J.Natn.Sci.Foundation Sri Lanka 2025 53 (4): 317 - 331
DOI: http://dx.doi.org/10.4038/jnsfsr.v53i4.12645

RESEARCH ARTICLE

Graph Theory

An upper bound for star chromatic index of simple connected sub-cubic graphs and applications

CLR Fernando* **and AMCUM Athapattu**
Department of Mathematics, Faculty of Science, University of Peradeniya, Peradeniya.

Submitted: 24 April 2025; Revised: 22 July 2025; Accepted: 29 July 2025

Abstract: This paper explores the star edge coloring of simple connected sub-cubic graphs, which is a more restricted way of edge coloring. The idea of star edge coloring was introduced by two mathematicians Liu and Deng in 2008, motivated by its vertex version. Since then, star edge coloring has been studied extensively by many researchers. Computing the star chromatic index $\chi'_{st}(G)$ for a graph G is a challenging problem, and finding an algorithm for it is an active area of research in graph theory. So, numerous studies have been conducted introducing upper bounds for the star chromatic index. One of the primary objectives of this research is to establish a better upper bound for the star chromatic index of simple connected sub-cubic graphs, partially answering the conjecture in Lužar et al., 2017, posed by Dvorak, et al. in 2013: "If G is a sub-cubic graph, then $\chi'_{st}(G) \leq 6$". The method of star edge coloring for 2-connected graphs, used in this paper mainly based on the decomposition of the graph into a matching and a 2-factor. That can also be expanded into real-world situations, like branching companies, and planning cities.

Keywords: 2-connected graphs, simple graphs, star chromatic index, star edge coloring, sub-cubic graphs.

INTRODUCTION

Graph theory provides the very foundation for modern mathematics, which furnishes a simple and appropriate framework through which problems containing relationships and connections between individuals or things could be modeled and resolved. Since its inception with the great Swiss mathematician Leonhard Euler's solution to the famous Koningsberg bridge problem in 1736, graph theory has developed into a cornerstone of modern mathematics, fostering both pure and applied mathematics.

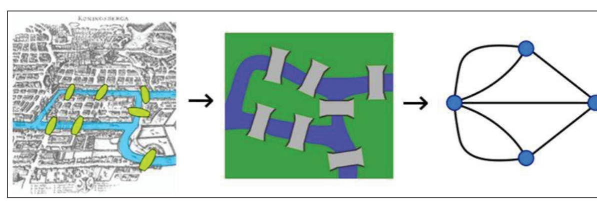


Figure 1: Graph Theory approach for the Koningsberg bridge problem

Definition 1. A simple graph G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set of distinct vertices and $E(G)$ is a set of edges with no multi-edges or loops.

Definition 2. A cycle is a closed path in which the only vertex that repeats is the starting vertex.

Definition 3. A graph is called connected if it cannot be expressed as a union of disjoint graphs.

* Corresponding author (lakshithaf@sci.pdn.ac.lk; https://orcid.org/0009-0004-5716-8231)

A graph is called 2-connected if the graph obtained by deleting a vertex with its incident edges is also connected.

Definition 4. A simple graph in which every vertex has the same degree is called regular.

Definition 5. A 3-regular graph is called cubic, and a graph with maximum degree 3 is called sub-cubic (Lužar et al., 2017).

Among the many topics that fall under the wide scope of graph theory, one of the most popular and significant is graph coloring, which is an assignment of "colors" to elements of a graph according to some given constraints. The interesting problems of graph coloring originated in a question concerning the coloring of regions on a map which led to the four-color problem. It was solved in 1976 and introduced the well-known four-color theorem by Appel, Haken, and Koch.

Theorem 1. (Four-Color Theorem) (Bona, 2016) Every simple planar graph is 4-colorable

Definition 6. An edge coloring of a graph is a function assigning colors to the edges of the graph in such a way that any two adjacent edges receive different colors (Verma & James, 2019).

The chromatic index of a graph G, $\chi(G)$ is the minimum k such that G is edge k-colorable.

A graph G is considered to be properly vertex (or edge) colored if two adjoining vertices (or edges) are colored uniquely. Star coloring introduced by Grunbaum (1973) is a path-based coloring in which no path of length 3 is bi-colored. The name "star coloring" was derived from the fact that in a star coloring, the induced sub-graphs formed by the vertices of any two colors have connected components that are star graphs.

Definition 7. A star coloring is a proper vertex coloring in which there are no bi-colored paths or cycles on more than 3 vertices. Star chromatic number, $\chi_{st}(G)$ is the minimum k such that G admits a k-star-coloring (Horacek, 2019).

Following this, Fertin et al. (2004) observed that in a star vertex coloring any bi-chromatic subgraph is in the form of a galaxy. Where a galaxy is a forest composing only of stars according to Gallian (2022). This coloring was introduced along with the acyclic coloring of graphs and several results of star coloring in comparison with the acyclic coloring of graphs can be noted in Albertson et al. (2004) and Brause et al. (2022). A similar concept to star coloring, but applied to the edges, was introduced by Liu and Deng (2007).

Definition 8. Let $f: E(G) \to [k]$ be a proper edge coloring of a graph G. If f is a star (vertex) coloring of $L(G)$, then f is a star edge coloring of G. (Horacek, 2019).

Equivalently, a proper edge coloring of a graph G is called a star edge coloring if there are neither bi-chromatic paths nor bi-chromatic cycles of length four. (Bezegova et al., 2013; Lužar et al., 2017)

The minimum number of colors for which G admits a star edge coloring is called the star chromatic index and it is denoted by $\chi'_{st}(G)$.

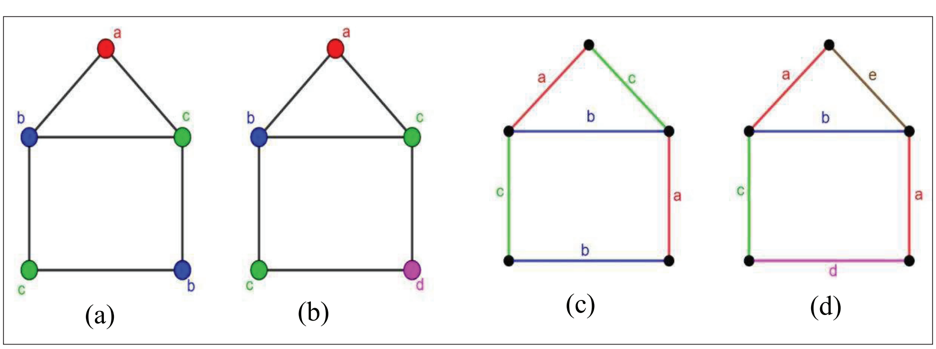


Figure 2: (a)Vertex coloring, (b)Star vertex coloring, (c)Edge coloring and (d)Star edge coloring of the House graph

In (a), the rectangle forms a cycle of 4 vertices which is bi-colored. Hence it is not a star vertex coloring. To obtain a star vertex coloring, the color of the bottom right vertex is changed as shown in (b). Hence, $\chi_{st}(G) = 4$.

In (c), there are two paths of length 4 which are bi-colored (the red-blue path and green-blue path). Hence it is not a star edge coloring. To obtain a star edge coloring, the colors of two edges are changed (one from each bi-colored path) as shown in (d). Hence $\chi'_{st}(G) = 5$.

Through the years, several results and theorems have emerged in said topic which can be seen in Bezegova et al. (2016), Lei and Shi (2020), Casselgren et al. (2021), and Evangeline Lydia and Vijaya Xavier Parthipan (2022), etc.

