versus robust or non-conventional measures, readers are referred to Irfan et al. (2021).
Remark 3.1.
The quantile levels 0.15, 0.25 and 0.35 are specifically chosen to capture the behavior of the estimator in the lower part of the conditional distribution. This allows for a detailed analysis of lower-tail characteristics, which is particularly useful in skewed populations. By inserting different values of $\widehat{\beta}^{\eta}_{(Q)} = \{Q = 0.15,\ 0.25\, and\, 0.35\}, \widehat{\beta}^{\eta}_{(Q)} \in \left\{\widehat{\beta}^{\eta}_{(Q)l}, \widehat{\beta}^{\eta}_{(Q)u}\right\}$ in equation (19), we obtain the following families of ratio estimators, utilizing quantile regression for median estimation under neutrosophic statistics.

(i) By substituting $\widehat{\beta}^{\eta}_{(0.15)} \in \left\{\widehat{\beta}^{\eta}_{(0.15)l}, \widehat{\beta}^{\eta}_{(0.15)u}\right\}$; The proposed family of ratio-type median estimators using quantile regression under neutrosophic statistics simplifies to:

$$\widehat{\mathbf{T}}_m^\eta = S_z^\eta \left[M_y^\eta \left(\frac{M_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta}{\widehat{M}_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta} \right)^{\frac{M_x^\eta \theta_i^\eta}{M_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta}} + \widehat{\beta}_{(0.15)}^\eta (M_x^\eta - \widehat{M}_x^\eta) \right] \tag{20}$$

(ii) By substituting $\widehat{\beta}_{(0.25)}^\eta \in \left\{ \widehat{\beta}_{(0.25)l}^\eta, \widehat{\beta}_{(0.25)u}^\eta \right\}$; The proposed family of ratio type median estimators using quantile regression under neutrosophic statistics simplifies to:

$$\widehat{\mathbf{T}}_m^\eta = S_z^\eta \left[M_y^\eta \left(\frac{M_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta}{\widehat{M}_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta} \right)^{\frac{M_x^\eta \theta_i^\eta}{M_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta}} + \widehat{\beta}_{(0.25)}^\eta (M_x^\eta - \widehat{M}_x^\eta) \right] \tag{21}$$

(iii) By substituting $\widehat{\beta}_{(0.35)}^\eta \in \left\{ \widehat{\beta}_{(0.35)l}^\eta, \widehat{\beta}_{(0.35)u}^\eta \right\}$; The proposed family of ratio type median estimators using quantile regression under neutrosophic statistics simplifies to:

$$\widehat{\mathbf{T}}_m^\eta = S_z^\eta \left[M_y^\eta \left(\frac{M_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta}{\widehat{M}_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta} \right)^{\frac{M_x^\eta \theta_i^\eta}{M_x^\eta \theta_i^\eta + R^\eta \vartheta_i^\eta}} + \widehat{\beta}_{(0.35)}^\eta (M_x^\eta - \widehat{M}_x^\eta) \right] \tag{22}$$

$$MSE\left(\widehat{\mathbf{T}}_{(Q)m}^\eta\right)_{\min} \approx \left[M_y^{\eta 2} - \frac{\Lambda_1^{\eta 2}}{2\Lambda_2^\eta} \right] \tag{23}$$

Where

$$\Delta_P^\eta = \frac{M_x^\eta\, \boldsymbol{\theta}_i^\eta}{M_x^\eta\, \boldsymbol{\theta}_i^\eta + R^\eta\, \boldsymbol{\vartheta}_i^\eta}, \quad \text{and}$$

,

$$S_z^\eta = \frac{M_y^{\eta 2} + \frac{\zeta^\eta}{2}\left[\left({\Delta_P^\eta}^4 + {\Delta_P^\eta}^3 \right) R^{\eta 2} \left(M_{Sx}^\eta \right)^2 - 2{\Delta_P^\eta}^2 R^\eta M_{Sxy}^\eta \right]}{\begin{array}{c} M_y^{\eta 2} + \zeta^\eta \Big[\left(M_{Sy}^\eta \right)^2 + \Big\{ 2{\Delta_P^\eta}^4 R^{\eta 2} + \left(\widehat{\beta}_{(Q)}^\eta \right)^2 \\ + {\Delta_P^\eta}^3 R^{\eta 2} + 2{\Delta_P^\eta}^2 \widehat{\beta}_{(Q)}^\eta R^\eta \Big\} \left(M_{Sx}^\eta \right)^2 - \left\{ 4{\Delta_P^\eta}^2 R^\eta + 2\widehat{\beta}_{(Q)}^\eta \right\} M_{Sxy}^\eta \Big] \end{array}}$$

Proof:

In order to demonstrate the relationship between equations (19) and (23), the proposed estimator $\widehat{T}_{(Q)m}^\eta$ is expressed in terms of the error components e_i^η as follows.

$$\widehat{T}^{\eta}_{(Q)m} = S^{\eta}_{z}\Big[\left(M^{\eta}_{y} + M^{\eta}_{y}e^{\eta}_{0}\right)\left(1+\Delta^{\eta}_{P}e^{\eta}_{1}\right)^{-\Delta^{\eta}_{P}} - \widehat{\beta}^{\eta}_{(Q)}M^{\eta}_{x}e^{\eta}_{1}\Big] \tag{24}$$

Subtracting M^{η}_{y} from equation (24) and expanding the results up to the first degree of approximation, we obtain:

$$\begin{aligned}\widehat{T}^{\eta}_{(Q)m} - M^{\eta}_{y} = \Big[\, & S^{\eta}_{z}M^{\eta}_{y} + S^{\eta}_{z}M^{\eta}_{y}e^{\eta}_{0} - S^{\eta}_{z}M^{\eta}_{y}\left(\Delta^{\eta}_{P}\right)^{2}e^{\eta}_{1} \\ & - S^{\eta}_{z}\widehat{\beta}^{\eta}_{(Q)}M^{\eta}_{x}e^{\eta}_{1} + \frac{1}{2}S^{\eta}_{z}M^{\eta}_{y}\left(\left(\Delta^{\eta}_{P}\right)^{4} + \left(\Delta^{\eta}_{P}\right)^{3}\right)\left(e^{\eta}_{1}\right)^{2} \\ & - S^{\eta}_{z}M^{\eta}_{y}\left(\Delta^{\eta}_{P}\right)^{2}e^{\eta}_{0}e^{\eta}_{1} - M^{\eta}_{y}\,\Big]\end{aligned} \tag{25}$$

