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RESEARCH ARTICLE

Mathematical Modelling

Optimal control strategies for diabetes population management using mathematical models.

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Abstract: Diabetes is a long-lasting medical condition that affects the entire world. In the present paper, we modify and analyze an existing mathematical model for diabetes population dynamics, considering control strategies. We presented two models, and the first model was developed with the assumption that the total population grows logistically. The second model was constructed by employing strategies in pursuit of controlling the growth of the diabetic population. Using Pontryagin's maximum principle, we characterized the optimal controls. The optimality systems were solved using the forward-backward sweep iteration with the fourth-order Runge-Kutta method. This shows that the incidence rate of pre-diabetes is a crucial factor in estimating the burden of diabetes. It can also be concluded that by applying control strategies such as awareness of diet plans and regular testing, the number of incidences of pre-diabetes decreases by nearly 45% and the number of diabetes incidences, with and without complications, declined by approximately 75% over the period of 50 years.

Keywords: Diabetes population dynamics, optimal control, Pontryagin's maximum principle.

INTRODUCTION

Diabetes is a chronic disease that substantially affects people's health, and the global pandemic affects all nations. The International Diabetes Federation estimates that 537 million adults worldwide had diabetes in 2021, and one person dies from it every five seconds. Therefore, approximately 17,280 adults die due to diabetes per day worldwide (International Federation of Diabetes, 2025). Diabetes is characterized by issues with the insulin hormone. The World Health Organization claims that diabetes develops when the pancreas does not produce enough insulin or when the body cannot effectively use the insulin that is produced in the pancreas. As a result of this fact, the level of glucose in the blood rises in general (WHO, 2025).

According to the World Health Organization (WHO), there are three main types of diabetes called Type 1, Type 2, and Gestational Diabetes. Type 1 diabetes arises when the pancreas does not produce insulin or does not produce a sufficient amount of insulin. Approximately 5-10% of all diabetes cases are Type 1. Type 2 diabetes occurs when the patient's body does not respond properly to insulin. The majority of diabetics (90–95%) have Type 2. Some people have "prediabetes," which means their blood sugar is high, but not high enough to be diagnosed with type 2 diabetes. Pre-diabetes increases the chances of developing type 2 diabetes. Gestational diabetes occurs when women are pregnant. Although gestational diabetes typically goes away after delivery, it increases the risk of type 2 diabetes in the future.

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Diabetes requires expensive, time-consuming treatment and care due to its chronic nature and serious complications, which have an impact on the whole of society. Therefore, a large number of research studies have been conducted using optimal control strategies to decrease the diabetes burden over the past years. In a related study, in 2020, Abdelfatah Kouidere and co-authors proposed a mathematical model in an effort to reduce the impact of behavioral influences and reduce the number of diabetics with treatable complications (Kouidere *et al.*, 2020). They employed control theory to characterize the optimal control strategies to manage these complications. The effectiveness of the suggested control strategy was demonstrated by the numerical simulation of the results. In 2014, M. Derouich and co-authors presented an optimal control approach to the dynamics of the diabetes population (Derouich *et al.*, 2014). They demonstrated the existence of an optimal control and then monitored the population density in each compartment using an implicit finite-difference method. According to their model, by using optimal control strategies, the number of diabetics with and without complications can be significantly decreased over the course of ten years. In the paper by Kouidere *et al.*, (2021), the authors have considered the negative impact of lifestyle on diabetic patients and presented a model for diabetic population dynamics. On the basis of age, they emphasized how socio-environmental factors affect diabetic patients negatively. Also, they presented an optimal strategy for implementing the best awareness campaigns aimed at protecting diabetes patients from the negative impact of a lifestyle that leads to complications. They used Pontryagin's maximum principle to classify the optimal controls, and an iterative approach was used to solve the optimality problem. MATLAB was utilized to complete the numerical simulation. The effectiveness of the suggested control strategies was demonstrated by the numerical simulation of the results.

T. T. Yusuf proposed a deterministic model for effective health management of diabetic patients using two different control strategies: careful management of patients with complications to keep them from developing further complications, and management of patients before complications arise (Yusuf, 2015). In order to determine the best combination of control strategies that will reduce the cost of putting the suggested strategies into practice, as well as the incidence of complications within the population, they formulated a fixed-time optimal control problem subject to the dynamics of the model. To prove the existence and uniqueness of the optimal control, they have used Pontryagin's maximum principle. Their findings demonstrated that those without complications should be given preference in cases where there are insufficient resources to treat all patients medically in order to keep them free from complications. The growing issue of the diabetes epidemic in Sri Lanka was investigated by P. Katulanda, M. H. R. Sheriff, and D. R. Matthews (Katulanda *et al.*, 2009). They claim that the largest study ever conducted on the prevalence of diabetes in Sri Lanka was released in 2005. It revealed a 14.2% prevalence in men and a 13.5% prevalence in women. They came to the conclusion that Sri Lanka is one of the nations with the highest diabetes prevalence rates in the world and that the data on the prevalence of diabetes in the country clearly shows an upward trend. At the time, an estimated 2.8 million adults in Sri Lanka had diabetes mellitus, with a considerable proportion remaining undiagnosed. A cross-sectional study was conducted on diabetes complications in Sri Lankan adults admitted to two government hospitals by U. Navaseelan and K. Judenimal (Navaseelan & Judenimal, 2017). This study found a significant number of diabetic patients with diabetes mellitus visiting diabetic clinics, in particular hospitals in the Eastern Province of Sri Lanka, as well as co-morbidities and complications related to their diabetes mellitus. Additionally, this study confirms that there is a paucity of patient information regarding diabetes prevention, complications, and treatment. It is unquestionably important to arm patients with the data they need to get the most out of their diabetes treatment.

METHODOLOGY

A Mathematical model for pre-diabetes (E), Diabetes without complications (D) and diabetes with complications (C).

We take into account the model formulated by A. Boutayeb *et al.* (Derouich *et al.*, 2014) by considering three compartments, Pre-diabetes (E), Diabetes free from complications (D) and Diabetes with complications (C).

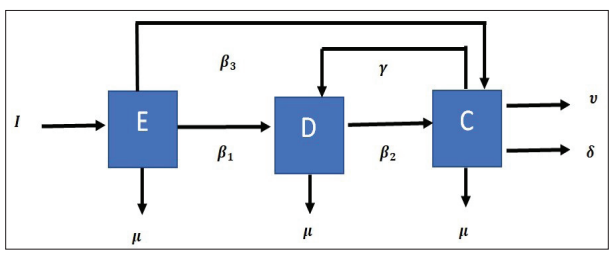

Figure 1: Flow diagram of *EDC* Model.

The flow diagram of the model (*EDC* model) is shown in Figure 1.