In 2007 and 2008, the pioneers of star edge coloring Liu & Deng worked on finding upper bounds for some types of graphs. Also, they gave an upper bound for any graph of the maximum degree Δ.

Theorem 2. For the path graph P_n $(n \geq 2)$ and the cycle graph C_n $(n \geq 3)$ (Lužar et al., 2017),

$$\chi'_{st}(P_n) = \begin{cases} 1, & n = 2 \\ 2, & n = 3,4 \\ 3, & n \geq 5 \end{cases} \quad \text{and} \quad \chi'_{st}(C_n) = \begin{cases} 3, & n \neq 5 \\ 4, & n = 5 \end{cases}$$

For any integer $n \geq 4$, the wheel graph W_n (resp. the fan graph F_n) is the n-vertex graph obtained by joining a vertex to each of the $n - 1$ vertices of the cycle graph C_{n-1} (resp. the path graph P_{n-1}). (Lei & Shi, 2020)

Theorem 3. For W_n $(n \geq 4)$ and F_n $(n \geq 4)$ (Lužar et al., 2017),

$$\chi'_{st}(W_n) = \begin{cases} n+2, & n = 5 \\ n+1, & n = 4,6,7 \\ n-1, & n \geq 8 \end{cases} \quad \text{and}$$

$$\chi'_{st}(F_n) = \begin{cases} n+1, & n = 5 \\ n, & n = 3,4,6,7 \\ n-1, & n \geq 8 \end{cases}$$

Theorem 4. Let G be a graph with maximum degree $\Delta \geq 7$. Then, $\chi'_{st}(G) \leq \lceil 16(\Delta - 1)^{3/2} \rceil$ (Liu & Deng, 2008).

Theorem 5. Every graph G with $\Delta = 4$ satisfies $\chi'_{st}(G) \leq 14$ (Wang et al., 2019).

In 2013, Dvorak et al. worked on finding an upper bound for star chromatic index of complete graphs and later by Melinder (2020) and Casselgren et al. (2021) developed that upper bound.

Theorem 6. The star chromatic index of the complete graph K_n satisfies (Dvořák et al., 2013).

$$2n(1 + O(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+O(1))\sqrt{\log n}}}{(\log n)^{1/4}}$$

Theorem 7. If G is a bipartite graph, then $\chi'_{st}(G) \leq \Delta^2 - \Delta + 1$ (Melinder, 2020).

Theorem 8. For any bipartite graph $K_{r,d}$ with $r, d \geq 1$,

$\chi'_{st}(K_{r,d}) \leq 15\lceil d/8 \rceil \lceil r/8 \rceil$ (Casselgren et al., 2021).

The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth (Lei & Shi, 2020).

Theorem 9. Let G be a sub-cubic planar graph with girth at least 7. If G has a perfect matching, then $\chi'_{st}(G) \leq 6$ (Casselgren et al., 2021).

In 2016, Bezegova et al. worked on star edge coloring of trees, and outerplanar graphs. A **tree** is a connected undirected graph with no cycles.

Theorem 10. Let *T* be a tree with maximum degree Δ. Then $\chi'_{st}(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor$ (Bezegova et al., 2013).

Theorem 11. Let *T* be a sub-cubic tree. Then $\chi'_{st}(G) \leq 4$ (Bezegova et al., 2013).

A graph G is outerplanar if and only if it contains no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$, where x denotes an edge of K_4 (Syszo, n.d.).

Theorem 12. Let G be an outer-planar sub-cubic graph. Then $\chi'_{st}(G) \leq 5$ (Bezegova et al., 2013).

A cactus is a connected graph *H* such that any two cycles in *H* have at most one vertex in common, and hence, in the case of sub-cubic graphs, any two cycles in *H* are vertex-disjoint.

A cactus is a (multi)graph in which every block is a cycle or an edge (Das, n.d.).

Theorem 13. If G is a cactus with maximum degree Δ, then $\chi'_{st}(T) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ (Omoomi et al., 2019).

In 2013, Dvorak et al. introduced the best known upper bound for star chromatic index of sub-cubic graphs, and referring to that in 2019, Luzar et al. obtained the best known upper bound for list star chromatic index of sub-cubic graphs. Those two theorems and their proofs are the main references used to find the new best upper bound for star chromatic index in this work.

Theorem 14. If G is a sub-cubic graph, then $\chi'_{st}(G) \leq 7$ (Dvořák et al., 2013).

Dvorak et al. (2013) observed that all sub-cubic graphs they have encountered could be colored using only 6 colors. So, they posed the following conjecture and problems.

Conjecture 1. If G is a sub-cubic graph, then $\chi'_{st}(G) \leq 6$.

Question 1. Is it true that $ch'_{st}(G) \leq 7$ for every sub-cubic graph G? (Perhaps even ≤ 6?)

Question 2. Is it true that $ch'_{st}(G) = \chi'_{st}(G)$ for every graph G?

The Question 1 is explicitly proved in (Lužar et al., 2017). The concepts they used and the observations we made are discussed below.

Theorem 15. If G is a sub-cubic graph, then $ch'_{st}(G) \leq 7$ (Lužar et al., 2017).

Theorem 16. Let G be a sub-cubic graph without bridges and with at most one vertex of degree at most 2. Then, there is a matching M that covers all vertices of degree 3 (Lužar et al., 2017).

Note that a set of edges that do not have common vertices is called a matching and a 2-factor graph is a subgraph of a given graph which contains disjoint cycles. A connector (aka bridge) is a single edge that connects two cycles.

The supergraph $\hat{G}$ of a 2-connected sub-cubic graph G is the graph obtained by introducing edges between pairs of vertices of degree 2 in G until it contains at most one vertex of degree 2 (multiple edges are valid but not loops) (Lužar et al., 2017).

Since there is always an even number of odd vertices in a graph, an eventual vertex v of degree 2 from the Theorem 16 is never covered by M' and hence, as an immediate corollary of the above we have that $\hat{G}$ contains a 2-factor F where, clearly, v is a part of a cycle. Connecting the cycles of F with the minimum number of edges of M such that the resulting graph is connected, we obtain a spanning cactus ψ of $\hat{G}$.

Theorem 17. A supergraph $\hat{G}$ of any 2-connected sub-cubic graph G can be decomposed into a spanning cactus ψ and a matching M' such that every vertex of $\hat{G}$ is contained in some cycle of ψ (Lužar et al., 2017).

Observation 1. Referring to the proof of Theorem 17, for any connected sub-cubic graph G, obtain the supergraph $\hat{G}$, decompose it into a spanning cactus ψ and a matching M'. Color the edges in M' so that edges at distance at most 2 receive distinct colors. Since there are at most 4 edges of M'at distance 2 from any $e \in E(M')$, at most 4 colors are needed to color M'. Hence, it is possible to color the cactus if there are 3 remaining colors after the matching is colored, as any cycle (except C_5) can be star edge colored with 3 colors. So, the method in (Lužar et al., 2017) is possible if there are 7 colors. But for 6 colors, this would not work as any cycle cannot be star edge colored with only 2 colors.