The MSE of $\widehat{T}^{\eta}_{(Q)m}$up to the first degree of approximation are given as:

$$\begin{aligned}MSE\left(\widehat{T}^{\eta}_{(Q)m}\right) \approx \Bigg[& \left(S^{\eta}_{z}M^{\eta}_{y} - M^{\eta}_{y}\right)^{2} + \left(S^{\eta}_{z}\right)^{2}\Bigg(\left(M^{\eta}_{y}\right)^{2}E\left(\left(e^{\eta}_{0}\right)^{2}\right) \\ & + \Big(2\left(\Delta^{\eta}_{P}\right)^{4}\left(M^{\eta}_{y}\right)^{2} + \left(\widehat{\beta}^{\eta}_{(Q)}\right)^{2}\left(M^{\eta}_{x}\right)^{2} + \left(\Delta^{\eta}_{P}\right)^{3}\left(M^{\eta}_{y}\right)^{2} \\ & + 2\left(\Delta^{\eta}_{P}\right)^{2}\widehat{\beta}^{\eta}_{(Q)}M^{\eta}_{y}M^{\eta}_{x}\Big)E\left(\left(e^{\eta}_{1}\right)^{2}\right) \\ & - \left(4\left(\Delta^{\eta}_{P}\right)^{4}\left(M^{\eta}_{y}\right)^{2} + 2\widehat{\beta}^{\eta}_{(Q)}M^{\eta}_{y}M^{\eta}_{x}\right)E\left(e^{\eta}_{0}e^{\eta}_{1}\right)\Bigg) \\ & - S^{\eta}_{z}\left(\left(\Delta^{\eta}_{P}\right)^{4}\left(M^{\eta}_{y}\right)^{2} + \left(\Delta^{\eta}_{P}\right)^{3}\left(M^{\eta}_{y}\right)^{2}\right)E\left(\left(e^{\eta}_{1}\right)^{2}\right) \\ & + 2\left(\Delta^{\eta}_{P}\right)^{2}\left(M^{\eta}_{y}\right)^{2}E\left(e^{\eta}_{0}e^{\eta}_{1}\right)\Bigg]\end{aligned} \tag{26}$$

Now, by differentiating equation (26) with respect to S^{η}_{z} and set it equal to zero, we obtain the value of S^{η}_{z} as shown above from the preceding equation (23)

$$\Lambda^{\eta}_{1} = M^{\eta 2}_{y} + \frac{\zeta^{\eta}}{2}\left[\left({\Delta^{\eta}_{P}}^{4} + {\Delta^{\eta}_{P}}^{3}\right)R^{\eta 2}{M^{\eta}_{Sx}}^{2} - 2{\Delta^{\eta}_{P}}^{2}R^{\eta}M^{\eta}_{Sxy}\right],$$

$$\begin{aligned}\Lambda^{\eta}_{2} = M^{\eta 2}_{y} + \zeta^{\eta}\Bigg[& {M^{\eta}_{Sy}}^{2} + \left\{2{\Delta^{\eta}_{P}}^{4}R^{\eta 2} + \left(\widehat{\beta}^{\eta}_{(Q)}\right)^{2} + {\Delta^{\eta}_{P}}^{3}R^{\eta 2} + 2{\Delta^{\eta}_{P}}^{2}\widehat{\beta}^{\eta}_{(Q)}R^{\eta}\right\}{M^{\eta}_{Sx}}^{2} \\ & - \left\{4{\Delta^{\eta}_{P}}^{2}R^{\eta} + 2\widehat{\beta}^{\eta}_{(Q)}\right\}M^{\eta}_{Sxy}\Bigg]\end{aligned}$$

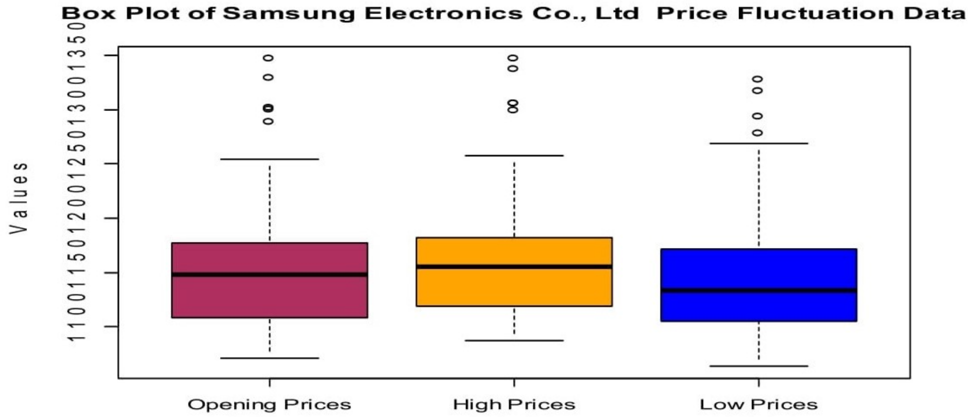

Figure 2: Boxplot of daily stock prices of Samsung Electronics

4 Practical Application in Stock Price Data

In a real-world application, we assess the efficiency of the proposed novel families of estimators under neutrosophic quantile regression by comparing their performance with other competitive estimators, using a dataset of real-world indeterminacy intervals. This data set comprises daily stock prices for Samsung Electronics from June 01, 2022, to July 29, 2022, as obtained from Yahoo Finance. Each trading day includes the opening, high, and low prices. The opening price, known and fixed at the start of each day, is used as an auxiliary variable and is not treated as a neutrosophic variable. In contrast, high and low prices vary during the day due to market fluctuations and are considered neutrosophic variables to account for natural uncertainty in the financial market. It is important to note that we did not construct any artificial intervals for the high or low prices; instead, we used the original recorded values available on the website. The descriptive measures of the data set are summarized in Table 4. We use mean squared error (MSE) as a key performance metric and, for a deeper understanding, we also examine the percentage decrease in MSE to compare the effectiveness of the proposed estimators. Figure 2 presents a box plot of the behavior of the stock price during the study period and shows the opening, high, and low prices, visually confirming the range and variability considered in our analysis.

Table 4 presents the descriptive statistics of population of daily stock prices of (Samsung Electronics 2022) from June 01, 2022 to July 29, 2022.

4.1 Important Findings:

We calculated the MSE_{min} of traditional and proposed neutrosophic quantile regression estimators as given in Tables 5 and 6. Figure 3 Shows MSE_{min} of the proposed and reviewed estimators. Some important observations are re-

Table 4: Data statistics

Population of Samsung Electronics 2022
$N=41$, $n_l^\eta=8$, n_u^η =8, $\mathbf{M_{xl}^\eta}$= **1149.5,** $\mathbf{M_{xu}^\eta}$= **1149.5,** $\mathbf{M_{yl}^\eta}$= **1133,**$\mathbf{M_{yu}^\eta}$= **1156.5,** $\mathbf{f_y^\eta(M_y^\eta)_l}$= **0.005798982,** $\mathbf{f_x^\eta(M_x^\eta)_l}$= **0.005660414,** $\mathbf{f_y^\eta(M_y^\eta)_u}$= **0.005795176,** $\mathbf{f_x^\eta(M_x^\eta)_u}$= **0.005660414,** $\rho_{yxl}^\eta=0.9512195$, $\rho_{yxu}^\eta=0.9512195$, $t_{ml}^\eta=1146.5$, $t_{mu}^\eta=1146.5$, h_{ll}^η =1142.625, h_{lu}^η =1142.625, d_{ml}^η =1156.778, d_{mu}^η =1156.778, q_{rl}^η =69, $q_{ru}^\eta=69$, $m_{rl}^\eta=1208.75$, m_{rl}^η =1208.75, $q_{dl}^\eta=34.5$, $q_{du}^\eta=34.5$, $q_{al}^\eta=1143.5$,$q_{au}^\eta=1143.5$,$\beta_{0.15l}^\eta=0.8454259$, $\beta_{0.15u}^\eta=0.9900596$, $\beta_{0.25l}^\eta=0.9501312$, $\beta_{0.25u}^\eta$ =0.9798851, $\beta_{0.35l}^\eta=0.9253247$, $\beta_{0.35u}^\eta$ =0.9792531

