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RESEARCH ARTICLE

Ordered Random Variables

On recursive computation for the moments of generalized order statistics for the Kumaraswamy family of distributions with characterizations

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Abstract: The generalized order statistics (GOS) is a unified model for random variables that are arranged in increasing order of magnitude. Several other models for increasingly arranged random variables appear as special cases of *GOS*. The distributional properties of generalized order statistics for specific baseline distributions have been studied by various authors. One specific domain of study in the context of generalized order statistics is to develop some recursive formulae to obtain moments of the distribution of generalized order statistics for any baseline distribution. The relations for the moments of generalized order statistics for families of distributions have not been explored much. This paper is based on some recursive relations to compute the single and joint moments of generalized order statistics for the Kumaraswamy family of distributions. These relations are useful to recursively compute the single and joint moments of generalized order statistics for any member of the Kumaraswamy family of distributions. We illustrate the results with examples using different baseline distributions. Additionally, we present some characterization results using single and joint moments of GOS.

Keywords: Characterizations, generalized order statistics, Kumaraswamy family of distributions, recurrence relations.

INTRODUCTION

In some practical situations, the data being studied is double-bounded, so a suitable double-bounded distribution is needed. Several such distributions are available in the literature. The beta distribution is one such distribution that can be used to model this type of data.

For a continuous random variable w, the probability density and cumulative distribution function of the beta distribution are

$$g(w)=\frac{1}{B(a,b)}w^{a-1}(1-w)^{b-1}\ ;\ 0<w<1\ ,\ (a,b)>0$$

and

$$G(w)=I_x(a,b)=\frac{B_x(a,b)}{B(a,b)}=\frac{1}{B(a,b)}\int_0^x v^{a-1}(1-v)^{b-1}dv,$$

where a and b are the shape parameters, $B(a,b)$, $B_x(a,b)$, and $I_x(a,b)$ are, the complete beta function, the incomplete beta function, and the incomplete beta function ratio, respectively, that are defined as

$$B(a,b)=\int_0^1 v^{a-1}(1-v)^{b-1}du\ ;\ B_x(a,b)=\int_0^x v^{a-1}(1-v)^{b-1}du\ ;\ I_x(a,b)=\frac{B_x(a,b)}{B(a,b)}.$$

One disadvantage of the beta distribution is that its cumulative distribution function (cdf) $G(w)$ is not in the compact form. A simple, yet powerful, alternative to the

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beta distribution for modeling of [0,1] bounded data is the Kumaraswamy distribution (Kumaraswamy, 1980). The probability density and cumulative distribution functions of the Kumaraswamy distribution are

$$g(w;a,b) = abw^{a-1}\left(1-w^a\right)^{b-1} \; ; 0<w<1 \; ; a,b>0 \qquad ...(1)$$

and $G(w) = 1-\left(1-w^a\right)^b \; ; 0<z<1 \; ; a,b>0$, ...(2)

respectively. The Kumaraswamy distribution has extensive applications in various domains of life, including hydrology. The distribution has also been used as an alternative to the beta distribution as its cumulative distribution function (cdf) is in a compact form (Jones, 2009).

The generalized order statistics (GOS) (Kamps, 1995) and the dual generalized order statistics (DGOS) (Burkschat *et al*., 2003) are two unified models to study the distributional properties arranged in increasing or decreasing order. The probability density function (pdf) of a single GOS and the joint pdf of two GOS for a sample of size n ~~is~~ are (Kamps, 1995)

$$f_{r:n,m,k}(w) = \frac{C_{r-1}}{(r-1)!} f(w)[\bar{F}(w)]^{\gamma_r - 1}\tau_m^{r-1}[F(w)]; w \in \mathbb{R}, \qquad ...(3)$$

and

$$f_{r,s:n,m,k}(w_1,w_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(w_1) f(w_2)\left[\bar{F}(w_1)\right]^m \tau_m^{r-1}\left[F(w_1)\right]$$
$$\times\left[\bar{F}(w_2)\right]^{\gamma_s - 1}\left[\varphi_m\left\{F(w_1)\right\} - \varphi_m\left\{F(w_2)\right\}\right]^{s-r-1}$$
$$-\infty < w_1 < w_2 < \infty \qquad ...(4)$$

respectively, where $\bar{F}(w) = 1-F(w)$, $\gamma_r = k+(n-r)(m+1)$, $C_{r-1} = \prod_{j=1}^{r}\gamma_j$

and

$$\varphi_m(v) = \begin{cases} -(1-v)^{m+1}/(m+1) & ; m \neq -1 \\ -\ln(1-v) & ; m = -1. \end{cases};$$

$$\tau_m(v) = \begin{cases} \left[1-(1-v)^{m+1}\right]/(m+1); & m \neq -1 \\ -\ln(1-v) & m = -1. \end{cases}$$