Algorithm for star edge coloring of a simple connected sub-cubic graph

Let Q_3 be the collection of simple connected sub-cubic graphs. For $G \in Q_3$,

- Step I: Obtain a collection of disjoint cycles $\mathcal{C}(G)$ (or a collection of disjoint cycles and paths) such that each vertex of G is contained in exactly one cycle (or path) $C \in \mathcal{C}(G)$. Let $\mathcal{M}(G)$ be the collection of remaining edges of G (which contains paths of length at most 2) such that $E(\mathcal{C}(G)) \cup E(\mathcal{M}(G)) = G$ and $E(\mathcal{C}(G)) \cap E(\mathcal{M}(G)) = \emptyset$.
- Step II: Color $e_{mi}, e_{mj} \in E(\mathcal{M}(G))$ such that if $d(e_{mi}, e_{mj}) \leq 2$, both edges get distinct colors, say α_k for $k \leq 6$.
- Step III:
  - ∗ Use the same α_k color of $e_{mi} \in E(\mathcal{M}(G))$ for $e_{ci} \in E(\mathcal{C}(G))$ where $d(e_{mi}, e_{ci}) \geq 3$.
  - ∗ Color the possible edges of $\mathcal{C}(G)$ using α_k colors such that $\nexists\, e_i, e_j \in E(G)$ with $d(e_i, e_j) \leq 2$ having the same color.
  - ∗ Color the remaining edges considering star edge coloring conditions using $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ colors. This is always possible according to the Theorem 2.

Algorithm for the coloring of $\mathcal{M}(G)$:

Define $f: E(\mathcal{M}(G)) \to \{\alpha_k\}$ where $1 \leq k \leq 6$.

- Let $f(e_{m1}) = \alpha_1$.
- If $d(e_{m1}, e_{m2}) \geq 3$, $f(e_{m2}) = \alpha_1$.
  - If $d(e_{m1}, e_{m3}) \geq 3$, $d(e_{m2}, e_{m3}) \geq 3$, $f(e_{m3}) = \alpha_1$.
    - If $d(e_{m1}, e_{m4}) \geq 3$, $d(e_{m2}, e_{m4}) \geq 3$, $d(e_{m3}, e_{m4}) \geq 3$, $f(e_{m4}) = \alpha_1$.
    - If $d(e_{m1}, e_{m4}) \leq 2$, $d(e_{m2}, e_{m4}) \geq 3$, $d(e_{m3}, e_{m4}) \geq 3$, $f(e_{m4}) = \alpha_2$.
    - If $d(e_{m1}, e_{m4}) \geq 3$, $d(e_{m2}, e_{m4}) \leq 2$, $d(e_{m3}, e_{m4}) \geq 3$, $f(e_{m4}) = \alpha_2$.
    - If $d(e_{m1}, e_{m4}) \geq 3$, $d(e_{m2}, e_{m4}) \geq 3$, $d(e_{m3}, e_{m4}) \leq 2$, $f(e_{m4}) = \alpha_2$.
    - If $d(e_{m1}, e_{m4}) \leq 2$, $d(e_{m2}, e_{m4}) \leq 2$, $d(e_{m3}, e_{m4}) \geq 3$, $f(e_{m4}) = \alpha_2$.
    - If $d(e_{m1}, e_{m4}) \leq 2$, $d(e_{m2}, e_{m4}) \geq 3$, $d(e_{m3}, e_{m4}) \leq 2$, $f(e_{m4}) = \alpha_2$.
    - If $d(e_{m1}, e_{m4}) \geq 3$, $d(e_{m2}, e_{m4}) \leq 2$, $d(e_{m3}, e_{m4}) \leq 2$, $f(e_{m4}) = \alpha_2$.
    - If $d(e_{m1}, e_{m4}) \leq 2$, $d(e_{m2}, e_{m4}) \leq 2$, $d(e_{m3}, e_{m4}) \leq 2$, $f(e_{m4}) = \alpha_2$.
  - If $d(e_{m1}, e_{m3}) \leq 2$, $d(e_{m2}, e_{m3}) \geq 3$, $f(e_{m3}) = \alpha_2$.
  - If $d(e_{m1}, e_{m3}) \geq 3$, $d(e_{m2}, e_{m3}) \leq 2$, $f(e_{m3}) = \alpha_2$.
  - If $d(e_{m1}, e_{m3}) \leq 2$, $d(e_{m2}, e_{m3}) \leq 2$, $f(e_{m3}) = \alpha_2$.
- If $d(e_{m1}, e_{m2}) \leq 2$, $f(e_{m2}) = \alpha_2$.
  - If $d(e_{m1}, e_{m3}) \geq 3$, $d(e_{m2}, e_{m3}) \geq 3$, $f(e_{m3}) = \alpha_1 \text{ or } \alpha_2$.
  - If $d(e_{m1}, e_{m3}) \leq 2$, $d(e_{m2}, e_{m3}) \geq 3$, $f(e_{m3}) = \alpha_2$.
  - If $d(e_{m1}, e_{m3}) \geq 3$, $d(e_{m2}, e_{m3}) \leq 2$, $f(e_{m3}) = \alpha_1$.
  - If $d(e_{m1}, e_{m3}) \leq 2$, $d(e_{m2}, e_{m3}) \leq 2$, $f(e_{m3}) = \alpha_3$.

An upper bound for star chromatic index of simple connected sub-cubic graphs

Here, we present the theorem regarding the new upper bound for the star chromatic index of simple connected sub-cubic graphs ($G \in \mathcal{Q}_3$) and develop an extended proof using the concepts in (Lužar et al., 2017).

Theorem 18. If G is a simple connected sub-cubic graph, then $\chi'_{st}(G) \leq 6$.

Definition 9. Define the distance between two edges $e_1, e_2 \in E(G)$ by $d(e_1, e_2) = N + 1$ where N is the number of edges between e_1 and e_2 in the shortest path.

Lemma 1. Let G_0 be the 3-regular graph of 10 vertices such that $v_i v_j \in E(G_0)$ for $v_i \in V(G_0)$ for $i = 0, 1, \ldots, 9$ whenever $j = \begin{cases} i+1, i+7, i+9 : i - even \\ i+1, i+3, i+9 : i - odd \end{cases}$ under $modulo$ 10. Then $\chi'_{st}(G) \leq 6$.

Proof: Let G_0 be the 3-regular graph with 10 vertices satisfying the condition; $v_i v_j \in E(G_0)$ *for* $v_i \in V(G_0)$ for $i = 0, 1, \ldots, 9$ whenever $j = \begin{cases} i+1, i+7, i+9 : i - even \\ i+1, i+3, i+9 : i - odd \end{cases}$ under $modulo$ 10.

Since $\deg(v) = 3$ for all $v \in V(G_0)$, G_0 is a supergraph. According to Theorem 17, G_0 can be decomposed into a cactus and a matching. Consider the decomposition of G_0 into the cycle $C(G_0)$ that contains each vertex, and the matching $M(G_0)$ consisting of five disjoint edges as shown in Figure 3. Observe that $d(e_i, e_j) = 2$ for all e_i, e_j in either $M(G_0)$ or $C(G_0)$ whenever $d(e_i, e_j) \neq 1$. For $M(G_0)$, follow the Step II in the previous algorithm. Since $d(e_i, e_j) = 2$ for all $e_i, e_j \in \mathrm{E}(\mathrm{M}(\mathrm{G}_0))$, assign five distinct colors $\{a, b, c, d, e\}$. Color the possible edges in $C(G_0)$ according to the Step III in the algorithm. While coloring, it is necessary to maintain the condition that $e_i, e_j \in \mathrm{E}(\mathrm{C}(\mathrm{G}_0))$ get distinct colors whenever $d(e_m, e_c) = 3$. Then, for the remaining five non-adjacent edges of $C(G_0)$ use a different color. This will give a star edge coloring for G_0 as shown in Figure 4.