The *EDC* model is presented in systems of equations 1.

$$\frac{dE}{dt} = I - (\mu + \beta_3 + \beta_1)E,$$

$$\frac{dD}{dt} = \beta_1 E - (\mu + \beta_2)D + \gamma C, \quad ...(1)$$

$$\frac{dC}{dt} = \beta_3 E + \beta_2 D - (\mu + \gamma + \upsilon + \delta)C.$$

Description of the model variables and parameters is given in Table 1. Unit of each of the model parameter is per year.

The parameter v represents the rate at which individuals with diabetes develop severe disabilities due to complications such as diabetic retinopathy (leading to blindness), diabetic neuropathy (resulting in pain and amputations), diabetic nephropathy (causing kidney failure), cardiovascular diseases (like heart attacks and strokes), diabetic foot diseases (leading to severe infections and amputations), and cognitive impairments (increasing the risk of dementia), critically impacting the population's health burden and the effectiveness of the optimal control strategy aimed at minimizing this burden through the objective functional $J(u)$.

The flow diagram of the model with control strategies is given in Figure 2. Here u is a control variable, in diabetes population models, such as lifestyle modification efforts, health education, early detection programmes, etc.

Table 1: Description of the model variables and parameters

Parameter / variable	Description
E	Cases of pre-diabetes
D	Cases of diabetes free from complications
C	Cases of diabetes with complications
$n = E + D + C$	The population size of diabetes and pre- diabetes
N	The total population size
I	Incidence of pre-diabetes
μ	Natural death rate
β_1	The rate of developing diabetes
β_2	The complication development rate in a diabetic person
β_3	The rate at which people with pre-diabetes develop complications
γ	Curing rate of complications
v	Severely disabling rate of diabetes with complications
δ	The death rate of people with diabetes complications

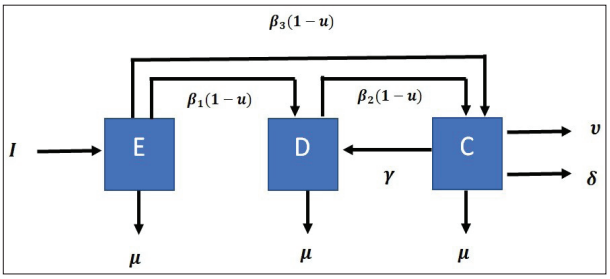

Figure 2: *EDC* model with control u.

$$\frac{dE}{dt} = I - \big(\mu + (\beta_3 + \beta_1)(1-u)\big)E,$$

$$\frac{dD}{dt} = \beta_1(1-u)E - \big(\mu + \beta_2(1-u)\big)D + \gamma C, \quad ...(2)$$

$$\frac{dC}{dt} = \beta_3(1-u)E + \beta_2(1-u)D - (\mu + \gamma + \upsilon + \delta)C,$$

For this optimization process, the objective functional has been constructed by considering the competitive nature of *D, C* with *u*. Managing diabetes at the population level presents a trade-off between intervention costs and health outcomes. Increasing control efforts, such as treatment, screening, or education, can reduce the number of people with diabetes and further with complications, but incur higher immediate costs. In contrast, minimal control reduces short-term costs but increases long-term health and economic burdens. Therefore, an optimal balance between these two concepts is needed to minimize the overall cost of maintaining diabetes within the population. Thus, in this study, the objective functional $J(u)$ is defined in the following manner:

$$J(u) = \int_0^T (D + C + Au^2)dt. \quad ...(3)$$

Where;

A - Positive weight that balances the size of terms,
T - Terminal time of the optimal control process.

Define the set of control by

$U = \{u(t)|u, \quad is\ measurable, \quad 0 \le u(t) \le 1, t \in [0,T]\}$

The aim is to characterize an optimal control $u^* \epsilon U$ satisfying:

$$J(u^*) = \min_{u \epsilon U} J(u).$$

We use a modified version of this mathematical model (Derouich *et al.*, 2014) to explain the dynamics of diabetes. We aim to forecast the impact of control strategies on the population over a long period. However, the model presented by Derouich *et al*. (2014) is not applicable for long-term forecasting because it does not take into account the growth of the total population over time. Additionally, our control strategies differ from those considered by them, to check whether we could get better results compared to the existing study.

Modification of the *EDC* model considering time-dependent incidence of Pre-diabetes *(I).*

In the model (equations 2), the incidence of pre-diabetes is constant throughout the considered time period. But if the considered time period is long, the incidence rate of pre-diabetes could not be constant as the total population grows as time increases. The study indicated about rising diabetes prevalence, with estimates of 2%–6% during the 1990s and 8%–15% in the 2000s–2010s (Prasan Rannan-Eliya *et al*., 2023). Based on the preliminary report of the *Census of Population and Housing 2024 Enumeration Stage*, (2025) , Sri Lanka's total population rose from 14.8 million in 1981 to 21.7 million in 2024, indicating a significant population increase over the period. Furthermore, there is compelling evidence from several studies indicating that the prevalence of prediabetes has significantly grown in Sri Lanka throughout the last two to three decades (Akhtar *et al.*, 2023). Hence, assuming that the incidence of pre- diabetes $I(t)$, at time *t,* is proportional to the total population size $N(t)$, at time *t*, is justifiable and can be used to modify the underlined mathematical model.

Therefore, the *EDC* model can be modified by replacing *I* by αN, where α is the proportional constant. That is,

$$I \propto N \Rightarrow I = \alpha N. \quad ...(4)$$

Moreover, the census data and demographic trends in Sri Lanka over the past few decades reflect the logistic growth model. The population growth rate averaged approximately 2.6% annually before the early 1970s, then declined steadily to below 1% by the early 2000s. This trajectory reflects the characteristic pattern of logistic growth, with an initial phase of rapid acceleration, followed by a peak growth period, and a subsequent deceleration as the population approaches a stable level (*Sri Lanka - Population, Migration, Density | Britannica*, 2025). The logistic growth model takes the following form (Stewart, 2012)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right). \quad ...(5)$$

Here, *r* is the growth rate, and *K* is the carrying capacity of the population. The solution of this model takes the form;

$$N(t) = \frac{K}{1 + e^{-r(t-t_0)}} \quad ...(6)$$

The future total population size in Sri Lanka can be forecasted by estimating the parameters of equation 6 using the available census data. The estimated parameters are $r = 0.0228$, $K = 3.9474 \times 10^7$, and $t_0 = 2007$.