Various distributional methods for modeling random variables arranged in ascending order appear as a special case of GOS. The ordinary order statistics appear as a special case when $m = 0$ and $k = 1$. The kth upper records appear as a special case when $m = -1$.Several textbooks, such as Ahsanullah and Nevzorov (2001) and Shahbaz *et al*. (2016), provide detailed discussions on GOS. Much of the research in this area focuses on deriving formulas for the recursive calculation of GOS moments for various baseline distributions. In addition, some authors have used GOS moments to obtain characterization results.

The recurrence relations for moments of GOS for the exponential distribution (Ahsanullah, 2000) and Rayleigh distribution (Mohsin *et al*., 2010) are some useful illustrations of the recurrence relations for moments of GOS. The development of the recurrence relations for moments of DGOS has also been an area of interest. Some examples of recurrence relations for the moments of dependent generalized order statistics for well-known distributions, such as the power function distribution, are given by Pawlas and Szynal (2001) and Athar and Faizan (2011). Kumar (2011) also provided recurrence relations for the moments of GOS from the Kumaraswamy distribution, along with some characterizations. In addition, a general formula showing the relationship between single and joint moments of GOS is available in Athar and Islam (2004), which is useful for developing recurrence relations for specific cases.

MATERIALS AND METHODS

The Kumaraswamy distribution has been used as a baseline distribution to propose a new family of distributions, known as the Kumaraswamy family of distributions (Cordeiro & Castro, 2011). The cdf and probability density function (pdf) of the Kumaraswamy generated (*Kum–G*) family of distributions are

$$F_G(w) = 1 - [1 - H^a(w;\boldsymbol{\xi})]^b; w \in \mathbb{R} \qquad ...(5)$$

and

$$f_G(w) = abh(w;\boldsymbol{\xi})[H(x;\boldsymbol{\xi})]^{a-1}[1 - H^a(w;\boldsymbol{\xi})]^{b-1};$$

$$w \in \mathbb{R}, \qquad \text{...(6)}$$

respectively, where $H(w;\xi)$ and $h(w;\xi)$ are the *cdf* and *pdf* of any baseline distribution, and ξ is the vector of parameters of the baseline distribution $H(x;\xi)$. It is to be noted that the *exponentiated–G* family of distributions; (Gupta *et al*., 1998; Nadarajah & Kotz, 2006), appears as a special case of the *Kum–G* family of distributions for $b = 1$.

The Kum–G family of distributions has gained the interest of many researchers, leading to the development of several new distributions based on different baseline distributions. A Kumaraswamy inverse Weibull distribution has been proposed by using the inverse Weibull distribution as a baseline distribution in (3) (Shahbaz *et al*., 2012). A Kumaraswamy power function distribution has been proposed by using the power function baseline distribution in the *Kum–G* family of distributions (Abdul-Moniem, 2017), whereas a Kumaraswamy log-logistic distribution has been proposed by using the log-logistic distribution in (3) (de Santana *et al*., 2012).

In this paper, some recurrence relations for moments of GOS for the *Kum–G* family of distributions have been obtained. These relations are obtained in the following sections.

Recurrence relation for the single moments

This section deals with the recurrence relation for single moments of GOS for the Kumaraswamy family of distributions. The recurrence relation is obtained by first noting the following relation between the density and distribution function of the *Kum–G* family of distributions

$$\bar{F}(w) = \frac{1}{ab} r_H^{-1}(w)\left[H^{-a}(w) - 1\right] f(w) \qquad \text{...(7)}$$

where $r_H(w) = h(w)/H(w)$ is the reversed hazard rate function of the baseline distribution $H(w)$. The relation (7) is useful in deriving a recurrence relation for moments of GOS for the Kumaraswamy family of distributions. The recurrence relation for single moments of GOS for the *Kum–G* distribution is derived in the following theorem.