Table 5: **$\mathbf{MSE_{min}}$ of traditional estimators for the dataset of Samsung Electronics**

Estimators	$\mathbf{MSE_{minl}}$	$\mathbf{MSE_{minu}}$	Estimators	$\mathbf{MSE_{minl}}$	$\mathbf{MSE_{minu}}$
$\widehat{\mathbf{M}}_{\mathbf{0}}^\eta$	747.9569	748.9397	$\widehat{\mathbf{M}}_{\mathbf{d3}}^\eta$	71.17003	71.26341
$\widehat{\mathbf{M}}_{\mathbf{R}}^\eta$	73.75616	75.93837	$\widehat{\mathbf{M}}_{\mathbf{d4}}^\eta$	71.03212	71.12101
$\widehat{\mathbf{M}}_{\mathbf{ex}}^\eta$	213.7852	220.1939	$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\ominus}$	70.83841	70.9106
$\widehat{\mathbf{M}}_{\mathbf{d0}}^\eta$	71.19163	71.28517	$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\bigoplus}$	70.99297	71.06741
$\widehat{\mathbf{M}}_{\mathbf{d1}}^\eta$	71.18769	71.28138	$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\bigotimes}$	70.79044	70.8435
$\widehat{\mathbf{M}}_{\mathbf{d2}}^\eta$	71.18768	71.28137	$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\circledast}$	70.28938	70.32231

Table 6: **$\mathbf{MSE_{min}}$ of proposed estimators for the dataset of Samsung Electronics data sets**

$\widehat{\mathbf{T}}_{\text{℗} m}^\eta$	$\mathbf{MSE}_{\min l}$	$\mathbf{MSE}_{\min u}$	$\widehat{\mathbf{T}}_{\circledR m}^\eta$	$\mathbf{MSE}_{\min l}$	$\mathbf{MSE}_{\min u}$	$\widehat{\mathbf{T}}_{\copyright m}^\eta$	$\mathbf{MSE}_{\min l}$	$\mathbf{MSE}_{\min u}$
$\widehat{\mathbf{T}}_{\square\mathbf{1}}^\eta$	17.81569	24.20015	$\widehat{\mathbf{T}}_{\circledR\mathbf{1}}^\eta$	21.7571	23.77418	$\widehat{\mathbf{T}}_{\copyright\mathbf{1}}^\eta$	20.7841	23.74786
$\widehat{\mathbf{T}}_{\square\mathbf{2}}^\eta$	17.81893	24.20415	$\widehat{\mathbf{T}}_{\circledR\mathbf{2}}^\eta$	21.76072	23.77815	$\widehat{\mathbf{T}}_{\copyright\mathbf{2}}^\eta$	20.78771	23.75182
$\widehat{\mathbf{T}}_{\square\mathbf{3}}^\eta$	17.81564	24.2001	$\widehat{\mathbf{T}}_{\circledR\mathbf{3}}^\eta$	21.75704	23.77413	$\widehat{\mathbf{T}}_{\copyright\mathbf{3}}^\eta$	20.78413	23.7478
$\widehat{\mathbf{T}}_{\square\mathbf{4}}^\eta$	17.81605	24.2006	$\widehat{\mathbf{T}}_{\circledR\mathbf{4}}^\eta$	21.7575	23.77462	$\widehat{\mathbf{T}}_{\copyright\mathbf{4}}^\eta$	20.78457	23.7483
$\widehat{\mathbf{T}}_{\square\mathbf{5}}^\eta$	17.81589	24.2004	$\widehat{\mathbf{T}}_{\circledR\mathbf{5}}^\eta$	21.75732	23.77443	$\widehat{\mathbf{T}}_{\copyright\mathbf{5}}^\eta$	20.7844	23.74811
$\widehat{\mathbf{T}}_{\square\mathbf{6}}^\eta$	17.87232	24.26992	$\widehat{\mathbf{T}}_{\circledR\mathbf{6}}^\eta$	21.8203	23.84328	$\widehat{\mathbf{T}}_{\copyright\mathbf{6}}^\eta$	20.8458	23.81691
$\widehat{\mathbf{T}}_{\square\mathbf{7}}^\eta$	17.81913	24.20439	$\widehat{\mathbf{T}}_{\circledR\mathbf{7}}^\eta$	21.76093	23.77838	$\widehat{\mathbf{T}}_{\copyright\mathbf{7}}^\eta$	20.78792	23.75205
$\widehat{\mathbf{T}}_{\square\mathbf{8}}^\eta$	**17.8151**	**24.19943**	$\widehat{\mathbf{T}}_{\circledR\mathbf{8}}^\eta$	**21.75644**	**23.77347**	$\widehat{\mathbf{T}}_{\copyright\mathbf{8}}^\eta$	**20.78354**	**23.74715**
$\widehat{\mathbf{T}}_{\square\mathbf{9}}^\eta$	17.81638	24.201	$\widehat{\mathbf{T}}_{\circledR\mathbf{9}}^\eta$	21.75787	23.77503	$\widehat{\mathbf{T}}_{\copyright\mathbf{9}}^\eta$	20.78493	23.7487
$\widehat{\mathbf{T}}_{\square\mathbf{10}}^\eta$	17.81842	24.20352	$\widehat{\mathbf{T}}_{\circledR\mathbf{10}}^\eta$	21.76015	23.77752	$\widehat{\mathbf{T}}_{\copyright\mathbf{10}}^\eta$	20.78716	23.7512
$\widehat{\mathbf{T}}_{\square\mathbf{11}}^\eta$	17.81515	24.20015	$\widehat{\mathbf{T}}_{\circledR\mathbf{11}}^\eta$	21.75649	23.77353	$\widehat{\mathbf{T}}_{\copyright\mathbf{11}}^\eta$	20.78359	23.7472
$\widehat{\mathbf{T}}_{\square\mathbf{12}}^\eta$	17.81653	24.20119	$\widehat{\mathbf{T}}_{\circledR\mathbf{12}}^\eta$	21.75804	23.77522	$\widehat{\mathbf{T}}_{\copyright\mathbf{12}}^\eta$	20.7851	23.74889

vealed from Tables 5 and 6 that are given as

(i) $\widehat{M}_{T6}^{\eta\circledast}$ performs more efficiently compared to the reviewing estimators i.e.$\widehat{M}_{T6}^{\eta\bigotimes}$, $\widehat{M}_{T6}^{\eta\bigoplus}$, $\widehat{M}_{T6}^{\eta\ominus}$, $\widehat{M}_{d4}^{\eta}$, $\widehat{M}_{d3}^{\eta}$, $\widehat{M}_{d2}^{\eta}$, $\widehat{M}_{d1}^{\eta}$, $\widehat{M}_{d0}^{\eta}$, $\widehat{M}_{ex}^{\eta}$, $\widehat{M}_{R}^{\eta}$,and $\widehat{M}_{0}^{\eta}$

(ii) All the suggested families of estimators $\widehat{\mathbf{T}}_{\square\mathbf{m}}^\eta$,$\widehat{\mathbf{T}}_{\circledR\mathbf{m}}^\eta$**and** $\widehat{\mathbf{T}}_{\copyright\mathbf{m}}^\eta$**have** MSE_{min} compared to all reviewing estimators.