The *EDC* model is modified by replacing *I* by αN is given in the system of equations 7.

$$\begin{aligned} \frac{dE}{dt} &= \alpha N - \big(\mu + (\beta_3 + \beta_1)(1-u)\big)E, \\ \frac{dD}{dt} &= \beta_1(1-u)E - \big(\mu + \beta_2(1-u)\big)D + \gamma C, \\ \frac{dC}{dt} &= \beta_3(1-u)E + \beta_2(1-u)D - (\mu + \gamma + \upsilon + \delta)C. \end{aligned} \quad ...(7)$$

Theorem 1.

The set,

$$\Omega = \left\{(E, D, C)\epsilon\mathbb{R}^3 / \ 0 \leq E, D, C \leq \frac{\alpha N}{\mu}\right\}$$

is positively invariant under system 7, thereby guaranteeing the existence and positivity of solutions within this region.

Proof: This has been proven by Derouich *et al.*, (2014) for the case when *N* is constant. However, the same process can further be used to obtain the result for the case when *N* is a variable quantity since it is a bounded quantity, as it will not exceed the carrying capacity.

Characterization of the optimal control

In order to derive the necessary conditions for optimal control, we can apply Pontryagin's maximum principle (Chambers *et al.*, 1965).

Theorem 2. *Given an optimal control u^* and solutions E^*, D^*, and C^* of the associated state system 7, there exist adjoint variables λ_1, λ_2, and λ_3 satisfying;*

$$\begin{aligned}
\lambda_1{}' &= (\lambda_1 - \lambda_2)(1-u^*)\beta_1 + (\lambda_1 - \lambda_3)(1-u^*)\beta_3 + \mu\lambda_1, \\
\lambda_2{}' &= -1 + \beta_2(1-u^*)(\lambda_2 - \lambda_3) + \mu\lambda_2, \\
\lambda_3{}' &= -1 + (\lambda_3 - \lambda_2)\gamma + \lambda_3(\mu + \nu + \delta).
\end{aligned} \quad ...(8)$$

With transversality conditions: $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$ *and the optimal control:*

$$u^* = min\left(1, max\left(0, \frac{1}{2A}[E^*\beta_1(\lambda_2 - \lambda_1) + E^*\beta_3(\lambda_3 - \lambda_1) + D^*\beta_2(\lambda_3 - \lambda_2)]\right)\right). \quad ...(9)$$

Proof: Hamiltonian *H,* at time *t,* is defined by,

$$H = D + C + Au^2 + \lambda_1 f_1(E^*, D^*, C^*) + \lambda_2 f_2(E^*, D^*, C^*) + \lambda_3 f_3(E^*, D^*, C^*).$$

Here;

$$f_1(E, D, C) = \alpha N - \big(\mu + (\beta_3 + \beta_1)(1-u)\big)E,$$

$$f_2(E, D, C) = \beta_1(1-u)E - \big(\mu + \beta_2(1-u)\big)D + \gamma C,$$

$$f_3(E, D, C) = \beta_3(1-u)E + \beta_2(1-u)D - (\mu + \gamma + \upsilon + \delta)C.$$

Optimality condition u^* is found such that *H* is optimized. The critical value u^* is given by the solution of the equation $\frac{dH}{du} = 0$. That is;

$$2Au + \lambda_1(\beta_1 + \beta_3)E + \lambda_2(-\beta_1 E + \beta_2 D) - \lambda_3(\beta_3 E + \beta_2 D) = 0,$$

$$u^* = \frac{1}{2A}[\beta_1 E^*(\lambda_2 - \lambda_1) + \beta_3 E^*(\lambda_3 - \lambda_1) + \beta_2 D^*(\lambda_3 - \lambda_2)]$$

The adjoint variables λ_1, λ_2 and λ_3 are obtained by solving the following system:

$$\begin{aligned}
\lambda_1{}' &= -\frac{dH}{dE} = (\lambda_1 - \lambda_2)(1-u^*)\beta_1 + (\lambda_1 - \lambda_3)(1-u^*)\beta_3 + \mu\lambda_1, \\
\lambda_2{}' &= -\frac{dH}{dD} = -1 + \beta_2(1-u^*)(\lambda_2 - \lambda_3) + \mu\lambda_2, \\
\lambda_3{}' &= -\frac{dH}{dC} = -1 + (\lambda_3 - \lambda_2)\gamma + \lambda_3(\mu + \nu + \delta).
\end{aligned} \quad ...(10)$$

provided with respect to conditions,

$$\rightarrow \lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0.$$

Modification of the *EDC* model considering two control strategies

Furthermore, we can modify system 1 by introducing new controls:

- u_1 - Awareness of diet plans.
- u_2 - Awareness of Regular testing for diabetes without complications.

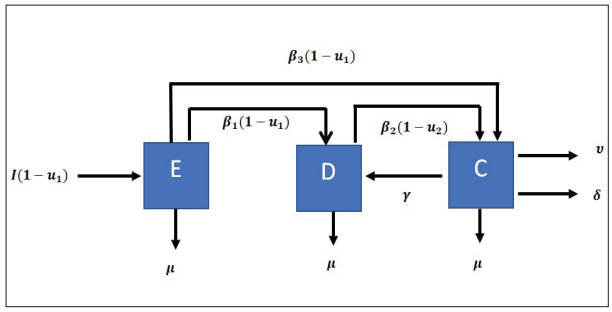

Figure 3: EDC model with two controls, u_1, u_2.

Based on the above controls, the proposed model is depicted in Figure 3.

$$\frac{dE}{dt} = I(1-u_1) - \big(\mu + (\beta_3+\beta_1)(1-u_1)\big)E,$$
$$\frac{dD}{dt} = \beta_1(1-u_1)E - \big(\mu + \beta_2(1-u_2)\big)D + \gamma C, \quad ...(11)$$
$$\frac{dC}{dt} = \beta_3(1-u_1)E + \beta_2(1-u_2)D - (\mu + \gamma + \upsilon + \delta)C.$$

The aim of this process is to minimize the following objective functional,

$$J(u_1,u_2) = \int_0^T (D + C + Au_1{}^2 + Bu_2{}^2)dt, \quad ...(12)$$

where;

A, B - Positive weights that balance the size of terms.
The objective is to characterize an optimal control $u_1{}^*, u_2{}^* \in U$ satisfying;

$$J(u_1{}^*, u_2{}^*) = \min_{u_1,u_2 \in U} J(u_1,u_2).$$

Existence and positivity of solutions

Theorem 3

The set $\Omega = \left\{(E,D,C)\epsilon\mathbb{R}^3 / \, 0 \le C,D,E \le \frac{I}{\mu}\right\}$ *is positively invariant under the system 11.*

Proof:

$$\frac{dE}{dt} = I(1-u_1) - \big(\mu + (\beta_3+\beta_1)(1-u_1)\big)E,$$
$$\frac{dE}{dt} \ge -\big(\mu + (\beta_3+\beta_1)(1-u_1)\big)E.$$

By the differential form of Gronwall's inequality:

This implies $E(t) > 0$.