Theorem–1: The single moments of GOS for the *Kum–G* family of distributions are related as

$$E\left(W^p_{r:n,m,k}\right) - E\left(W^p_{r-1:n,m,k}\right) = \frac{p}{ab\gamma_r}\left[E\left\{\Phi\left(W_{r:n,m,k}\right)\right\} - E\left\{\Delta\left(W_{r:n,m,k}\right)\right\}\right], \qquad \text{...(8)}$$

where $\Phi(w) = w^{p-1} r_H^{-1}(w) H^{-a}(w)$ and $\Delta(w) = w^{p-1} r_H^{-1}(w)$.

Proof: Consider the following relation for the single moments of GOS for any parent distribution, (Athar & Islam, 2004),

$$E\left(W^p_{r:n,m,k}\right) - E\left(W^p_{r-1:n,m,k}\right) = \frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} w^{p-1}\left[\bar{F}(w)\right]^{\gamma_r} \tau_m^{r-1}\left[F(w)\right] dw \quad \text{or}$$

$$E\left(W^p_{r:n,m,k}\right) - E\left(W^p_{r-1:n,m,k}\right) = \frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} w^{p-1}\bar{F}(w)\left[\bar{F}(w)\right]^{\gamma_r - 1} \tau_m^{r-1}\left[F(w)\right] dw. \qquad \text{...(9)}$$

Now, using (7) in (9), we have

$$E\left(W^p_{r:n,m,k}\right) - E\left(W^p_{r-1:n,m,k}\right) = \frac{pC_{r-1}}{ab\gamma_r (r-1)!}\int_{-\infty}^{\infty} w^{p-1}\left[r_H^{-1}(w)\left\{H^{-a}(w) - 1\right\}\right] f(w) \times\left[\bar{F}(w)\right]^{\gamma_r - 1} \tau_m^{r-1}\left[F(w)\right] dw$$

or $$E\left(W_{r:n,m,k}^{p}\right)-E\left(W_{r-1:n,m,k}^{p}\right)=\frac{pC_{r-1}}{ab\gamma_r(r-1)!}\int_{-\infty}^{\infty}\left[w^{p-1}r_H^{-1}(w)H^{-a}(w)\right]f(w)\left[\bar{F}(w)\right]^{\gamma_r-1}$$
$$\times\tau_m^{r-1}\left[F(w)\right]dw-\frac{pC_{r-1}}{ab\gamma_r(r-1)!}\int_{-\infty}^{\infty}\left[w^{p-1}r_H^{-1}(w)\right]f(w)\left[\bar{F}(w)\right]^{\gamma_r-1}\tau_m^{r-1}\left[F(w)\right]dw$$

or $$E\left(W_{r:n,m,k}^{p}\right)-E\left(W_{r-1:n,m,k}^{p}\right)=\frac{p}{ab\gamma_r}\left[E\left\{\Phi\left(W_{r:n,m,k}\right)\right\}-E\left\{\Delta\left(W_{r:n,m,k}\right)\right\}\right] ,$$

where $\Phi(w)=w^{p-1}r_H^{-1}(w)H^{-a}(w)$ and $\Delta(w)=w^{p-1}r_H^{-1}(w)$, then the theorem is complete.

Since the exponentiated–*G* family of distributions appears as a special case of the *Kum–G* family of distributions for $b = 1$, the recurrence relation for single moments of GOS for the *exponentiated–G* family of distributions can be readily written from (8). The recurrence relations for single moments of special cases of GOS can also be easily written from (8). For example, the recurrence relation for single moments of upper record values is obtained by using $m = -1$ in (8) and is given as

$$E\left(W_{K(r)}^{p}\right)-E\left(W_{K(r-1)}^{p}\right)=$$
$$\frac{p}{abk}\left[E\left\{\Phi\left(W_{K(r)}\right)\right\}-E\left\{\Delta\left(W_{K(r)}\right)\right\}\right] \quad ...(10)$$

The relation (8) can be used to obtain the recurrence relation for other special cases.

Recurrence relation for the joint moments

In this section, we have obtained the recurrence relation for the joint moments of GOS for the *Kum–G* family of distributions. The recurrence relation is obtained in the following theorem.