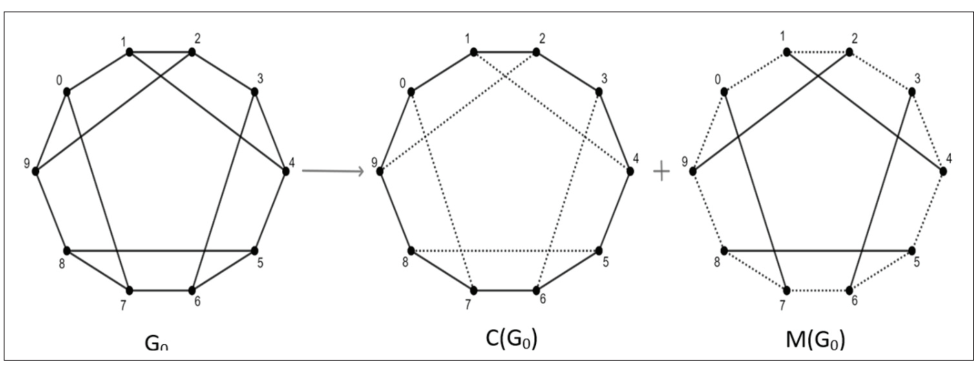


Figure 3: Decomposition of G_0 into the cycle $C(G_0)$ and the matching $M(G_0)$

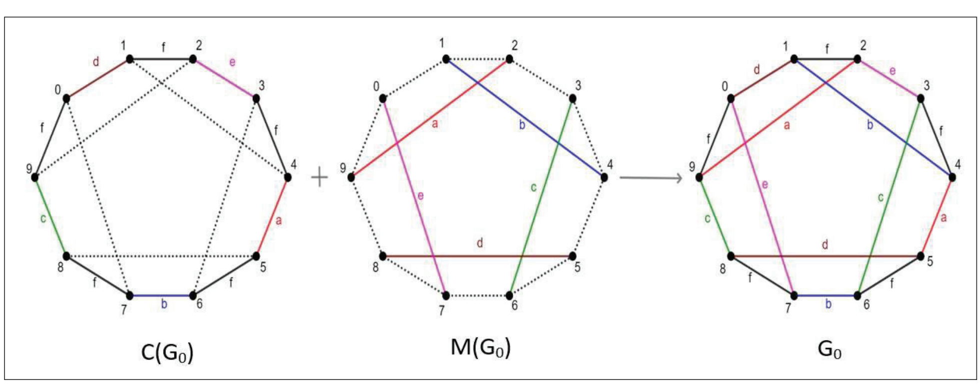


Figure 4: Star edge coloring of the cycle $C(G_0)$ and the matching $M(G_0)$

Corollary 1. The graph $G_0 - e$ where $e \in E(G_0)$ has $\chi'_{st}(G_0 - e) \leq 6$.

Proof: This can be considered as two cases as shown in the Figure 5.

Let $G_0 - e = \begin{cases} G_1: e \in E(\mathrm{M}(G_0)) \\ G_2: e \in E(\mathrm{C}(G_0)) \end{cases}$. Then the result is obvious by the Lemma 1.

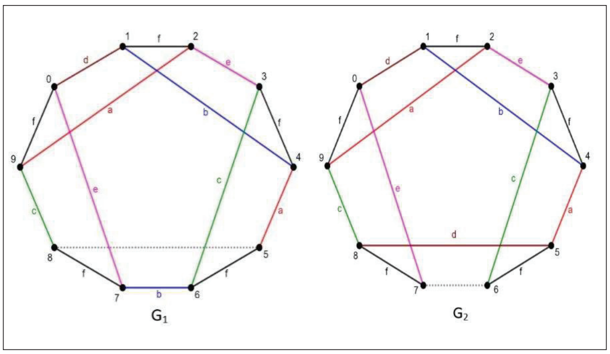


Figure 5: Star edge coloring of G_1 and G_2

Lemma 2. Let G_{1i} be the simple 2-connected sub-cubic graph obtained by adding i vertices to G_1 for $i = 1,2,...,n$. Then $\chi'_{st}(G_{1i}) \leq 6$ for all i.

Proof: Consider as cases by adding finite number of vertices with their adjacent edges to the graph G_1 so that the resulting graph $G_{1i} \in Q_3$ where it has at most two vertices $v_1, v_2 \in V(G_{1i})$ with $deg(v_1), deg(v_2) \leq 2$. i.e. obtain a supergraph or a graph with as many edges as possible under the required conditions.

Let G_{11} be the graph obtained by adding a vertex u to the graph G_1. In order to obtain a simple 2-connected sub-cubic graph, there should be two edges that connect u to G_1 as shown in Figure 6. Color the newly added two edges using the colors of the edges of G_1 at distance 3.

Note 1. If $d(e_1, e_2) = 3$ for $e_1, e_2 \in E(G_0)$, then they are colored with the same color from the list $\{a, b, c, d, e\}$. So, for an edge $e_E \in E(G_{1(i+1)})$ where $e_E \notin E(G_{1i})$, there is always an edge $e \in E(G_{1i})$ which is colored either from the list above or the color f and that guarantees a color for e_E.

For the succeeding cases, after G_{1i} is obtained, break it at the degree 2 vertex to get $G_{1(i+1)}$. So, G_{12} is obtained by breaking G_{11} at u and adding an edge $u - v$. There is only one new edge ($u - v$) to color and by the Note 1, there is always a color either from an edge at distance 3 or from a non-adjacent edge which does not violate the star edge coloring conditions. See the Figure 6. For the consequent two cases; G_{13} and G_{14}, the coloring is shown in Figure 7.

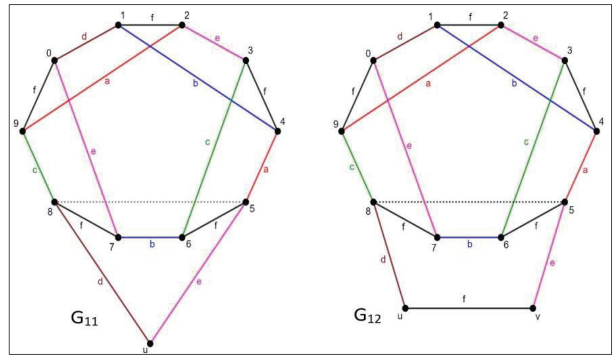


Figure 6: Star edge coloring of G_{11} and G_{12}

When coloring G_{11}, newly added edges should be colored with the colors of the edges at distance 3. If the edge$-5u$ is colored first, there are two possibilities: d or e. If d is chosen for the edge$-5u$, there will be no color for the edge$-8u$ from an edge at distance 3. So, the only possible color for the edge$-5u$ is e, and the edge$-8u$ should be colored with d.