Assuming that ∃, some time $t_* > 0$ such that $D(t_*) = 0$, other variables are positive and $D(t) > 0$ for $t \in [0, t_*]$. By the second equation of the system 11:

$$\frac{dD}{dt} = \beta_1(1-u_1)E - \big(\mu + \beta_2(1-u_2)\big)D + \gamma C,$$
$$\frac{dD}{dt} + (\mu+\beta_2)D = \beta_1(1-u_1)E + \beta_2 u_2 D + \gamma C.$$

Multiplying both sides by $e^{(\mu+\beta_2)t}$,

$$\frac{d}{dt}\big(De^{(\mu+\beta_2)t}\big) = e^{(\mu+\beta_2)t}[\beta_1(1-u_1)E + \beta_2 u_2 D + \gamma C].$$

Integrating this from 0 to t_*,

$$\int_0^{t_*} \frac{d}{dt}\big(De^{(\mu+\beta_2)t}\big) = \int_0^{t_*} e^{(\mu+\beta_2)t}[\beta_1(1-u_1)E + \beta_2 u_2 D + \gamma C]dt,$$

We obtain the result

$$D(t_*) = e^{-(\mu+\beta_2)t_*}\left[D(0) + \int_0^{t_*} e^{(\mu+\beta_2)t}[\beta_1(1-u_1)E + \beta_2 u_2 D + \gamma C]dt\right].$$

$D(t_*) > 0$, which contradicts $D(t_*) = 0$. Consequently, $D(t) > 0 \; \forall t \in [0,T]$.

By the third equation of the system 11:

$$\frac{dC}{dt} = \beta_3(1-u_1)E + \beta_2(1-u_2)D - (\mu+\gamma+\upsilon+\delta)C,$$

$$\frac{dC}{dt} \ge -(\mu+\gamma+\upsilon+\delta)C \quad (\because E(t) > 0 \; and \; D(t) > 0).$$

By the differential form of Gronwall's inequality,

$$C(t) = C(0).exp(-(\mu+\gamma+\nu+\delta)t) > 0,$$

This implies; $C(t) > 0$.

By adding equations of the system 11, we obtain:

$$\frac{dn}{dt} = I - \mu n - (\upsilon+\delta)C \le I - \mu n,$$

where $n = E + D + C$.

By integrating, we obtain:

$$n \le \frac{I}{\mu} - \left(\frac{I}{\mu} - n(0)\right)e^{-\mu t},$$

This implies $n(t) \le \frac{I}{\mu}$.

Theorem 4

The system 11 that satisfies a given initial condition $(E(0),D(0),C(0)) \in \Omega$ *has a unique solution.*

Proof: Let;

$$X = \begin{pmatrix} E \\ D \\ C \end{pmatrix}, \text{ and } \psi(X) = X_t = \begin{pmatrix} \frac{dE}{dt} \\ \frac{dD}{dt} \\ \frac{dC}{dt} \end{pmatrix}.$$

Therefore, the system 11 can be rewritten as follows:

$$\psi(X) = X_t = AX + B,$$

where,

$$A = \begin{pmatrix} -[\mu + (\beta_3 + \beta_1)(1-u_1)) & 0 & 0 \\ \beta_1(1-u_1) & -[\mu + \beta_2(1-u_2)] & \gamma \\ \beta_3(1-u_1) & \beta_2(1-u_2) & -(\mu + \gamma + \nu + \delta) \end{pmatrix},$$

and

$$B = \begin{pmatrix} I(1-u_1) \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$\| \psi(X_1) - \psi(X_2) \| = \| (AX_1 + B) - (AX_2 + B) \|,$$

$$\| \psi(X_1) - \psi(X_2) \| = \| AX_1 - AX_2 \|,$$

$$\| \psi(X_1) - \psi(X_2) \| \leq \| A \| . \| (X_1 - X_2) \|.$$

Therefore, the function ψ is uniformly Lipschitz continuous. Hence, from the definition of the controls $u_1(t), u_2(t)$ and the limitations $E(t) \geq 0, D(t) \geq 0$ *and* $C(t) \geq 0$, it can be concluded that the solution of the system 11 exists (Birkhoff & Rota, 1989).

Existence of an optimal control pair

Lemma 1. *There exist optimal solutions* $(u_1^*, u_2^*) \in U$ *such that* $J(u_1^*, u_2^*) = \min_{u_1, u_2 \in U} J(u_1, u_2)$ *for the optimal control problem indicated in system 11.*

Proof: By using Fleming and Rishel result (Fleming & Rishel, 2012) and performing the following steps, it is possible to obtain the existence of optimal control.

- According to theorems 3 and 4, the set of controls and associated state variables are non-empty.
- $J(u_1, u_2) = \int_0^T (D + C + Au_1^2 + Bu_2^2)dt$ is convex in u_1 and u_2.
- By definition, the control space U is convex and closed.
- The integrand in the objective functional, $(D + C + Au_1^2 + Bu_2^2)$ is obviously convex in $U \times U$.
- There exist constants a, b, c *such that* $(D + C + Au_1^2 + Bu_2^2) \geq a + b|u_1|^\alpha + c|u_2|^\alpha$, where $\alpha > 1$.

Characterization of an optimal control pair

The necessary conditions for the optimal control pair arise from Pontryagin's maximum principle (Chambers *et al*., 1965).

Theorem 5.

Given an optimal control (u_1^*, u_2^*) *and solutions* E^*, D^* and C^* *of the associated state system 11, there exist adjoint variables* λ_1, λ_2 and λ_3 *satisfying:*

$$\begin{aligned} \lambda_1' &= \lambda_1\mu + \beta_3(1-u_1^*)(\lambda_1 - \lambda_3) + \beta_1(1-u_1^*)(\lambda_1 - \lambda_2), \\ \lambda_2' &= -1 + \lambda_2\mu + \beta_2(1-u_2^*)(\lambda_2 - \lambda_3), \\ \lambda_3' &= -1 + \lambda_3(\mu + \nu + \delta) + \gamma(\lambda_3 - \lambda_2), \end{aligned} \quad \ldots(13)$$

With transversality conditions $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$ *and the optimal control pair*

$$\begin{aligned} u_1^* &= min\left(1, max\left(0, \frac{1}{2A}[I\lambda_1 + E\beta_1(\lambda_2 - \lambda_1) + E\beta_3(\lambda_3 - \lambda_1)]\right)\right), \\ u_2^* &= min\left(1, max\left(0, \frac{D}{2B}[\beta_2(\lambda_3 - \lambda_2)]\right)\right) \end{aligned} \quad \ldots(14)$$