Theorem–2: The joint moments of GOS for the *Kum–G* family of distributions are related as

$$E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}X_{s-1:n,m,k}^{p_2}\right)=$$
$$\frac{p_2}{ab\gamma_s}\left[E\left\{\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)\right\}\right.$$
$$\left.-E\left\{\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)\right\}\right], \quad ...(11)$$

where
$$\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2) \text{ and } \Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=x_1^{p_1}x_2^{p_2-1}r_H^{-1}(w_2).$$

Proof: Consider the following relation (Athar and Islam, 2004) for the joint moments of *GOS* for any parent distribution

$$E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s-1:n,m,k}^{p_2}\right)=\frac{p_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{w_1}^{\infty}w_1^{p_1}w_2^{p_2-1}f(w_1)\left[\bar{F}(w_1)\right]^m$$
$$\times\tau_m^{r-1}\left[F(w_1)\right]\left[\varphi_m\left\{F(w_2)\right\}-\varphi_m\left\{F(w_1)\right\}\right]^{s-r-1}\left[\bar{F}(w_2)\right]^{\gamma_s}dw_2dw_1,$$

$$E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s-1:n,m,k}^{p_2}\right)=\frac{p_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{w_1}^{\infty}w_1^{p_1}w_2^{p_2-1}f(w_1)$$

or $$\times\left[\bar{F}(w_1)\right]^m\tau_m^{r-1}\left[F(w_1)\right]\left[\varphi_m\left\{F(w_2)\right\}-\varphi_m\left\{F(w_1)\right\}\right]^{s-r-1}\times\bar{F}(w_2)\left[\bar{F}(w_2)\right]^{\gamma_s-1}dw_2dw_1. \quad ...(12)$$

Now, using (7) in (12), we have

$$E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s:n,m,k}\right)-E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s-1:n,m,k}\right)=\frac{qC_{s-1}}{ab\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{w_1}^{\infty}w_1^{p_1}w_2^{p_2-1}\Big[r_H^{-1}(w_2)$$
$$\times\left\{H^{-a}(w_2)-1\right\}\Big]f(w_1)f(w_2)\left[\bar F(w_1)\right]^m\tau_m^{r-1}\left[F(w_1)\right]$$
$$\times\left[\varphi_m\{F(w_2)\}-\varphi_m\{F(w_1)\}\right]^{s-r-1s-r-1}\left[\bar F(w_2)\right]^{\gamma_s-1}dw_2dw_1.$$

or

$$E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s:n,m,k}\right)-E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s-1:n,m,k}\right)=\frac{p_2C_{s-1}}{ab\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{w_1}^{\infty}\left[w_1^{p}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2)\right]$$
$$\times f(w_1)f(w_2)\left[\bar F(w_1)\right]^m\tau_m^{r-1}\left[F(w_1)\right]\left[\varphi_m\{F(w_2)\}-\varphi_m\{F(w_1)\}\right]^{s-r-1s-r-1}$$
$$\times\left[\bar F(w_2)\right]^{\gamma_s-1}dw_2dw_1-\frac{p_2C_{s-1}}{ab\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{w_1}^{\infty}\left[w_1^{p}w_2^{p_2-1}r_H^{-1}(w_2)\right]$$
$$\times f(w_1)f(w_2)\left[\bar F(w_1)\right]^m\tau_m^{r-1}\left[F(w_1)\right]\left[\varphi_m\{F(w_2)\}-\varphi_m\{F(w_1)\}\right]^{s-r-1s-r-1}$$
$$\times\left[\bar F(w_2)\right]^{\gamma_s-1}dw_2dw_1$$

or

$$E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s:n,m,k}\right)-E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s-1:n,m,k}\right)=\frac{p_2}{ab\gamma_s}\left[E\left\{\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)\right\}-E\left\{\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)\right\}\right],$$

where $\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2)$ and $\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)$, then the theorem is complete.

The recurrence relation for the joint moments of GOS for the *exponentiated–G* family of distributions can be readily obtained from (11) by using $b = 1$.

Examples

The recurrence relations for single and joint moments of *GOS* for some members of the *Kum–G* family of distributions have been derived in this section.

Kumaraswamy power function distribution

The Kumaraswamy power function distribution (Abdul-Moniem, 2017) is obtained by using the following cdf $H(w)=\left(\frac{w}{\lambda}\right)^{\theta}$ and pdf $h(w)=\frac{\theta}{\lambda}\left(\frac{w}{\lambda}\right)^{\theta-1}$ in (3) and (4), where $0<w<\lambda, \theta\geq 1, \lambda>0.$

Also, for the power function distribution, we have $r_H(w)=h(w)/H(w)=\theta/w$. Now, the recurrence relations for single and joint moments of *GOS* for the Kumaraswamy power function distribution can be obtained by using (8) and (11). For this, we first see that for the power function distribution, we have

$$\Phi(w)=w^{p-1}r_H^{-1}(w)H^{-a}(w)=\lambda^{a\theta}w^{p-a\theta}/\theta$$

and $\Delta(w)=w^{p-1}r_H^{-1}(w)=w^{p}/\theta$.