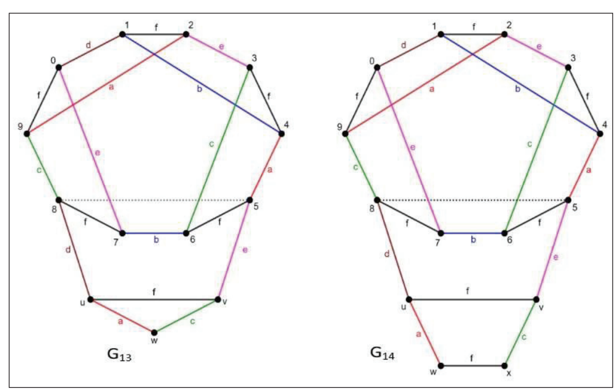


Figure 7: Star edge coloring of G_{13} and G_{14}

Note that, i vertices were added to G_1 in such a way that the resulting graph $G_{1i} \in Q_3$ has at most two vertices $v_1, v_2 \in V(G_{1i})$ such that $\deg(v_1), \deg(v_2) \leq 2$. There, $G_{1(i+1)}$ is obtained by modifying G_{1i}. But there can be few possibilities for G_{1i} according to the way the newly added i vertices are connected. For example, consider G_{16}.

See the Figure 8 which depicts some possible ways to connect newly added 6 vertices to G_1 to obtain simple 2-connected sub-cubic graphs where it has at most two vertices $v_1, v_2 \in V(G_{16})$ such that $\deg(v_1), \deg(v_2) \leq 2$. In a similar way explained above, such cases can also be handled.

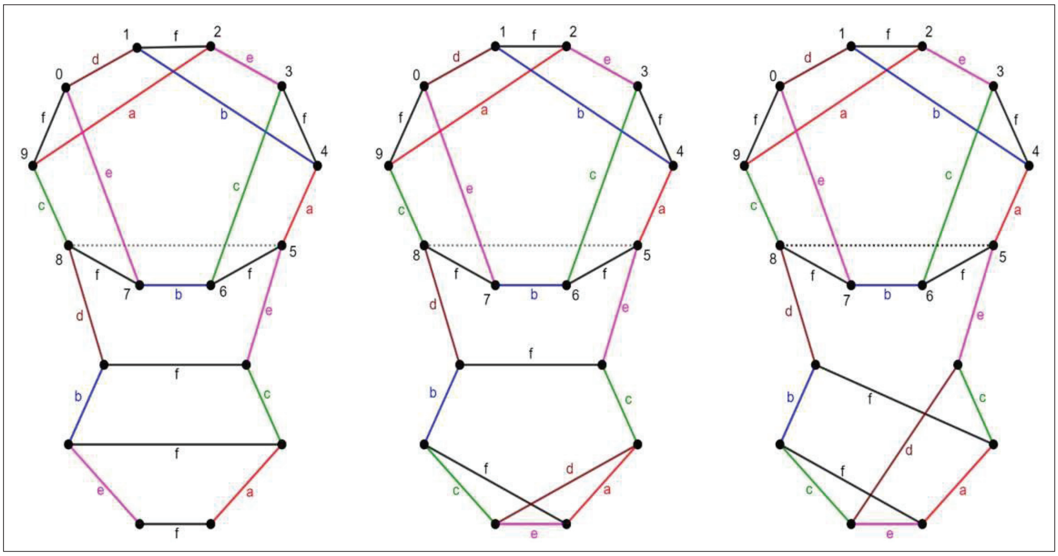


Figure 8: Star edge coloring of some possibilities of G_{16}

Suppose the graph $G_{1i} \in \mathcal{Q}_3$ obtained by adding *i* vertices to G_1 is having more than two vertices $v_m \in V(G_{1i})$ with $\deg(v_m) \leq 2$. Clearly such graphs are subgraphs of the above considered G_{1i}'s. So, those are obviously star edge colorable using 6 colors.

Hence, this expansion can be done for finitely many vertices as far as the resulting graph $G_{1i} \in \mathcal{Q}_3$. So, $\chi'_{st}(G_{1i}) \leq 6$ holds for all $i = 1,2,...,n$.

Lemma 3. Let G_{2i} be the graph obtained by adding *i* vertices to G_2 such that $G_{2i} \in \mathcal{Q}_3$ for all *i* = 1,2,...,*n*. Then $\chi'_{st}(G_{2i}) \leq 6$ for all *i*.

Proof: This can be proved in a similar manner to Lemma 2.

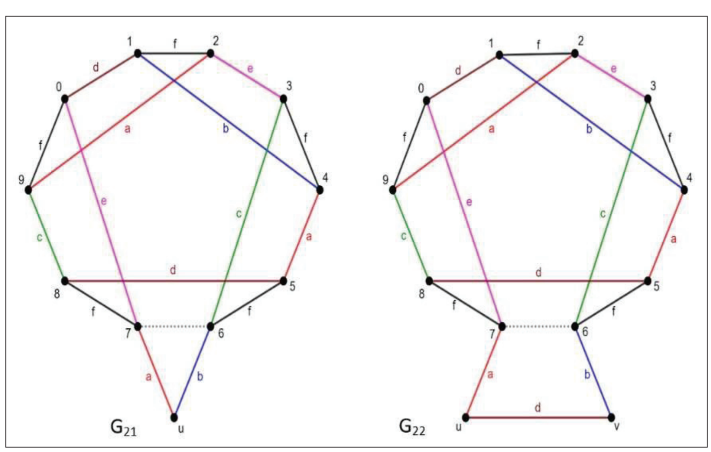


Figure 9: Star edge coloring of G_{20} and G_{22}

When coloring G_{21}, newly added edges should be colored with the colors of the edges at distance 3. Since the edge$-6u$ can only be colored with the color b, the edge$-7u$ should be colored with the color a.

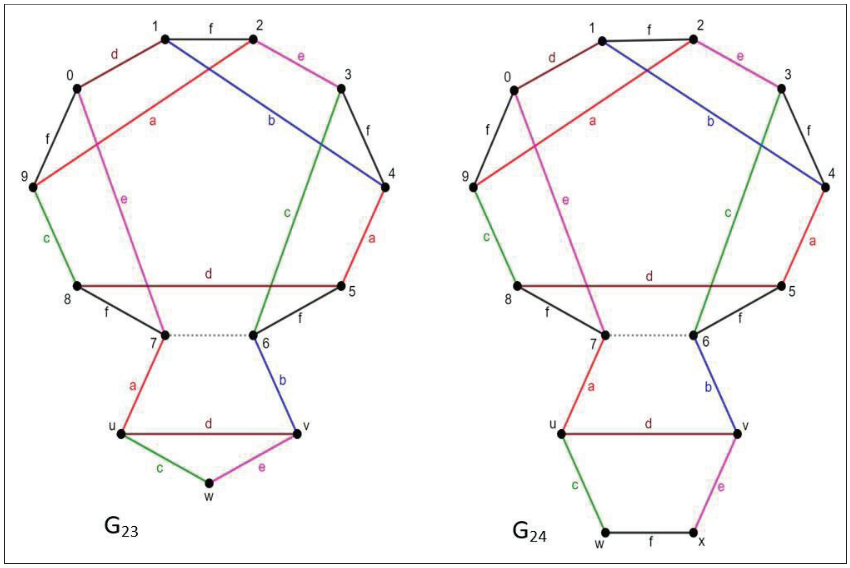


Figure 10: Star edge coloring of G_{23} and G_{24}

Now we present the proof of the main theorem; Theorem 18.