Proof: the Hamiltonian H at time t is defined by:

$$H = D + C + Au_1^2 + Bu_2^2 + \lambda_1 f_1(E^*, D^*, C^*) + \lambda_2 f_2(E^*, D^*, C^*) + \lambda_3 f_3(E^*, D^*, C^*).$$

Here,

$$\begin{aligned} f_1(E, D, C) &= I(1-u_1) - \big(\mu + (\beta_3 + \beta_1)(1-u_1)\big)E, \\ f_2(E, D, C) &= \beta_1(1-u_1)E - \big(\mu + \beta_2(1-u_2)\big)D + \gamma C, \\ f_3(E, D, C) &= \beta_3(1-u_1)E + \beta_2(1-u_2)D - (\mu + \gamma + \upsilon + \delta)C. \end{aligned}$$

The optimality condition yields the optimal control:

$$\frac{\partial H}{\partial u_1} = 0 \Rightarrow 2Au_1 - I\lambda_1 + \lambda_1[(\beta_1 + \beta_3)E] + \lambda_2[-\beta_1 E] + \lambda_3[-\beta_3 E] = 0,$$

$$\frac{\partial H}{\partial u_1} = 0 \Rightarrow u_1^* = \frac{1}{2A}[I\lambda_1 + E\beta_1(\lambda_2 - \lambda_1) + E\beta_3(\lambda_3 - \lambda_1)].$$

$$\frac{\partial H}{\partial u_2} = 0 \Rightarrow 2Bu_2 + \lambda_2\beta_2 D - \lambda_3\beta_2 D,$$

$$\frac{\partial H}{\partial u_2} = 0 \Rightarrow u_2^* = \frac{D}{2B}[\beta_2(\lambda_3 - \lambda_2)].$$

We can obtain the adjoint variables λ_1, λ_2 and λ_3 by the following system:

$$\lambda_1' = -\frac{dH}{dE} = \beta_1(1-u_1^*)(\lambda_1 - \lambda_2) + \beta_3(1-u_1^*)(\lambda_1 - \lambda_3) + \mu\lambda_1,$$

$$\lambda_2' = -\frac{dH}{dD} = -1 + \beta_2(1-u_2^*)(\lambda_2 - \lambda_3) + \mu\lambda_2,$$

$$\lambda_3{}' = -\frac{dH}{dC} = -1 + (\lambda_3 - \lambda_2)\gamma + \lambda_3(\mu + \nu + \delta). \quad ...(15)$$

$$\rightarrow \lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0.$$

Numerical simulation

Using a fourth-order Runge-Kutta scheme, we numerically solved the resulting optimality systems 7 and 11. This method involves a two-step process. It begins by forward solving the state equations using an initial guess for the control parameters u_1 and u_2. As the forward solution progresses, the controls are continuously updated based on control equations 9 or 14. In parallel, the method solves adjoint equations backwards initialized with transversality conditions until results converge; this process is continued iteratively. This process is called Forward Backward Sweep Method.

There are many types of tests available to obtain the convergence behavior of the numerical solutions (Bellonietal,1993; Lenhart & Workman, 2007). We considered the evolution of the size between two consecutive solution with respect to the number of iterations. Suppose that $X = (E, D, C)$ and $U = (u_1, u_2)$ are the state and control vectors. X_{k+1}, U_{k+1} denote the vectors of estimated values of the current iteration and X_k, U_k are those of the previous iteration. Then we define,

$$\|X_{k+1} - X_k\| = \sum_{i=1}^{N+1} |X_{k+1}(i) - X_k(i)|$$

and

$$\|U_{k+1} - U_k\| = \sum_{i=1}^{N+1} |U_{k+1}(i) - U_k(i)|$$

In order to define the stopping criteria for the optimization algorithm, define

$$test_X = \delta\|X_{k+1}\| - \|X_{k+1} - X_k\| \text{ and } test_U = \delta\|U_{k+1}\| - \|U_{k+1} - U_k\|$$

for some $\delta > 0$. Then we stop the algorithm if, $\min(test_X, test_U)$ is sufficiently small. The utilized optimization algorithm with the forward-backward sweep method is presented below.

Algorithm 1 Optimization algorithm with forward backward sweep method

Input: Initial state X_0, initial control U_0, tolerance (tol), set parameters.
Output: Optimal control U^*, optimal state X^*
Set $k = 0$
Test = *min* $(test_X, test_U)$ (initially set as a constant value.)
while Test $\geq$ *tol*

1. Solve the state system (given U_k)
2. Solve the adjoint system
3. Compute the cost functional
4. Update U_{k+1} using 14 or 9
5. Compute *Test* = *min* $(test_X, test_U)$
6. Set $k = k+1$

end
$U^* = U_k$
$X^* = X_k$

The appropriate parameter values are selected for the numerical simulation, and the year 2007 is used as the starting point for the simulations.

The weight constants are chosen in the following way because of the difference in scales between the control and state functions: For the *EDC* model with variable incidence of pre-diabetes, A = 3,550,000 and for the *EDC* model with two controls, A = 4,550,000 and B = 1,550,000. Also, initial states are chosen as E(0) = 1,710,456, D(0) = 148,428 and $C(0)$ = 103,899 which are the relevant values for Sri Lanka in the year 2007.

Table 2: Parameter values used in numerical simulation

Parameter	Value year^{-1}	Description
μ	0.0059	Natural death rate
v	0.005	Severely disabling rate of diabetes with complications
γ	0.05	Curing rate of complications
δ	0.059	The death rate of people with diabetes complications
β_1	0.03	Diabetes developing probability
β_2	0.0067	Complication developing probability of a diabetic person
β_3	0.0005	The probability of prediabetes getting complications
I	565,440	Incidence of pre-diabetes

RESULTS AND DISCUSSION

EDC model with variable incidence of pre-diabetes (I)

Control strategies and the compartment sizes of the proposed model were estimated using the numerical solutions of the optimal control problem. The results of the current model indicate that the incidence rate I may not behave as a constant for a long time period. Figures 4, 5, and 6 illustrate the variation in the population sizes of individuals with diabetes (with and without complications) and those with pre-diabetes, under both constant incidence rate and incidence rate varying according to the logistic growth of the population. Figure 7 presents the variation in the control for both scenarios. Figure 8 shows the behaviour of the objective functional when the incidence rate is constant, while Figure 9 presents it when the incidence rate varies according to the logistic growth of the population.