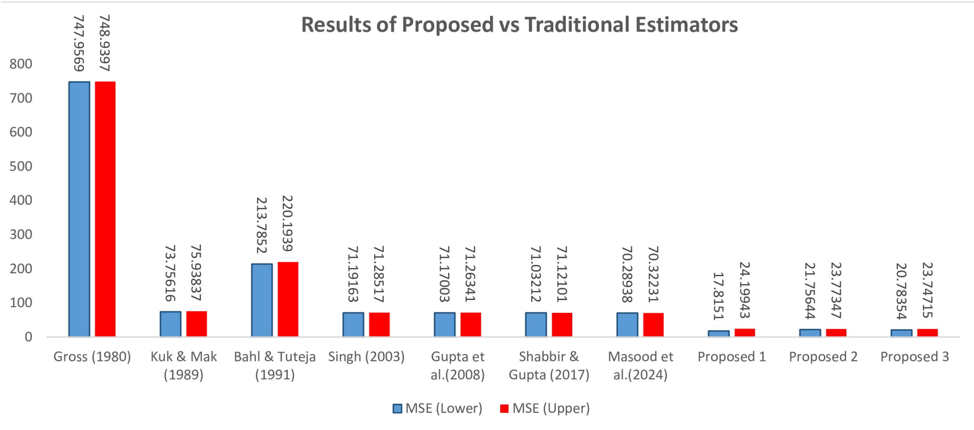

Figure 3: $\mathbf{MSE}_{\min}$ of all proposed and reviewing estimators.

(iii) A deep insight of Table 5 that $\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{8}}$, $\widehat{\mathbf{T}}^{\eta}_{®\mathbf{8}}$ **and** $\widehat{\mathbf{T}}^{\eta}_{©\mathbf{8}}$ outperforms compared to all proposed estimators

To gain a deeper understanding of median estimation through quantile regression within neutrosophic statistics, we introduce a new metric called P_D^{η} (Percentage Decrease in Mean Squared Error), hereafter referred to as $MSE_{P_D^{\eta}}$ Abid et al. (2021). The values of $MSE_{P_D^{\eta}}$ are presented in Table 7-9. An estimator with a higher $MSE_{P_D^{\eta}}$ is consider superior when compared to traditional estimators. The formula for calculating $MSE_{P_D^{\eta}}$ is given as:

$$MSE_{P_D^{\eta}} = \frac{MSE_T - MSE_P}{MSE_T}$$

Where MSE_T denoted by traditional estimator and MSE_P represents proposed estimator.

The results presented in Tables 7 to 9 highlight the efficiency of the proposed estimator through the $MSE_{P_D^{\eta}}$ (Percentage Decrease in Mean Squared Error) metric. The bold values within these tables represent the highest $MSE_{P_D^{\eta}}$ scores, clearly demonstrating the superior performance of our estimator compared to traditional methods. Overall, these findings confirm the robustness and practical advantages of the proposed estimator across various scenarios and data conditions.

The key insights from the analysis of Tables 7–9 can be summarized as follows.

1. By comparing all proposed families of estimators $\widehat{T}^{\eta}_{(Q)m}$ with traditional estimators, it is observed that the proposed families

Table 7: $\mathbf{MSE}_{\mathbf{P}_{\mathbf{D}}^{\eta}}$ of proposed Neutrosophic estimators $\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{m}}$

m	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{0}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{R}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{ex}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d0}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d1}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d2}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d3}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d4}}$	$\widehat{\mathbf{M}}^{\eta\ominus}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\oplus}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\otimes}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\circledast}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\circledcirc}_{\mathbf{T6}}$
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{1l}}$	97.6180	75.8451	91.9090	74.9750	74.9736	74.9736	74.9674	74.9188	74.8502	74.905	74.8332	74.6538	74.5436
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{1u}}$	96.7687	68.1318	88.6801	66.0516	66.0498	66.0498	66.0412	65.9732	65.8723	65.9476	65.8399	65.5868	65.4363
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{2l}}$	97.6176	75.8407	91.9076	74.9704	74.9690	74.9690	74.9628	74.9142	74.8456	74.9004	74.8286	74.6491	74.5389
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{2u}}$	96.7682	68.1265	88.6782	66.0460	66.0442	66.0442	66.0356	65.9676	65.8666	65.9419	65.8343	65.5811	65.4305
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{3l}}$	97.6180	75.8452	91.9091	74.9750	74.9737	74.9737	74.9675	74.9189	74.8503	74.9050	74.8332	74.6538	74.5436
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{3u}}$	96.7687	68.1319	88.6801	66.0517	66.0499	66.0499	66.0413	65.9733	65.8724	65.9476	65.8400	65.5868	65.4363
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{4l}}$	97.6180	75.8446	91.9089	74.9745	74.9731	74.9731	74.9669	74.9183	74.8497	74.9044	74.8326	74.6532	74.5430
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{4u}}$	96.7686	68.1312	88.6799	66.0510	66.0492	66.0492	66.0406	65.9726	65.8717	65.9469	65.8393	65.5861	65.4356
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{5l}}$	97.6180	75.8448	91.909	74.9747	74.9733	74.9733	74.9671	74.9185	74.8499	74.9047	74.8329	74.6535	74.5433
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{5u}}$	96.7687	68.1315	88.6800	66.0512	66.0494	66.0494	66.0409	65.9729	65.8719	65.9472	65.8396	65.5864	65.4359
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{6l}}$	97.6105	75.7683	91.8833	74.8954	74.8940	74.8940	74.8878	74.8391	74.7703	74.8252	74.7532	74.5732	74.4626
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{6u}}$	96.7594	68.0399	88.6475	65.9537	65.9519	65.9519	65.9433	65.8751	65.7739	65.8494	65.7415	65.4876	65.3366
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{7l}}$	97.6176	75.8404	91.9075	74.9702	74.9688	74.9688	74.9626	74.9139	74.8453	74.9001	74.8283	74.6489	74.5386
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{7u}}$	96.7681	68.1262	88.6781	66.0456	66.0438	66.0438	66.0353	65.9673	65.8663	65.9416	65.834	65.5807	65.4302
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{8l}}$	**97.6181**	**75.8459**	**91.9093**	**74.9758**	**74.9744**	**74.9744**	**74.9682**	**74.9196**	**74.8510**	**74.9058**	**74.8340**	**74.6546**	**74.5444**
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{8u}}$	**96.7688**	**68.1328**	**88.6804**	**66.0526**	**66.0508**	**66.0508**	**66.0422**	**65.9742**	**65.8733**	**65.9486**	**65.841**	**65.5878**	**65.4373**
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{9l}}$	97.6179	75.8442	91.9087	74.9740	74.9726	74.9726	74.9664	74.9178	74.8492	74.9040	74.8322	74.6528	74.5426
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{9u}}$	96.7686	68.1307	88.6797	66.0504	66.0486	66.0486	66.0400	65.9720	65.8711	65.9464	65.8387	65.5856	65.4350
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{10l}}$	97.6177	75.8414	91.9078	74.9711	74.9698	74.9697	74.9635	74.9149	74.8463	74.9011	74.8293	74.6499	74.5396
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{10u}}$	96.7682	68.1274	88.6785	66.0469	66.0450	66.0450	66.0365	65.9685	65.8675	65.9428	65.8352	65.5820	65.4314
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{11l}}$	97.6181	75.8458	91.9093	74.9757	74.9744	74.9744	74.9681	74.9195	74.851	74.9057	74.8339	74.6545	74.5443
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{11u}}$	96.7687	68.1318	88.6801	66.0516	66.0498	66.0498	66.0412	65.9732	65.8723	65.9476	65.8399	65.5868	65.4363
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{12l}}$	97.6179	75.8440	91.9087	74.9738	74.9724	74.9724	74.9662	74.9176	74.8490	74.9038	74.8320	74.6526	74.5423
$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{12u}}$	96.7686	68.1304	88.6796	66.0501	66.0483	66.0483	66.0398	65.9718	65.8708	65.9461	65.8385	65.5853	65.4348