Proof : Consider cases as shown in the following diagram:

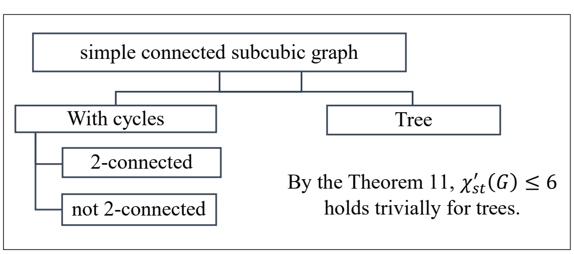


Figure 11: Flow of the proof of the theorem 18

First, suppose that $G \in Q_3$ is 2-connected. Referring to Theorems 15 and 16, if G is a supergraph then it can be decomposed into a matching $\mathcal{M}(G)$ and a collection of disjoint cycles (2-factor) $\mathcal{C}(G)$ such that each vertex of G is contained in exactly one $C \in \mathcal{C}(G)$. If G is not a supergraph, then there is more than one vertex of degree 2 and there are no vertices of degree 1. So, G can be decomposed into a collection of paths of length at most 2; $\mathcal{M}(G)$ and a collection of disjoint cycles and paths; $\mathcal{C}(G)$ such that every vertex of G is contained exactly in one cycle or path of $\mathcal{C}(G)$. That is, if $G \in Q_3$, then it can be decomposed into a collection of paths of length at most 2; $\mathcal{M}(G)$ and a collection of disjoint cycles and paths; $\mathcal{C}(G)$ such that $G = \mathcal{C}(G) \cup \mathcal{M}(G)$ and $E(\mathcal{C}(G)) \cap E(\mathcal{M}(G)) = \emptyset$.

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the set of distinct six colors that are available. In this method, first, a star edge coloring for $\mathcal{M}(G)$ should be given, and then referring to that, a star edge coloring for $\mathcal{C}(G)$ can be obtained. Any $G \in Q_3$ can be categorized considering its' $\mathcal{M}(G)$ as follows;

- Case I: $d(e_i, e_j) \geq 3$ for all $e_i, e_j \in E(\mathcal{M}(G))$
  Use the color α_1 for all $e_i \in E(\mathcal{M}(G))$.Since five colors available for the edges in $\mathcal{C}(G)$, referring to Theorem 2, it is always possible to star edge color. Therefore, $\chi'_{st}(G) \leq 6$.
- Case II: $d(e_i, e_j) = 2$ for $i,j = 1,2$ and $d(e_i, e_j) \geq 3$ for all $i, j \neq 1,2$ where $e_i, e_j \in E(\mathcal{M}(G))$
  Use α_1 and α_2 for the edges e_1 and e_2. Then all the other edges in $\mathcal{M}(G)$ can be colored by either α_1 or α_2. By the Theorem 2, the cycles and paths in $\mathcal{C}(G)$ can be colored by the remaining 4 colors. Therefore, $\chi'_{st}(G) \leq 6$.

- Case III: $d(e_i, e_j) = 2$ for i,j = 1,2,3 and $d(e_i, e_j) \geq 3$ for all $i,j \neq 1,2,3$ where $e_i, e_j \in E(\mathcal{M}(G))$
  Use three distinct colors α_1, α_2 and α_3 for e_1, e_2 and e_3. Use any of those colors to color the remaining edges in $\mathcal{M}(G)$. Three more colors are remaining to color $\mathcal{C}(G)$. Referring to Theorem 2, if $C_5 \notin \mathcal{C}(G)$, then $\mathcal{C}(G)$ can be star edge colored. If $C_5 \in \mathcal{C}(G)$, then consider the three possibilities shown in Figure 12.

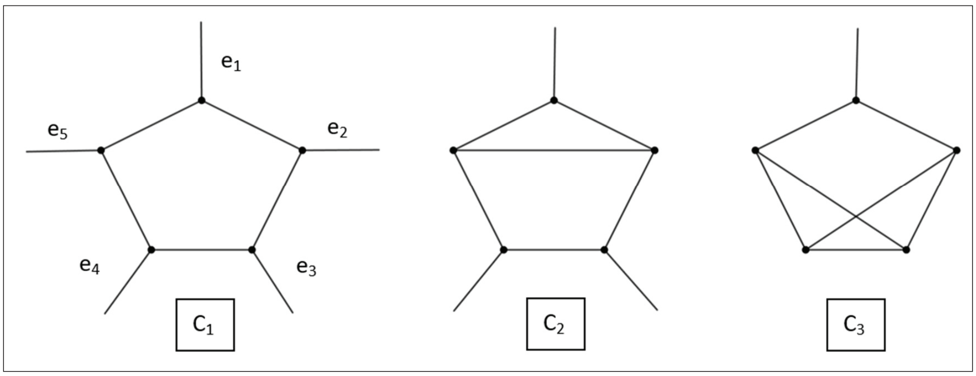


Figure 12: Possible arrangements of edges in $M(G)$ with regard to the C_5

In C_1, $d(e_i, e_{i+1}) = 2$ and $d(e_i, e_{i+2}) = 3$ where $e_i \in E(\mathcal{M}(G))$ for i,j = 1,2,3. Hence, the pairs of edges e_1, e_4 and e_3, e_5 can be colored using two colors and another color should be used for e_2. So only three colors are needed to color the edges in $\mathcal{M}(G)$. Note that, the only edge in $\mathcal{M}(G)$ that is colored with a unique color, say α, is e_2. Also note that, for each $e \in E(\mathcal{M}(G))$,there is an edge $z \in C_5$ such that $d(e,z) = 3$. The edge $z \in C_5$ such that $d(e_2, z) = 3$ receives the same color α. Since already there were three remaining colors for the cycle C_5, now with the color α, there are four colors available for the cycle. Hence it is star edge colorable by the Theorem 2, and consequently, C_1 is star edge colorable with six colors. Note that in C_2 and C_3, there are less edges in $\mathcal{M}(G)$, hence similarly, it can be shown that, the cycle C_5 in both C_2 and C_3 has four available colors. So, even if $C_5 \in \mathcal{C}(G)$, the graph G is still star edge colorable with six colors. Therefore, whether or not $C_5 \in \mathcal{C}(G)$, $\chi'_{st}(G) \leq 6$.

- Case IV: $d(e_i, e_j) = 2$ for i,j = 1,2,3,4 and $d(e_i, e_j) \geq 3$ for all $i,j \neq 1,2,3,4$ where $e_i, e_j \in E(\mathcal{M}(G))$

There are four edges of $\mathcal{M}(G)$ such that all four are at distance 2 from each other. So, use four distinct colors; α_1, α_2, α_3 and α_4 for them. And all the other edges in $\mathcal{M}(G)$ can be colored using any of those four colors. Note that previously, in the Corollary 1 and Lemmas 2, 3 all the possibilities for G having four edges in $\mathcal{M}(G)$ at distance 2 from each other are covered. So, by referring to those, $\chi'_{st}(G) \leq 6$ can be shown.

- Case V: $d(e_i, e_j) = 2$ for i,j = 1,2,3,4,5 where $e_i, e_j \in E(\mathcal{M}(G))$

Note that if there are five edges at distance 2 from each other, since only sub-cubic graphs are considered, the only graph that can be obtained is the 3-regular graph of 10 vertices. One way of illustrating it is given in the Lemma 1, where $d(e_i, e_j) = 2$ for all non-adjacent $e_i, e_j \in E(G)$ and any other 3-regular graph of 10 vertices may have less constraints. So, referring to Lemma 1, any such graph is having $\chi'_{st}(G) \leq 6$.

Throughout all the five cases, all the simple 2-connected sub-cubic graphs were covered, and shown that for any such graph, the result holds.