Using the above expressions in (8), the recurrence relation for single moments of GOS for the Kumaraswamy power function distribution is obtained as

$$E\left(W^{p}_{r:n,m,k}\right)-E\left(W^{p}_{r-1:n,m,k}\right)=\frac{p}{ab\gamma_r}\left[\frac{\lambda^{a\theta}}{\theta}E\left(W^{p-a\theta}_{r:n,m,k}\right)-\frac{1}{\theta}E\left(W^{p}_{r:n,m,k}\right)\right]$$

or $$\mu^{p}_{r:n,m,k}-\mu^{p}_{r-1:n,m,k}=\frac{p}{ab\theta\gamma_r}\left(\lambda^{a\theta}\mu^{p-a\theta}_{r:n,m,k}-\mu^{p}_{r:n,m,k}\right)$$

or $$\mu^{p}_{r:n,m,k}=\frac{1}{ab\theta\gamma_r+p}\left(p\lambda^{a\theta}\mu^{p-a\theta}_{r:n,m,k}+ab\theta\gamma_r\mu^{p}_{r-1:n,m,k}\right), \quad ...(13)$$

where $\mu^{p}_{r:n,m,k}=E\left(W^{p}_{r:n,m,k}\right)$, etc.

Again, for the power function distribution, we have

$$\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2)$$
$$=\lambda^{a\theta}w_1^{p}w_2^{q-a\theta}/\theta$$

and $\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}\left(w_2\right)=w_1^{p_1}w_2^{p_2}/\theta$.

Using the above expressions in (11), the recurrence relation for the joint moments of GOS for the Kumaraswamy power function distribution is obtained as

$$E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s-1:n,m,k}^{p_2}\right)=$$

$$\frac{p_2}{ab\theta\gamma_s}\left[\lambda^{a\theta}E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2-a\theta}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)\right]$$

or $\mu_{r,s:n,m,k}^{p_1,p_2}-\mu_{r,s-1:n,m,k}^{p_1,p_2}=\dfrac{p_2}{ab\theta\gamma_s}\left(\lambda^{a\theta}\mu_{r,s:n,m,k}^{p,q-a\theta}-\mu_{r,s:n,m,k}^{p,q}\right)$

or $\mu_{r,s:n,m,k}^{p_1,q_2}=\dfrac{1}{ab\theta\gamma_s+p_2}\left(p_2\lambda^{a\theta}\mu_{r,s:n,m,k}^{p_1,p_2-a\theta}+ab\theta\gamma_s\mu_{r,s-1:n,m,k}^{p_1,p_2}\right)$, ...(14)

where $\mu_{r,s:n,m,k}^{p_1,p_2}=E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)$, etc. The relations (13) and (14) are the same as obtained earlier (Khan *et al*., 2018).

Kumaraswamy reflected exponential distribution

The Kumaraswamy reflected exponential distribution is obtained by using the following cdf and pdf in (3) and (4)

$$H(w)=\exp\left(\frac{w}{\theta}\right)\ \&\ h(w)=\frac{1}{\theta}\exp\left(\frac{w}{\theta}\right);\ w<0\ ,\ \theta>0.$$

Also, for the reflected exponential distribution, we have $r_H(w)=h(w)/H(w)=1/\theta$. Now, the recurrence relations for single and joint moments of GOS for the Kumaraswamy reflected exponential distribution can be obtained by using (8) and (11). For this, we first see that for the reflected exponential distribution, we have

$$\Phi(w)=w^{p-1}r_H^{-1}(w)H^{-a}(w)=\theta w^{p-1}\exp\left(-\frac{aw}{\theta}\right)$$

$$=\theta\sum_{j=0}^{\infty}\frac{(-1)^j a^j}{j!\theta^j}w^{p+j-1}$$

and $\Delta(w)=w^{p-1}r_H^{-1}(w)=\theta w^{p-1}$.