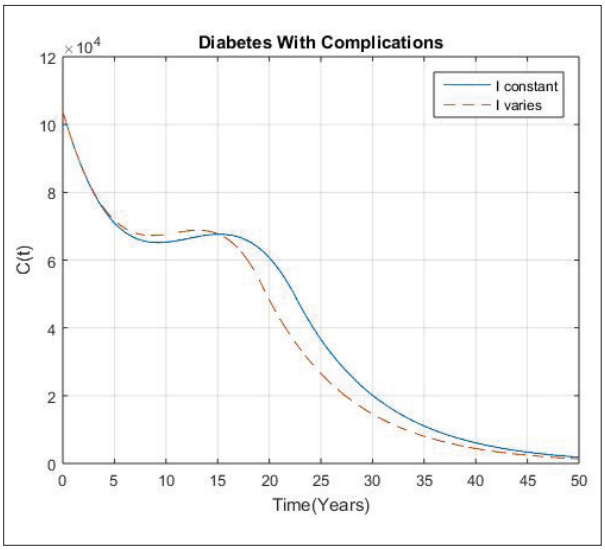

Figure 4: The evolution of the number of diabetes incidences with complications, with constant I and variable incidence rate (I)

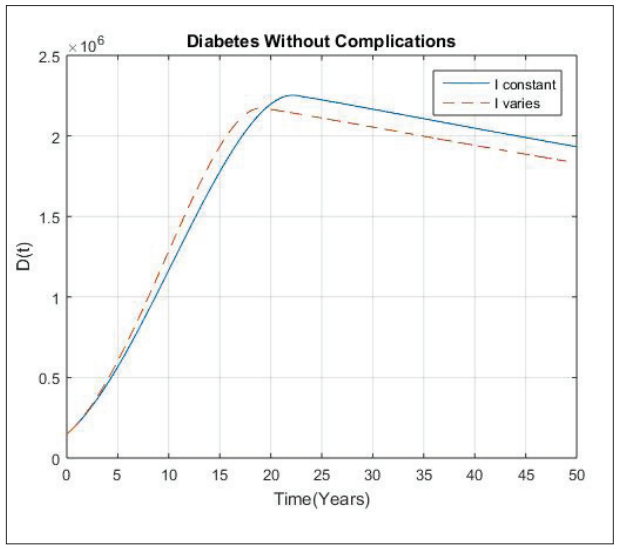

Figure 5: The evolution of the number of diabetes incidences, without complications with constant and variable incidence rate (I).

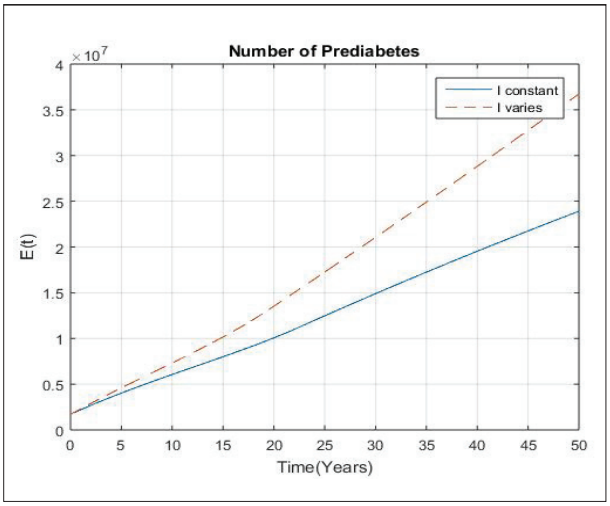

Figure 6: The evolution of the number of pre-diabetes incidences with constant and variable incidence rate (*I*).

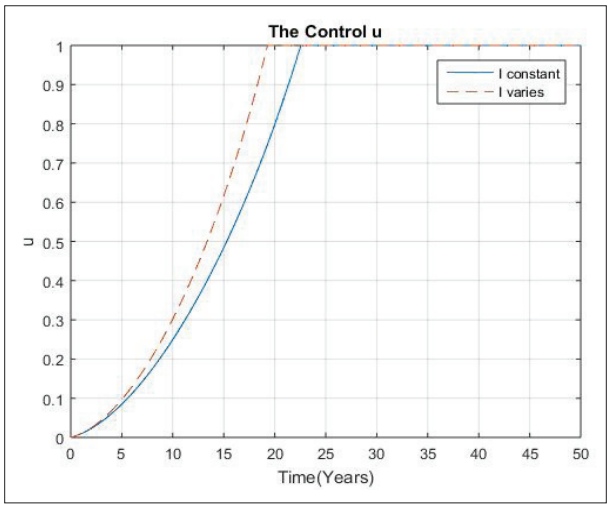

Figure 7: The behaviour of the control with constant and variable incidence rate (*I*).

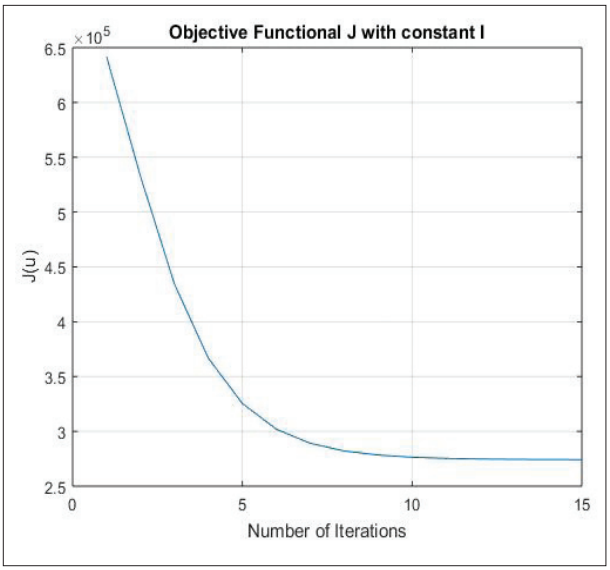

Figure 8: Behaviour of the objective functional against the number of iterations, when the incident rate is constant.

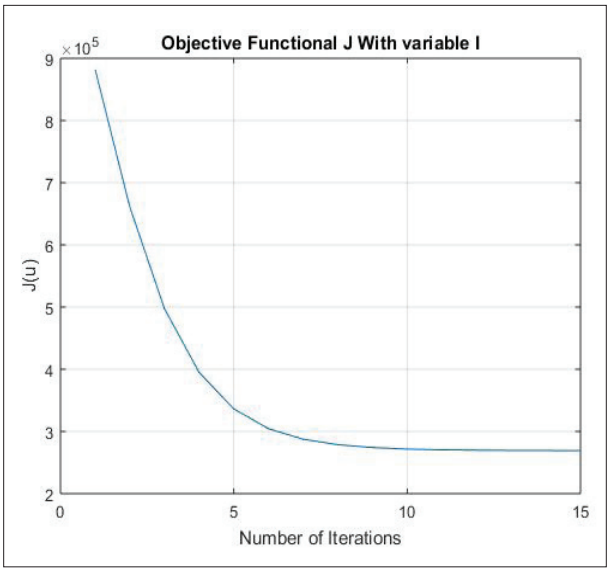

Figure 9: Behaviour of the objective functional against the number of iterations, when the incident rate varies.