Finally, consider the simple connected sub-cubic graph having cycles but not 2-connected. i.e. the graph has at least a single bridge (aka connectors) which connects two blocks. So, in this case separate the blocks by deleting the bridge. Then as shown in above, treat the blocks as separate simple 2-connected sub-cubic graphs. Once the blocks are colored, then use a color which is in a non-adjacent edge for the connector such that the star edge conditions are not violated. So, concluding all the cases, it is proved that for any simple connected sub-cubic graph G, $\chi'_{st}(G) \leq 6$, as required.

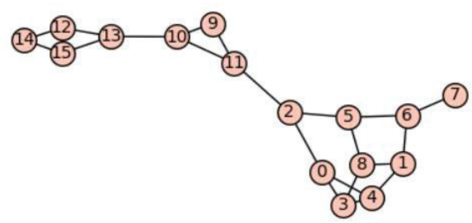
DISCUSSION

The method of star edge coloring introduced earlier can be verified using the Sagemath software. A random simple connected sub-cubic graph can be obtained by Sagemath software using the code provided in the Appendix section. After the graph is colored using the algorithm, it can be observed that the previously introduced upper bound for the star chromatic index holds for any simple connected sub-cubic graphs. Also, by giving the adjacency matrix to the software, one can obtain a suitable graph as shown in the following two examples:

Example 1.

```
M = Matrix([[0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0], [0,0,0,0,1,0,1,0,1,0,0,0,0,0,0,0],
[1,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0], [1,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0], [1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,1,0,0,0,1,0,1,0,0,0,0,0,0,0], [0,1,0,0,0,1,0,1,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0],
[0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,0,0,1,0,1,0,1,0,0],
[0,0,1,0,0,0,0,0,0,1,1,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1], [0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,1],
[0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1], [0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0]])

G2= Graph(M)
show(G2)
```



Code 1 : Sagemath code and output

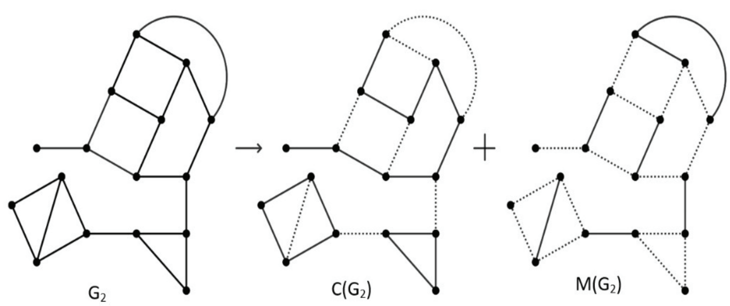


Figure 13: Decomposition of the graph G_2

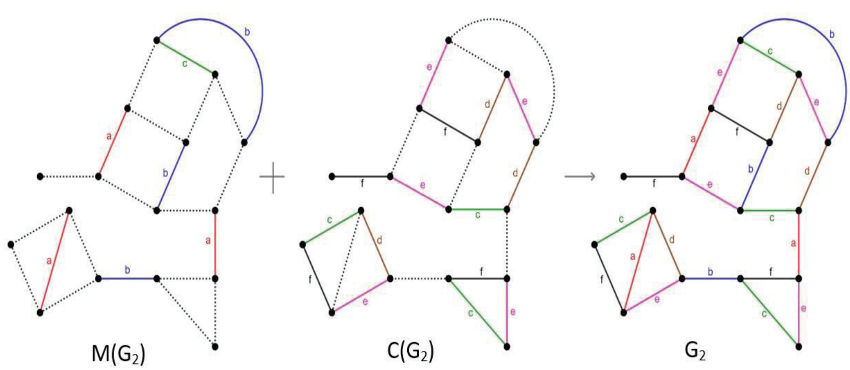
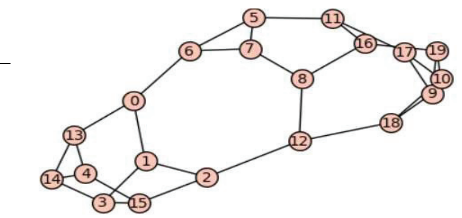


Figure 14: Coloring of the graph G_2

Example 2. Without using an adjacency matrix, a random graph that satisfies the simple, connected, sub-cubic conditions can be obtained by giving the coordinates as shown below.

```
G3 = { 0:[1,6,13], 1:[0,2,3], 2:[1,12,15], 3:[1,14,15], 4:[13,14,15], 5:[6,7,11], 6:[0,5,7], 7:[5,6,8],
8:[7,12,16], 9:[17,18,19], 10:[17,18,19], 11:[16,17,5], 12:[2,8,18], 13:[0,4,14], 14:[3,4,13],
15:[2,3,4], 16:[8,11,19], 17:[9,10,11], 18:[9,10,12], 19:[9,10,16]}

Graph(G3).show()
```



Code 2 : Sagemath code and output

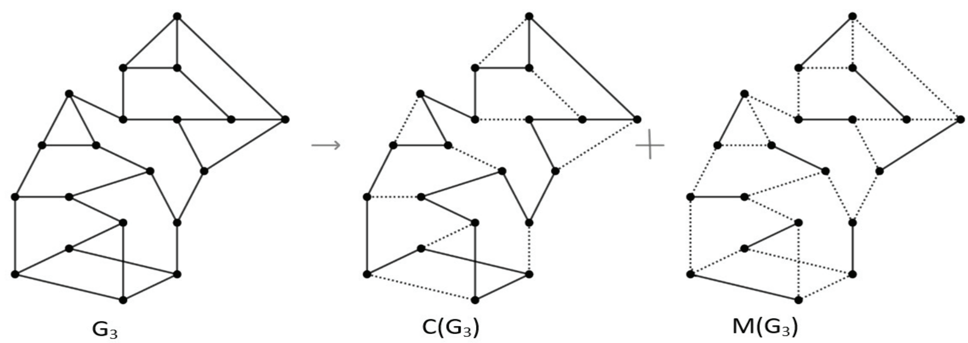


Figure 15: Decomposition of the graph G_3

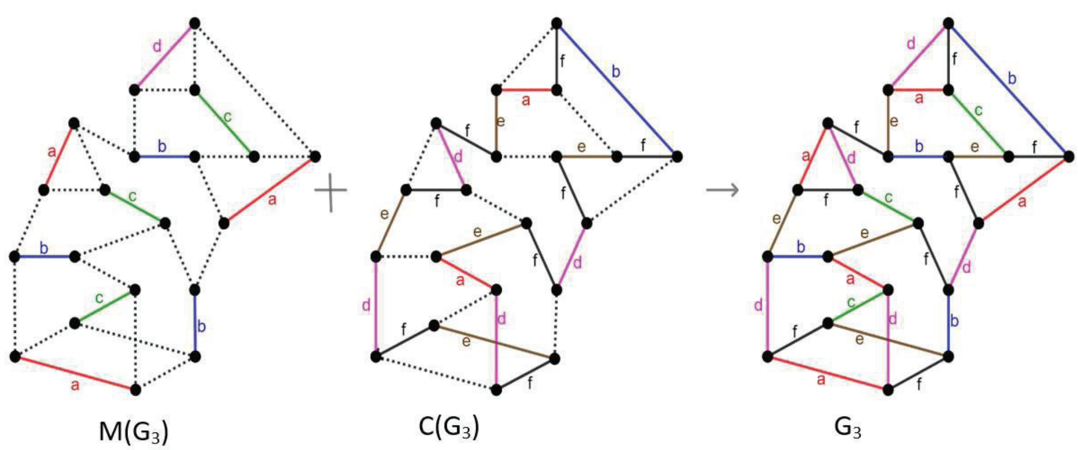


Figure 16: Coloring of the graph G_3

Applications

In real time applications, coloring is used in scheduling, network designing, clustering, etc. For example, in an art gallery for placing the security cameras so that the full gallery is covered coloring is used.