Table 8: $\mathbf{MSE}_{\mathbf{P}_{\mathbf{D}}^{\eta}}$ of proposed Neutrosophic estimators $\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{m}}$

m	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{0}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{R}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{ex}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d0}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d1}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d2}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d3}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d4}}$	$\widehat{\mathbf{M}}^{\eta\ominus}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\oplus}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\otimes}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\circledast}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\circledcirc}_{\mathbf{T6}}$
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{1l}}$	97.0911	70.5013	90.1191	69.4386	69.4369	69.4369	69.4294	69.3700	69.2863	69.3531	69.2654	69.0464	68.9118
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{1u}}$	96.8256	68.6927	88.8794	66.6491	66.6474	66.6474	66.639	66.5722	66.4730	66.5469	66.4412	66.1925	66.0446
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{2l}}$	97.0906	70.4964	90.1174	69.4336	69.4319	69.4319	69.4243	69.3649	69.2811	69.3480	69.2603	69.0412	68.9066
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{2u}}$	96.8250	68.6875	88.8775	66.6436	66.6418	66.6418	66.6334	66.5666	66.4674	66.5414	66.4356	66.1869	66.0390
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{3l}}$	97.0911	70.5013	90.1191	69.4387	69.4370	69.4370	69.4294	69.3701	69.2863	69.3532	69.2655	69.0464	68.9118
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{3u}}$	96.8256	68.6928	88.8794	66.6492	66.6474	66.6474	66.6390	66.5722	66.4731	66.5470	66.4413	66.1926	66.0447
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{4l}}$	97.0910	70.5007	90.1189	69.4381	69.4364	69.4364	69.4288	69.3695	69.2857	69.3526	69.2649	69.0458	68.9112
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{4u}}$	96.8255	68.6922	88.8792	66.6485	66.6468	66.6467	66.6383	66.5715	66.4724	66.5463	66.4406	66.1919	66.0440
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{5l}}$	97.0911	70.5010	90.1190	69.4383	69.4366	69.4366	69.4291	69.3697	69.2859	69.3528	69.2651	69.0460	68.9114
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{5u}}$	96.8255	68.6924	88.8792	66.6488	66.6470	66.6470	66.6386	66.5718	66.4726	66.5466	66.4409	66.1921	66.0443
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{6l}}$	97.0826	70.4156	90.0904	69.3499	69.3482	69.3482	69.3406	69.2810	69.1970	69.2641	69.1762	68.9564	68.8215
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{6u}}$	96.8164	68.6018	88.8470	66.5522	66.5504	66.5504	66.5420	66.4750	66.3756	66.4497	66.3437	66.0942	65.946
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{7l}}$	97.0906	70.4961	90.1173	69.4333	69.4316	69.4316	69.4240	69.3646	69.2808	69.3477	69.2600	69.0409	68.9063
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{7u}}$	96.8250	68.6872	88.8774	66.6433	66.6415	66.6415	66.6331	66.5663	66.4671	66.5410	66.4353	66.1865	66.0386
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{8l}}$	**97.0912**	**70.5022**	**90.1194**	**69.4396**	**69.4379**	**69.4379**	**69.4303**	**69.3709**	**69.2872**	**69.3540**	**69.2664**	**69.0473**	**68.9127**
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{8u}}$	**96.8257**	**68.6937**	**88.8797**	**66.6501**	**66.6484**	**66.6484**	**66.6400**	**66.5732**	**66.4740**	**66.548**	**66.4422**	**66.1935**	**66.0457**
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{9l}}$	97.0910	70.5002	90.1187	69.4376	69.4359	69.4359	69.4283	69.3689	69.2852	69.3520	69.2644	69.0453	68.9107
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{9u}}$	96.8255	68.6916	88.8790	66.6480	66.6462	66.6462	66.6378	66.5710	66.4718	66.5458	66.4400	66.1913	66.0434
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{10l}}$	97.0907	70.4971	90.1177	69.4344	69.4327	69.4327	69.4251	69.3657	69.2819	69.3488	69.2611	69.0420	68.9074
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{10u}}$	96.8251	68.6884	88.8778	66.6445	66.6427	66.6427	66.6343	66.5675	66.4683	66.5423	66.4365	66.1877	66.0399
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{11l}}$	97.0912	70.5021	90.1193	69.4395	69.4378	69.4378	69.4302	69.3709	69.2871	69.3540	69.2663	69.0472	68.9126
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{11u}}$	96.8257	68.6936	88.8797	66.6501	66.6483	66.6483	66.6399	66.5731	66.4739	66.5479	66.4421	66.1934	66.0456
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{12l}}$	97.091	70.5000	90.1186	69.4373	69.4356	69.4356	69.4280	69.3687	69.2849	69.3518	69.2641	69.0450	68.9104
$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{12u}}$	96.8254	68.6914	88.8789	66.6477	66.6459	66.6459	66.6375	66.5707	66.4715	66.5455	66.4398	66.1910	66.0432

Table 9: $\mathbf{MSE}_{\mathbf{P}_{\mathbf{D}}^{\eta}}$ of Proposed Neutrosophic estimators $\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{m}}$