Using the above expressions in (8), the recurrence relation for single moments of GOS for the Kumaraswamy reflected exponential distribution is now obtained as

$$E\left(W_{r:n,m,k}^{p}\right)-E\left(W_{r-1:n,m,k}^{p}\right)=$$

$$\frac{p\theta}{ab\gamma_r}\left[\sum_{j=0}^{\infty}\frac{(-1)^j a^j}{j!\theta^j}E\left(W^{p+j-1}\right)-E\left(W_{r:n,m,k}^{p-1}\right)\right]$$

or

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=\frac{p\theta}{ab\gamma_r}\left[\sum_{j=0}^{\infty}\frac{(-1)^j a^j}{j!\theta^j}\mu_{r:n,m,k}^{p+j-1}-\mu_{r:n,m,k}^{p-1}\right].$$

...(15)

Again, for the reflected exponential distribution, we have

$$\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}\left(w_2\right)H^{-a}\left(w_2\right)$$

$$=\theta w_1^{p_1}w_2^{p_2-1}\exp\left(-\frac{aw_2}{\theta}\right)$$

$$=\theta\sum_{j=0}^{\infty}\frac{(-1)^j a^j}{j!\theta^j}w_1^{p_1}w_2^{p_2+j-1}$$

and

$$\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}\left(w_2\right)=\theta w_1^{p_1}w_2^{p_2-1}.$$

Using the above expressions (11), the recurrence relation for the joint moments of GOS for the Kumaraswamy reflected exponential distribution is obtained as

$$E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s-1:n,m,k}^{p_2}\right)=$$

$$\frac{p_2\theta}{ab\gamma_s}\left[\sum_{j=0}^{\infty}\frac{(-1)^j a^j}{j!\theta^j}E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2+j-1}\right)\right.$$

$$\left.-E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2-1}\right)\right],$$

or

$$\mu_{r,s:n,m,k}^{p_1,p_2}-\mu_{r,s-1:n,m,k}^{p_1,p_2}=$$

$$\frac{p_2\theta}{ab\gamma_s}\left[\sum_{j=0}^{\infty}\frac{(-1)^j a^j}{j!\theta^j}\mu_{r,s:n,m,k}^{p_1,p_2+j-1}-\mu_{r,s:n,m,k}^{p_1,p_2-1}\right].$$

...(16)

The recurrence relations for single and joint moments of *GOS* for the exponentiated reflected exponential distribution can be easily obtained from (15) and (16) by using $b = 1$.

Kumaraswamy inverse Weibull distribution

The Kumaraswamy inverse Weibull distribution (Shahbaz *et al*., 2012) is obtained by using the following cdf and pdf in (3) and (4)

$$H(w)=\exp\left(-\alpha w^{-\beta}\right) \;\&\; h(w)=\alpha\beta w^{-(\beta+1)}\exp\left(-\alpha w^{-\beta}\right)$$

$w,\alpha,\beta>0$.

Also, for the inverse Weibull distribution, we have $r_H(w)=h(w)/H(w)=\alpha\beta w^{-(\beta+1)}$. Now, the recurrence relations for single and joint moments of GOS for the Kumaraswamy inverse Weibull distribution can be obtained by using (8) and (11). For this, we first see that for the inverse Weibull distribution, we have

$$\Phi(w)=w^{p-1}r_H^{-1}(w)H^{-a}(w)=\frac{1}{\alpha\beta}w^{p+\beta}\exp\left(a\alpha w^{-\beta}\right)$$

$$=\frac{1}{\alpha\beta}\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}w^{p-\beta(j-1)}$$

and $\Delta(w)=w^{p-1}r_H^{-1}(w)=w^{p+\beta}/\alpha\beta$.

Using the above formulae in (8), the recurrence relation for single moments of GOS for the Kumaraswamy inverse Weibull distribution is obtained as

$$E\left(W_{r:n,m,k}^{p}\right)-E\left(W_{r-1:n,m,k}^{p}\right)=$$

$$\frac{p}{ab\alpha\beta\gamma_r}\left[\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}E\left(W^{p-\beta(j-1)}\right)-E\left(W_{r:n,m,k}^{p+\beta}\right)\right]$$

or $\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=$

$$\frac{p}{ab\alpha\beta\gamma_r}\left[\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}\mu_{r:n,m,k}^{p-\beta(j-1)}-\mu_{r:n,m,k}^{p+\beta}\right]$$

or $\mu_{r:n,m,k}^{p+\beta}=$

$$\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}\mu_{r:n,m,k}^{p-\beta(j-1)}-\frac{ab\alpha\beta\gamma_r}{p}\left(\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}\right). \quad ...(17)$$