The observation of the presented model reveals that with a constant incident rate, the number of incidences of pre-diabetes, diabetes without complications, and diabetes with complications will reach 23.91 million, 19.33 million and 1839 in 50 years, respectively. Under the same parameter values, but with variable incident rate, the population of people with pre-diabetes, diabetes without complications, and diabetes with complications will reach 36.73 million, 18.32 million, and 1338 in 50 years, respectively. Moreover, with a variable incidence of pre-diabetes (*I*), the control u reaches its maximum value within 19 years, while it takes 23 years for a constant incidence of pre-diabetes (*I*).

EDC Model with two controls

In this section, we present the results of our second model, which demonstrates how pre-diabetes and diabetes (with and without complications) vary under the influence of two control variables. We also examine the behaviour of the objective functional and the control strategies u_1 and u_2.

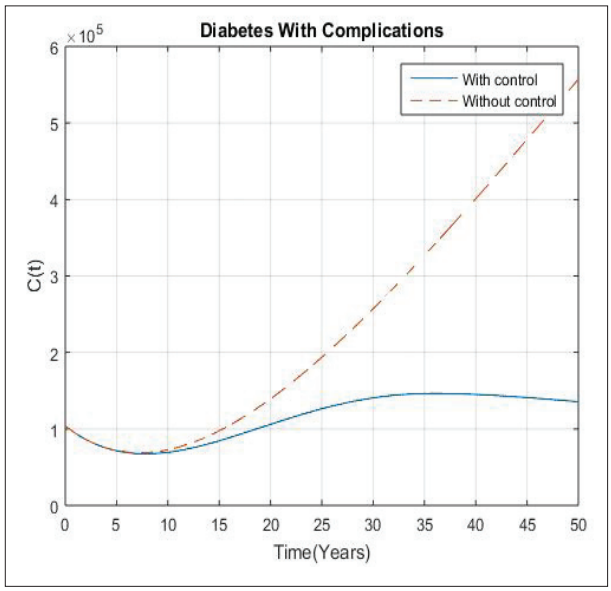

Figure 10: The evolution of the number of incidences of diabetes with complications, with and without controls.

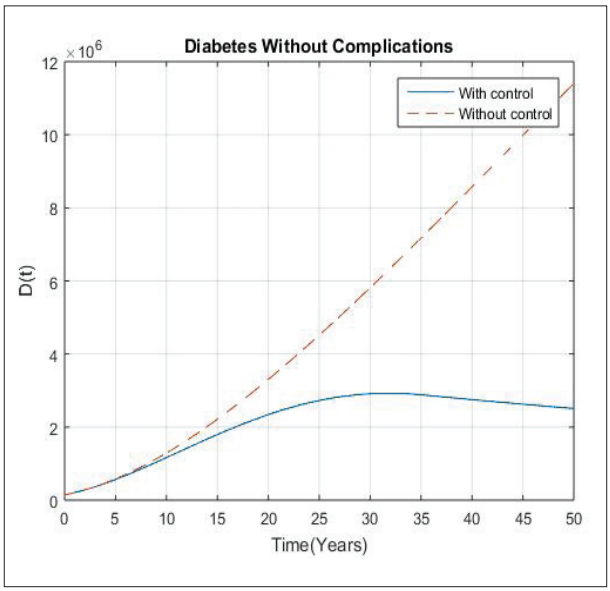

Figure 11: The evolution of the number of incidences of diabetes without complications, with and without controls.

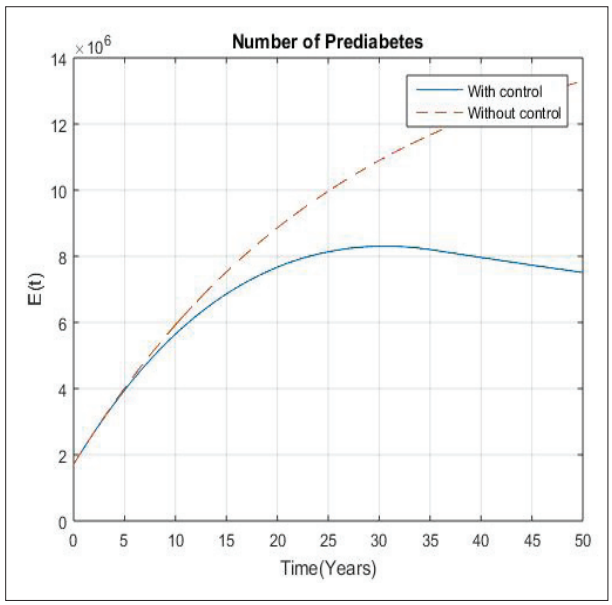

Figure 12: The evolution of the number of incidences of pre-diabetes, with and without controls.

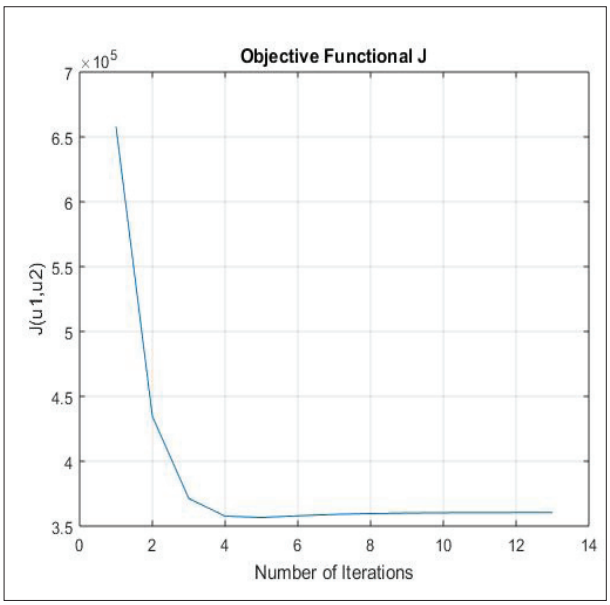

Figure 13: The behaviour of the objective functional $J(u_1, u_2)$.

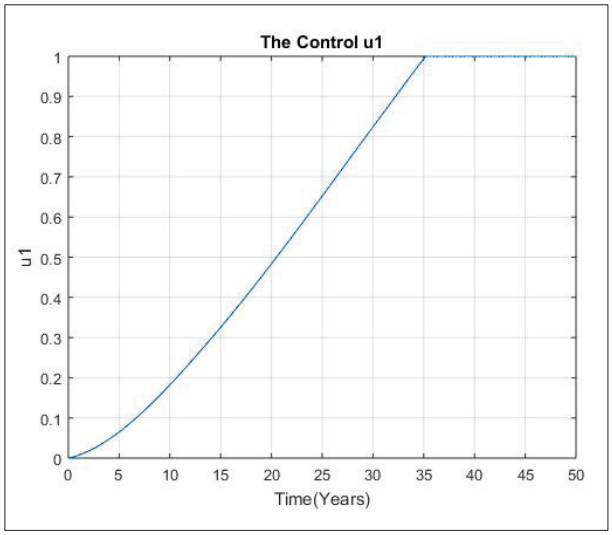

Figure 14: The behaviour of the control u_1.