m	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{0}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{R}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{ex}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d0}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d1}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d2}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d3}}$	$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d4}}$	$\widehat{\mathbf{M}}^{\eta\ominus}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\oplus}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\otimes}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\circledast}_{\mathbf{T6}}$	$\widehat{\mathbf{M}}^{\eta\circledcirc}_{\mathbf{T6}}$
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{1l}}$	97.2212	71.8204	90.5609	70.8053	70.8036	70.8036	70.7964	70.7397	70.6597	70.7236	70.6398	70.4305	70.302
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{1u}}$	96.8291	68.7274	88.8917	66.6861	66.6843	66.6843	66.6759	66.6092	66.5101	66.5840	66.4784	66.2299	66.0822
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{2l}}$	97.2207	71.8156	90.5593	70.8003	70.7987	70.7987	70.7914	70.7347	70.6547	70.7186	70.6348	70.4255	70.2969
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{2u}}$	96.8286	68.7222	88.8898	66.6805	66.6787	66.6787	66.6703	66.6036	66.5045	66.5784	66.4728	66.2243	66.0766
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{3l}}$	97.2212	71.8204	90.5609	70.8053	70.8037	70.8037	70.7965	70.7398	70.6598	70.7236	70.6399	70.4306	70.3020
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{3u}}$	96.8291	68.7275	88.8917	66.6862	66.6844	66.6844	66.6760	66.6093	66.5102	66.5841	66.4785	66.2300	66.0823
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{4l}}$	97.2211	71.8198	90.5607	70.8047	70.8031	70.8031	70.7959	70.7392	70.6591	70.7230	70.6393	70.43	70.3014
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{4u}}$	96.8290	68.7268	88.8915	66.6855	66.6837	66.6837	66.6753	66.6086	66.5095	66.5834	66.4778	66.2293	66.0816
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{5l}}$	97.2211	71.8201	90.5608	70.805	70.8033	70.8033	70.7961	70.7394	70.6594	70.7233	70.6395	70.4302	70.3016
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{5u}}$	96.8291	68.7271	88.8916	66.6857	66.6839	66.6839	66.6756	66.6088	66.5098	66.5836	66.4780	66.2296	66.0819
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{6l}}$	97.2129	71.7368	90.5329	70.7187	70.7170	70.7170	70.7098	70.6529	70.5727	70.6367	70.5527	70.3428	70.2139
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{6u}}$	96.8199	68.6365	88.8594	66.5892	66.5874	66.5874	66.5790	66.5121	66.4127	66.4868	66.3809	66.1317	65.9836
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{7l}}$	97.2207	71.8153	90.5592	70.8000	70.7984	70.7984	70.7911	70.7344	70.6544	70.7183	70.6345	70.4252	70.2966
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{7u}}$	96.8285	68.7219	88.8897	66.6802	66.6784	66.6784	66.6700	66.6033	66.5042	66.5781	66.4725	66.2240	66.0762
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{8l}}$	**97.2212**	**71.8212**	**90.5612**	**70.8062**	**70.8045**	**70.8045**	**70.7973**	**70.7406**	**70.6606**	**70.7245**	**70.6407**	**70.4314**	**70.3029**
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{8u}}$	**96.8292**	**68.7284**	**88.8920**	**66.6871**	**66.6853**	**66.6853**	**66.6769**	**66.6102**	**66.5111**	**66.5850**	**66.4794**	**66.2309**	**66.0833**
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{9l}}$	97.2211	71.8194	90.5606	70.8042	70.8026	70.8026	70.7953	70.7386	70.6586	70.7225	70.6387	70.4294	70.3009
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{9u}}$	96.8290	68.7263	88.8913	66.6849	66.6831	66.6831	66.6747	66.6080	66.5089	66.5828	66.4772	66.2287	66.0810
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{10l}}$	97.2208	71.8163	90.5596	70.8011	70.7995	70.7995	70.7922	70.7355	70.6555	70.7194	70.6356	70.4263	70.2977
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{10u}}$	96.8290	68.7263	88.8913	66.6849	66.6831	66.6831	66.6747	66.6080	66.5089	66.5828	66.4772	66.2287	66.0810
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{11l}}$	97.2212	71.8212	90.5612	70.8061	70.8045	70.8045	70.7972	70.7405	70.6605	70.7244	70.6406	70.4313	70.3028
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{11u}}$	96.8292	68.7283	88.8920	66.6870	66.6852	66.6852	66.6768	66.6101	66.5110	66.5849	66.4793	66.2309	66.0832
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{12l}}$	97.2210	71.8191	90.5605	70.8040	70.8024	70.8024	70.7951	70.7384	70.6584	70.7223	70.6385	70.4292	70.3006
$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{12u}}$	96.829	68.7261	88.8912	66.6846	66.6828	66.6828	66.6745	66.6077	66.5087	66.5825	66.4769	66.2285	66.0808

of estimators $\widehat{T}^{\eta}_{(Q)m}$ demonstrate greater efficiency, as indicated by lower MSE values.

2. The estimator $(\widehat{T}^{\eta}_{\square 8l}, \widehat{T}^{\eta}_{\square 8u})$, $(\widehat{T}^{\eta}_{®8l}, \widehat{T}^{\eta}_{®8u})$ and $(\widehat{T}^{\eta}_{©8l}, \widehat{T}^{\eta}_{©8u})$ has a larger $MSE_{P^{\eta}_{D}}$ by comparing all other proposed estimators.

4.2 Simulation Study

We evaluated the performance of the proposed and traditional estimators through a Monte Carlo simulation study based on the real-life data set described earlier. The population data, consisting of daily stock prices, exhibit a right-skewed distribution, which reflects typical financial market behavior. The statistical software R is utilized to perform the simulation analysis. The following steps are followed to conduct the simulation study:

(i) A simple random sample without replacement (SRSWOR) of size $n^{\eta} \in \{n^{\eta}_{l}, n^{\eta}_{u}\} =$10 and 15 is selected from the population N
(ii) Using the sample data from Step (i), the MSE of both the traditional and proposed estimators is calculated.

(iii)Using the sample data from Step (i), the MSE of both the traditional and proposed estimators is calculated.

(iv) This process generates 35,000 MSE or minimum MSE values for all the existing and proposed estimators.

(v) The average of these 35,000 values represents the MSE for each estimator.

Table 10 and 11 indicate that

(i) The estimators $\widehat{T}^{\eta}_{(Q)m}$ outperforms compared to all traditional estimators, including$\widehat{M}^{\eta}_{0}$, $\widehat{M}^{\eta}_{R}, \widehat{M}^{\eta}_{ex}, \widehat{M}^{\eta}_{di}(i = 0, 1, 2, 3, 4)$.

(ii) Additionally, as the sample size increases, the minimum MSE of the proposed estimators' decreases.

(iii) Additionally, as the sample size increases, the minimum MSE of the proposed estimators' decreases.
It is concluded that the generalized estimator demonstrates exceptional performance, particularly in the presence of extreme values.

Figure 4 presents a simple bar chart showing that as the sample size increases, the mean squared error (MSE) decreases. This pattern confirms that the proposed estimator becomes more accurate and reliable with larger sample sizes.

Table 10: **MSE**$_{\min}$ of Simulation study of traditional and proposed Neutrosophic estimators