Again, for the inverse Weibull distribution, we have

$$\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2)$$

$$=\frac{1}{\alpha\beta}w_1^{p_1}w_2^{p_2+\beta}\exp\left(-a\alpha w_2^{-\beta}\right)$$

and

$$\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)=w_1^{p_1}w_2^{p_2+\beta}/\alpha\beta.$$

Using the above expressions in (11), the recurrence relation for the joint moments of GOS for the Kumaraswamy reflected exponential distribution is obtained as

$$E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s-1:n,m,k}^{p_2}\right)=$$

$$\frac{p_2}{ab\alpha\beta\gamma_s}\left[\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}E\left\{W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2-\beta(j-1)}\right\}-E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2+\beta}\right)\right]$$

or $\mu_{r,s:n,m,k}^{p_1,p_2}-\mu_{r,s-1:n,m,k}^{p_1,p_2}=$

$$\frac{p_2}{ab\alpha\beta\gamma_s}\left[\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}\mu_{r,s:n,m,k}^{p_1,p_2-\beta(j-1)}-\mu_{r,s:n,m,k}^{p_1,p_2+\beta}\right]$$

or $\mu_{r,s:n,m,k}^{p_1,p_2+\beta}=$

$$\sum_{j=0}^{\infty}\frac{a^j\alpha^j}{j!}\mu_{r,s:n,m,k}^{p_1,p_2-\beta(j-1)}-\frac{ab\alpha\beta\gamma_s}{p_2}\left(\mu_{r,s:n,m,k}^{p_1,p_2}-\mu_{r,s-1:n,m,k}^{p_1,p_2}\right). \quad ...(18)$$

The relations (17) and (18) are useful to derive the relations for single and joint moments of GOS for the Kumaraswamy inverse exponential and Kumaraswamy inverse Rayleigh distributions. For example, using $\beta = 2$ in (17) and (18), the relations for single and joint moments of GOS for the inverse Rayleigh distribution are obtained.

Kumaraswamy log-logistic distribution

The Kumaraswamy log-logistic distribution (de Santana *et al*., 2012) is obtained by using the following cdf and pdf of the log-logistic distribution in (3) and (4)

$$H(w)=w^{\theta}\left(1+w^{\theta}\right)^{-1} \;\&\; h(w)=\theta w^{\theta-1}\left(1+w^{\theta}\right)^{-2}\;;\; x,\theta>0.$$

The relations for single and joint moments of *GOS* for the Kumaraswamy log-logistic distribution can be obtained by using (8) and (11). For this, we first see that for the log-logistic distribution, we have

$$\Phi(w)=w^{p-1}r_H^{-1}(w)H^{-a}(w)=\frac{1}{\theta}w^{p-a\theta}\left(1+w^{\theta}\right)^{a+1}$$

$$=\frac{1}{\theta}\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\,\Gamma(a-j+2)}w^{p-\theta(j-1)}$$

and $\Delta(w)=w^{p-1}r_H^{-1}(w)=\left(w^{p}+w^{p+\theta}\right)/\theta$.

Using the above formulae in (8), the recurrence relation for single moments of GOS for the Kumaraswamy log-logistic distribution is obtained as

$$E\left(W_{r:n,m,k}^{p}\right)-E\left(W_{r-1:n,m,k}^{p}\right)=\frac{p}{ab\theta\gamma_r}\left[\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}E\left(W_{r:n,m,k}^{p-\theta(j-1)}\right)-\left\{E\left(W_{r:n,m,k}^{p}\right)+E\left(W_{r:n,m,k}^{p+\theta}\right)\right\}\right]$$

or $ab\theta\gamma_r\mu_{r:n,m,k}^{p}-ab\theta\gamma_r\mu_{r-1:n,m,k}^{p}=p\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}\mu_{r-1:n,m,k}^{p-\theta(j-1)}-p\mu_{r:n,m,k}^{p}-p\mu_{r:n,m,k}^{p+\theta}$

or $\mu_{r:n,m,k}^{p+\theta}=\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}\mu_{r-1:n,m,k}^{p-\theta(j-1)}-\left(\frac{ab\theta\gamma_r}{p}+1\right)\mu_{r:n,m,k}^{p}+\frac{ab\theta\gamma_r}{p}\mu_{r-1:n,m,k}^{p}$. ...(19)

Again, for the log-logistic distribution, we have

$$\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2)=\frac{1}{\theta}w_1^{p_1}w_2^{p_2-a\theta}\left(1+w_2^{\theta}\right)^{a+1}=\frac{1}{\theta}\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}w_1^{p_1}w_2^{p_2-\theta(j-1)}$$

and $\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)=\left(w_1^{p_1}w_2^{p_2}+w_1^{p_1}w_2^{p_2+\theta}\right)/\theta$.