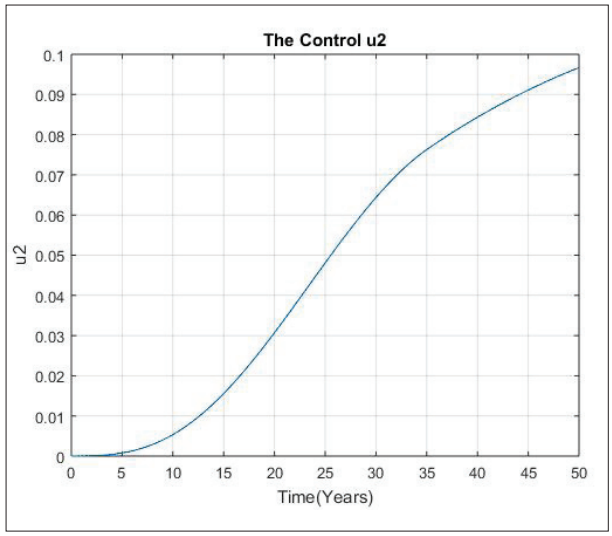

Figure 15: The behaviour of the control u_2.

In this model, we observed that the number of incidences of pre-diabetes, diabetes without complications, and diabetes with complications will reach 13.29 million, 11.4 million, and 0.5572 million, respectively, in 50 years without optimal control. Under the same parameter values, but with optimal control, the population of patients with pre-diabetes, diabetes without complications, and diabetes with complications will reach 7.5 million, 2.508 million, and 0.1356 million in 50 years, respectively. According to Figures 14 and 15, the control u_1 should receive more attention than the control u_2 during the considered time period. Further, the control u_1 should be conducted at full capacity after 35 years of period.

Error Analysis of the Solution

In this section, an error analysis of the solution is conducted by evaluating the errors and convergence rates of both the state and control variables at different step sizes to check the convergence of the proposed simulation algorithm. The obtained results are shown in Table 3, Table 4, and Figure 16. The analysis shows that as the step size decreases, the errors for both the state and control variables significantly decrease, indicating improved accuracy of the algorithm. The tables also show the logarithmic ratios of these errors, which further demonstrate the increasing order of convergence. This consistent improvement in accuracy with smaller step sizes highlights the effectiveness and reliability of the solution method employed.

Table 3: Error and order of convergence of state variables (h = 0.4)

Step size	$\|X_h\|$	$\frac{\|X_h - X_{h/2}\|}{\|X_h\|}$	$\frac{\|X_h - X_{h/2}\|}{\|X_{h/2} - X_{h/4}\|}$	$\log_2\left(\frac{\|X_h - X_{h/2}\|}{\|X_{h/2} - X_{h/4}\|}\right)$
h	0.3836×10^{9}	0.115888×10^{-4}	4.3947	2.1344
$h/2$	0.7653×10^{9}	0.013085×10^{-4}	3.2190	1.6740
$h/4$	1.5285×10^{9}	0.002052×10^{-4}	3.0245	1.5959
$h/8$	3.0550×10^{9}	0.000342×10^{-4}	6.5338	2.6891
$h/16$	6.1080×10^{9}	0.000026×10^{-4}		
$h/32$	0.1221×10^{9}			

Table 4: Error and order of convergence of control variables

Step size	$\|U_h\|$	$\frac{\|U_h - U_{h/2}\|}{\|U_h\|}$	$\frac{\|U_h - U_{h/2}\|}{\|U_h\|}$	$\log_2\left(\frac{\|U_h - U_{h/2}\|}{\|U_{h/2} - U_{h/4}\|}\right)$
h	40.7886	0.7328×10^{-5}	5.1170	2.3549
$h/2$	81.3029	0.0708×10^{-5}	4.1384	1.9424
$h/4$	162.3316	0.0123×10^{-5}	2.5669	1.3578
$h/8$	324.3892	0.0025×10^{-5}	7.4125	2.7519
$h/16$	648.5043	0.0001×10^{-5}		
$h/32$	1296.7344			

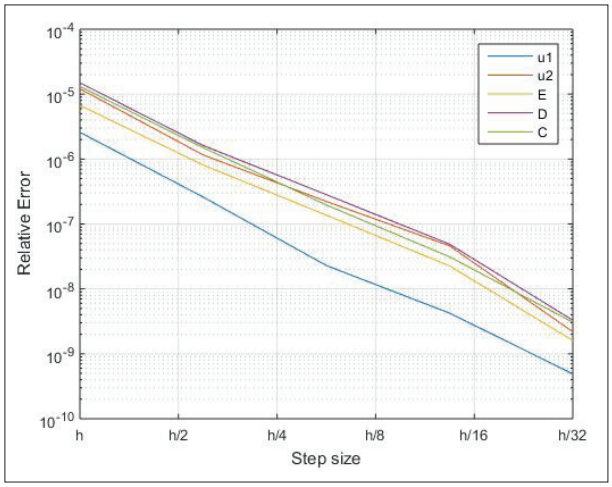

Figure 16: Variation of Relative Errors of State Variables and Controls against Step Size.

CONCLUSION

The incidence of pre-diabetes (E), diabetes without complications (D) and diabetes with complications (C) is a huge burden for the healthcare sectors in the world. This paper proposes two mathematical models considering the above-mentioned subclasses (*EDC*) and two diabetes control strategies. In the first model, we considered two cases, *EDC* model with a constant incidence rate and *EDC* model with variable incidence rate. In the case of variable incidence rates, it is assumed that the incidence rate is proportional to the time-dependent total population size. In the second model, we considered *EDC* model with two control strategies, which are awareness programmes by raising the consciousness of the importance of following a diet plan and maintaining good health care, and regular testing and awareness programmes for diabetes without complications. We observed that under control strategies, the population sizes of patients with pre-diabetes, diabetes without complications, and diabetes with complications decreased by 44%, 78%, and 76% respectively, after the end of 50 years. Therefore, the numerical simulation of the obtained results showed the effectiveness of the proposed control strategies. Since diabetes is a worldwide problem with different rates of incidence, prevalence, mortality, and morbidity, the current mathematical models can be further improved for various contexts based on the values of the parameters and other appropriate control strategies.

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