n = 10			n = 10			n = 10			n = 10		
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{i}}$	**MSE$_{\text{minL}}$**	**MSE$_{\text{minU}}$**	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{m}}$	**MSE$_{\text{minL}}$**	**MSE$_{\text{minU}}$**	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{m}}$	**MSE$_{\text{minL}}$**	**MSE$_{\text{minU}}$**	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{m}}$	**MSE$_{\text{minL}}$**	**MSE$_{\text{minU}}$**
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{0}}$	562.101	562.839	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{1}}$	10.0619	13.6677	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{1}}$	12.2879	13.4271	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{1}}$	11.7384	13.4123
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{R}}$	55.4288	57.0688	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{2}}$	10.0637	13.6700	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{2}}$	12.2900	13.4294	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{2}}$	11.7404	13.4145
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{ex}}$	165.479	160.662	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{3}}$	10.0619	13.6677	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{3}}$	12.2879	13.4271	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{3}}$	11.7384	13.4122
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d0}}$	53.4993	53.5697	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{4}}$	10.0621	13.6680	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{4}}$	12.2881	13.4274	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{4}}$	11.7387	13.4125
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d1}}$	53.4993	53.5697	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{5}}$	10.0620	13.6679	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{5}}$	12.2880	13.4273	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{5}}$	11.7386	13.4124
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d2}}$	53.4993	53.5697	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{6}}$	10.0939	13.7071	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{6}}$	12.3236	13.4662	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{6}}$	11.7732	13.4513
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d3}}$	53.4893	53.5596	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{7}}$	10.0638	13.6701	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{7}}$	12.2901	13.4295	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{7}}$	11.7405	13.4146
$\widehat{\mathbf{M}}^{\eta}_{\mathbf{d4}}$	53.4115	53.4791	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{8}}$	**10.0615**	**13.6673**	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{8}}$	**12.2876**	**13.4267**	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{8}}$	**11.7381**	**13.4119**
$\widehat{\mathbf{M}}^{\eta\ominus}_{\mathbf{T6}}$	53.3020	53.3603	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{9}}$	10.0623	13.6682	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{9}}$	12.2884	13.4276	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{9}}$	11.7389	13.4127
$\widehat{\mathbf{M}}^{\eta\oplus}_{\mathbf{T6}}$	53.3893	53.4488	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{10}}$	10.0634	13.6696	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{10}}$	12.2896	13.4290	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{10}}$	11.7401	13.4142
$\widehat{\mathbf{M}}^{\eta\otimes}_{\mathbf{T6}}$	53.2749	53.3223	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{11}}$	10.0616	13.6673	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{11}}$	12.2876	13.4268	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{11}}$	11.7381	13.4119
$\widehat{\mathbf{M}}^{\eta\circledast}_{\mathbf{T6}}$	52.9918	53.0278	$\widehat{\mathbf{T}}^{\eta}_{\square\mathbf{12}}$	10.0624	13.6683	$\widehat{\mathbf{T}}^{\eta}_{\circledR\mathbf{12}}$	12.2885	13.4277	$\widehat{\mathbf{T}}^{\eta}_{\copyright\mathbf{12}}$	11.739	13.4129

Table 11: $\mathbf{MSE}_{\min}$ of Simulation study of traditional and proposed Neutrosophic estimators

n = 15			n = 15			n = 15			n = 15		
$\widehat{\mathbf{M}}_{\mathbf{i}}^{\eta}$	$\mathbf{MSE}_{\mathbf{minL}}$	$\mathbf{MSE}_{\mathbf{minU}}$	$\widehat{\mathbf{T}}_{\square\mathbf{m}}^{\eta}$	$\mathbf{MSE}_{\mathbf{minL}}$	$\mathbf{MSE}_{\mathbf{minU}}$	$\widehat{\mathbf{T}}_{\circledR\mathbf{m}}^{\eta}$	$\mathbf{MSE}_{\mathbf{minL}}$	$\mathbf{MSE}_{\mathbf{minU}}$	$\widehat{\mathbf{T}}_{\copyright\mathbf{m}}^{\eta}$	$\mathbf{MSE}_{\mathbf{minL}}$	$\mathbf{MSE}_{\mathbf{minU}}$
$\widehat{\mathbf{M}}_{\mathbf{0}}^{\eta}$	314.293	314.706	$\widehat{\mathbf{T}}_{\square\mathbf{1}}^{\eta}$	3.14575	4.2730	$\widehat{\mathbf{T}}_{\circledR\mathbf{1}}^{\eta}$	3.84170	4.1978	$\widehat{\mathbf{T}}_{\copyright\mathbf{1}}^{\eta}$	3.6699	4.1932
$\widehat{\mathbf{M}}_{\mathbf{R}}^{\eta}$	30.9924	31.9094	$\widehat{\mathbf{T}}_{\square\mathbf{2}}^{\eta}$	3.14632	4.2737	$\widehat{\mathbf{T}}_{\circledR\mathbf{2}}^{\eta}$	3.84234	4.1985	$\widehat{\mathbf{T}}_{\copyright\mathbf{2}}^{\eta}$	3.6705	4.1939
$\widehat{\mathbf{M}}_{\mathbf{ex}}^{\eta}$	92.5259	89.8329	$\widehat{\mathbf{T}}_{\square\mathbf{3}}^{\eta}$	3.1457	4.2730	$\widehat{\mathbf{T}}_{\circledR\mathbf{3}}^{\eta}$	3.8416	4.1978	$\widehat{\mathbf{T}}_{\copyright\mathbf{3}}^{\eta}$	3.6699	4.1932
$\widehat{\mathbf{M}}_{\mathbf{d0}}^{\eta}$	29.9141	29.9535	$\widehat{\mathbf{T}}_{\square\mathbf{4}}^{\eta}$	3.1458	4.2731	$\widehat{\mathbf{T}}_{\circledR\mathbf{4}}^{\eta}$	3.8417	4.1979	$\widehat{\mathbf{T}}_{\copyright\mathbf{4}}^{\eta}$	3.6699	4.1933
$\widehat{\mathbf{M}}_{\mathbf{d1}}^{\eta}$	29.9141	29.9535	$\widehat{\mathbf{T}}_{\square\mathbf{5}}^{\eta}$	3.1457	4.2731	$\widehat{\mathbf{T}}_{\circledR\mathbf{5}}^{\eta}$	3.8417	4.1979	$\widehat{\mathbf{T}}_{\copyright\mathbf{5}}^{\eta}$	3.6699	4.1932
$\widehat{\mathbf{M}}_{\mathbf{d2}}^{\eta}$	29.9141	29.9535	$\widehat{\mathbf{T}}_{\square\mathbf{6}}^{\eta}$	3.1557	4.2854	$\widehat{\mathbf{T}}_{\circledR\mathbf{6}}^{\eta}$	3.8528	4.2100	$\widehat{\mathbf{T}}_{\copyright\mathbf{6}}^{\eta}$	3.6807	4.2054
$\widehat{\mathbf{M}}_{\mathbf{d3}}^{\eta}$	29.9110	29.9503	$\widehat{\mathbf{T}}_{\square\mathbf{7}}^{\eta}$	3.1463	4.2738	$\widehat{\mathbf{T}}_{\circledR\mathbf{7}}^{\eta}$	3.8423	4.1986	$\widehat{\mathbf{T}}_{\copyright\mathbf{7}}^{\eta}$	3.6705	4.1939
$\widehat{\mathbf{M}}_{\mathbf{d4}}^{\eta}$	29.8867	29.9251	$\widehat{\mathbf{T}}_{\square\mathbf{8}}^{\eta}$	**3.1456**	**4.2729**	$\widehat{\mathbf{T}}_{\circledR\mathbf{8}}^{\eta}$	**3.8415**	**4.1977**	$\widehat{\mathbf{T}}_{\copyright\mathbf{8}}^{\eta}$	**3.6697**	**4.1930**
$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\ominus}$	29.8524	29.8880	$\widehat{\mathbf{T}}_{\square\mathbf{9}}^{\eta}$	3.1458	4.2732	$\widehat{\mathbf{T}}_{\circledR\mathbf{9}}^{\eta}$	3.8418	4.1980	$\widehat{\mathbf{T}}_{\copyright\mathbf{9}}^{\eta}$	3.6700	4.1933
$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\oplus}$	29.8797	29.9157	$\widehat{\mathbf{T}}_{\square\mathbf{10}}^{\eta}$	3.1462	4.2736	$\widehat{\mathbf{T}}_{\circledR\mathbf{10}}^{\eta}$	3.8422	4.1984	$\widehat{\mathbf{T}}_{\copyright\mathbf{10}}^{\eta}$	3.6704	4.1938
$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\otimes}$	29.844	29.8761	$\widehat{\mathbf{T}}_{\square\mathbf{11}}^{\eta}$	3.1456	4.2729	$\widehat{\mathbf{T}}_{\circledR\mathbf{11}}^{\eta}$	3.8415	4.1977	$\widehat{\mathbf{T}}_{\copyright\mathbf{11}}^{\eta}$	3.6698	4.1931
$\widehat{\mathbf{M}}_{\mathbf{T6}}^{\eta\circledast}$	29.7554	29.784	$\widehat{\mathbf{T}}_{\square\mathbf{12}}^{\eta}$	3.1459	4.2732	$\widehat{\mathbf{T}}_{\circledR\mathbf{12}}^{\eta}$	3.8418	4.1980	$\widehat{\mathbf{T}}_{\copyright\mathbf{12}}^{\eta}$	3.6700	4.1934