Suppose a clothing brand wants to find the best places to initiate their new branches in a region so that anyone in that neighborhood can visit their outlets. Due to the high capital cost, they can't open a branch in each city. What they want to do is to find places to open new outlets so that anyone from neighboring cities can reach. The area can be illustrated by a graph, where each vertex represents a city and each edge represents a road that connects two cities. Depending on the population and available locations to initiate a new branch in each road, roads can be weighted and then, it can be star edge colored using the introduced algorithm. We can say that the company can open a new outlet in each road with that same color.

To obtain a sub-cubic graph, either the Kruskal's algorithm or the Prim's algorithm can be used with the condition $\deg(v) \leq 2$ for all $v \in V(G)$ where G is the spanning graph with the highest weight. In this application, Kruskal's algorithm is used as it selects the highest weighted edges randomly.

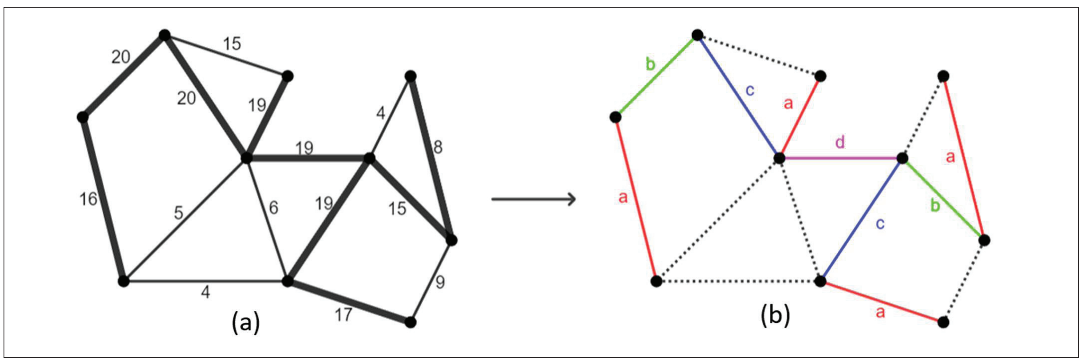


Figure 17: Star edge coloring applied to branching of a clothing brand

(a) The graph represents the area considered. Each vertex denotes a city and an edge denotes a road. The roads are weighted depending on the population and available locations for the shop in each road. The Kruskal's algorithm is used, with the sub-cubic condition is not violated.

(b) The selected graph is star edge colored using the introduced algorithm. The company can initiate new branches on the pink(d) colored edge(road) so that anyone from any city in the region can reach this outlet crossing at most 2 cities, hence this is the most cost effective choice of location to open a new branch.

CONCLUSION

In conclusion, this research has partially answered the conjecture: "If G is a sub-cubic graph, then $\chi'_{st}(G) \leq 6$ " which was posed by Dvorak et al. in 2013. It was successfully proved that above conjecture is true for any simple connected sub-cubic graph. Also, we have introduced a method to obtain a star edge coloring for any simple connected sub-cubic graph.

Our findings not only contribute to the theoretical understanding of graph coloring but also have potential implications for practical applications in areas such as network design and scheduling.

This method of coloring fails when there are loops or multi-edges in the graph. But as observed, we couldn't find any sub-cubic graph that cannot be star edge colored with 6 colors. Hence, it remains as a future direction for this research to prove that $\chi'_{st}(G) \leq 6$ for sub-cubic graphs with loops or multi-edges.

REFERENCES

Bezegová, L., Lužar, B., Mockovčiaková, M., Soták, R., & Škrekovski, R. (2013a). Star edge coloring of some classes of graphs. *Journal of Graph Theory*, 81(1), 73-82. https://doi.org/10.1002/jgt.21862

Bona, M. (2016). *Walk Through Combinatorics: An Introduction To Enumeration And Graph Theory* (4th ed.). *World*

Scientific Publishing Company.

Casselgren, C. J., Granholm, J. B., & Raspaud, A. (2021a). On star edge colorings of bipartite and subcubic graphs. *Discrete Applied Mathematics*, 298, 21–33. https://doi.org/10.1016/j.dam.2021.03.007

Dvořák, Z., Mohar, B., & Šámal, R. (2013). Star chromatic index. *Journal of Graph Theory*, 72(3), 313–326. https://doi.org/10.1002/jgt.21644

Horacek, K. M. (2019). Edge colorings of graphs on surfaces and star edge colorings of Edge colorings of graphs on surfaces and star edge colorings of sparse graphs sparse graphs. *Graduate Thesis, Dessertations, and Problem Reports.* https://doi.org/10.33915/etd.3910

Das, K. (n.d.). Cactus Graphs and Some Algorithms. https://doi.org/10.48550/arXiv.1408.4005

Liu, Xinsheng & Deng, Kai. (2008). An upper bound on the star chromatic index of graphs with $\Delta >= 7$. *Journal of Lanzhou University. Natural Sciences.* 44. https://www.researchgate.net/publication/265945355_An_upper_bound_on_the_star_chromatic_index_of_graphs_with_7

Lužar, B., Mockovčiaková, M., & Soták, R. (2017a). Note on list star edge-coloring of subcubic graphs. *Journal of Graph Theory*, 90(3), 304–310. https://doi.org/10.1002/jgt.22402

Melinder, V. (2020). Upper bounds on the star chromatic index for bipartite graphs. *Linköping University, Department of Mathematics, Mathematics and Applied Mathematics.*. https://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-164938

Omoomi, B., Dastjerdi, M. V., & Yektaeian, Y. (2019). Star edge coloring of Cactus graphs. http://arxiv.org/abs/1911.02343

Verma, S., & James, M. (2019). Chromatic Index, Total Coloring and Diameter of Pan and Tadpole Graphs. *Journal of Computer and Mathematical Sciences*, *10*(6), 1294–1301. https://www.researchgate.net/publication/334820747_Chromatic_Index_Total_Coloring_and_Diameter_of_Pan_and_Tadpole_Graphs

Wang, Y., Wang, Y., & Wang, W. (2019). Star edge-coloring of graphs with maximum degree four. *Applied Mathematics and Computation*, *340*, 268–275. https://doi.org/10.1016/j.amc.2018.08.035

Appendix : The Sagemath code formulated to generate a random simple connected sub-cubic graph is included below.

```
from sage.graphs.graph_generators import graphs
def get_subcubic_connected_graph(vertices, seed=None):
    ""Generate a simple connected graph with maximum degree 3.
    Parameters:
        vertices (int): The number of vertices in the graph.
        seed (int, optional): Random seed for reproducibility.
    Returns:
        Graph: A simple connected graph with maximum degree 3.""
import random
random.seed(seed) # Set seed for reproducibility if provided
while True:
    # Generate a random graph with edges added randomly
    G = graphs.RandomGNP(vertices, 0.3, seed=random.randint(0, 10**6))
    for v in G.vertices():
        while G.degree(v) > 3:
            G.delete_edge(G.random_edge())
    # Check for connectivity
    if G.is_connected() and max(G.degree()) <= 3:
        return G
    # Generate a simple connected subcubicgraph
vertices = 10
subcubic_graph = get_subcubic_connected_graph(vertices)
subcubic_graph.show()
```