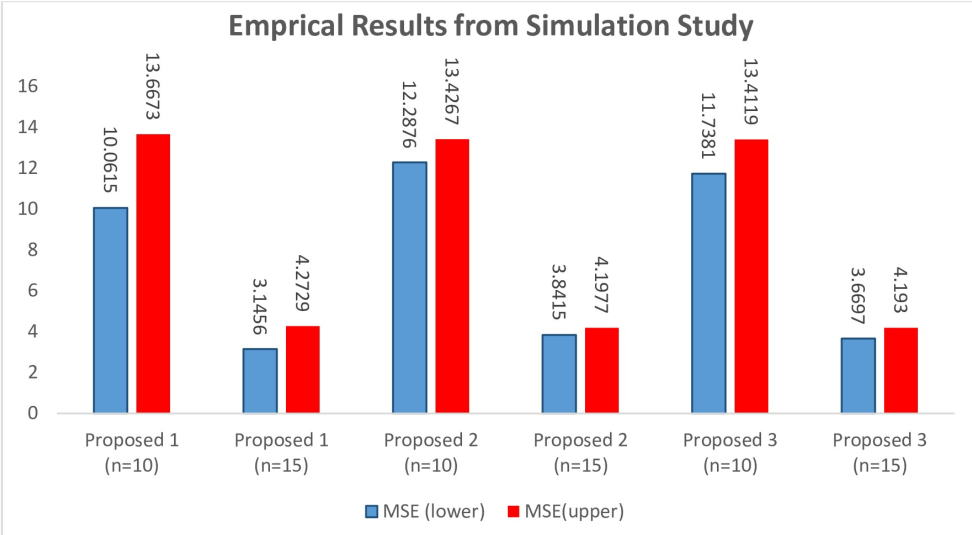

Figure 4: Graphical representation of simulation study results comparing proposed and traditional estimators

5 Summary of Findings

In this study, we introduced a generalized ratio estimator for the population median based on quantile regression within the neutrosophic statistical framework. The proposed estimator, denoted as $\widehat{T}^{\eta}_{(Q)m}$ demonstrated superior performance over existing traditional estimators across various evaluation metrics. In particular, it achieved consistently lower Mean Squared Error (MSE), especially as sample sizes increased. The method also exhibited strong robustness against outliers, offering a more reliable measure of central tendency in datasets characterized by extreme values or inherent variability.

To validate its practical utility, we applied the proposed estimator to real-world stock price data from Samsung Electronics Co. Ltd. and supported our findings through simulation studies. These applications underline the estimator's effectiveness in handling imprecise and uncertain data, a key advantage of the neutrosophic approach. Overall, this research contributes meaningfully to robust median estimation, offering a versatile solution for complex data environments encountered in finance and other applied fields.

Data Availability Statement

The data supporting the findings of this study are fully available within the manuscript.

Declaration of Competing Interests The authors affirm that there are no financial interests or personal affiliations that could potentially impact the integrity of the research presented in this paper.

References

Abid, M., Sherwani, R. A., Tahir, M., Nazir, HZ, and Riaz, M. (2021) 'An improved and robust class of variance estimator', *Scientia Iranica*, 28(6): 3589–3601.

Bahl, S. and Tuteja, R. K. (1991) 'Ratio and product type exponential estimators', *Journal of Information and Optimization Sciences*, 12(1): 159–164.

Francisco, C. A. and Fuller, W. A. (1991), 'Quantile estimation with a complex survey design', *The Annals of Statistics*, 19(1): 454–469.

Gross, S. (1980) 'Median estimation in sample surveys', in *Proceedings of the Section on Survey Research Methods* (Vol. 1814184). American Statistical Association: Ithaca, NY.

Gupta, S., Shabbir, J. and Ahmad, S. (2008) 'Estimation of median in two-phase sampling using two auxiliary variables', *Communications in Statistics - Theory and Methods*, 37(11): 1815–1822.

Hussain, M. A., Javed, M., Zohaib, M., Shongwe, S. C., Awais, M., Zaagan, A. A. and Irfan, M. (2024) 'Estimation of population median using bivariate auxiliary information in simple random sampling', *Heliyon*, 10(7).

Irfan, M., Javed, M. and Lin, Z. (2018) 'Efficient ratio-type estimators of finite population mean based on correlation coefficient', *Scientia Iranica*, 25(4): 2361–2372.

Irfan, M., Javed, M., Shongwe, S. C., Zohaib, M. and Haider Bhatti, S. (2021) 'Estimation of population median under robust measures of an auxiliary variable', *Mathematical Problems in Engineering*, 2021(1): 4839077.

Koenker, R. and Hallock, K. F. (2001) 'Quantile regression', *Journal of Economic Perspectives*, 15(4): 143–156.

Kuk, A. Y. and Mak, T. K. (1989) 'Median estimation in the presence of auxiliary information', *Journal of the Royal Statistical Society: Series B (Methodological)*, 51(2): 261–269.

Masood, S., Ibrar, B., Shabbir, J., Shokri, A. and Movaheedi, Z. (2024) 'Estimating neutrosophic finite median employing robust measures of the auxiliary variable', *Scientific Reports*, 14(1): 10255. DOI: 10.1038/s41598-024-10255

Shabbir, J. and Gupta, S. (2017) 'A generalized class of difference type estimators for population median in survey sampling', *Hacettepe Journal of Mathematics and Statistics*, 46(5): 1015–1028.

Shahzad, U., Hanif, M., Sajjad, I. and Anas, M. M. (2022) 'Quantile regression-ratio-type estimators for mean estimation under complete and partial auxiliary information', *Scientia Iranica*, 29(3): 1705–1715.

Singh, S., Joarder, A. H. and Tracy, D. S. (2001) 'Median estimation using double sampling', *Australian and New Zealand Journal of Statistics*, 43(1): 33–46.

Singh, S. (2003) *Advances Sampling Theory and Applications: How Michael Selected Amy*, Volumes I and II. Kluwer Academic Publishers: The Netherlands.

Smarandache, F. (1998) *Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis*.

Samsung Electronics Co., L. (2022) 'SMSN.IL Stock Price, News, Quote & History', Yahoo Finance. Available at: `https://finance.yahoo.com/quote/SMSN.IL/history?p=SMSN.IL` (Accessed: 31 July 2025).

Zohaib, M., Latif, W. and Alam, M. (2024) 'On enhanced ratio-type estimators using quantile regression for finite population mean under robustness and empirical validation', *Iran Journal of Science*. DOI: 10.1007/s40995-024-01700-1