Using the above expressions in (11), the relation for the joint moments of GOS for the Kumaraswamy reflected exponential distribution is obtained as

$$\begin{aligned}E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)-E\left(W_{r:n,m,k}^{p_1}W_{s-1:n,m,k}^{p_2}\right)=\frac{p_2}{ab\theta\gamma_s}&\left[\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2-\theta(j-1)}\right)\right.\\&\left.-\left\{E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2}\right)+E\left(W_{r:n,m,k}^{p_1}W_{s:n,m,k}^{p_2+\theta}\right)\right\}\right]\end{aligned}$$

or $ab\theta\gamma_s\mu_{r,s:n,m,k}^{p_1,p_2}-ab\theta\gamma_s\mu_{r,s-1:n,m,k}^{p_1,p_2}=p_2\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}\mu_{r,s:n,m,k}^{p_1,p_2-\theta(j-1)}-q\mu_{r,s:n,m,k}^{p_1,p_2}-q\mu_{r,s:n,m,k}^{p_1,p_2+\theta}$

or $\mu_{r,s:n,m,k}^{p_1,p_2+\theta}=\sum_{j=0}^{\infty}\frac{\Gamma(a+2)}{j!\Gamma(a-j+2)}\mu_{r,s:n,m,k}^{p_1,p_2-\theta(j-1)}-\left(\frac{ab\theta\gamma_s}{p_2}+1\right)\mu_{r,s:n,m,k}^{p_1,p_2}+\frac{ab\theta\gamma_s}{p_2}\mu_{r,s-1:n,m,k}^{p_1,p_2}$. ...(20)

The relations (19) and (20) can be used to obtain the recurrence relations for single and joint moments for the special cases of GOS for the Kumaraswamy log-logistic distributions.

Kumaraswamy pareto distribution

The Kumaraswamy Pareto distribution (Bourguignon *et al*., 2012) is obtained by using the following cdf and pdf of the Pareto distribution in (3) and (4)

$$H(w)=1-\left(\frac{\beta}{w}\right)^{k}\ \&\ h(w)=\frac{k\beta^k}{w^{k+1}}\ ;\ w\geq\beta\ ,\ \beta,k>0.$$

The recurrence relations for single and joint moments of GOS for the Kumaraswamy Pareto distribution can be obtained by using (8) and (11). For this, we first see that for the Pareto distribution, we have

$$\Phi(w)=w^{p-1}r_H^{-1}(w)H^{-a}(w)=\frac{1}{k\beta^k}w^{p+k}\left[1-(w/\beta)^{-k}\right]^{-(a-1)}=\frac{1}{k\beta^k}\sum_{j=0}^{\infty}(-1)^j\beta^{kj}\frac{\Gamma(a+r-1)}{r!\Gamma(a-1)}w^{p-k(j-1)}$$

and $\Delta\left(w\right)=w^{p-1}r_{H}^{-1}\left(w\right)=\dfrac{w^{p+k}}{k\beta^{k}}\left[1-\left(w/\beta\right)^{-k}\right]=\dfrac{1}{k\beta^{k}}\left(w^{p+k}-\beta^{k}w^{p}\right).$

Using the above expressions in (8), the relation for the single moments of GOS for the Kumaraswamy log-logistic distribution is obtained as

$$E\left(W_{r:n,m,k}^{p}\right)-E\left(W_{r-1:n,m,k}^{p}\right)=\frac{p}{abk\beta^{k}\gamma_{r}}\left[\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\frac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}E\left(W_{r:n,m,k}^{p-k(j-1)}\right)-\left\{E\left(W_{r:n,m,k}^{p+k}\right)-\beta^{k}E\left(W_{r:n,m,k}^{p}\right)\right\}\right]$$

or $abk\beta^{k}\gamma_{r}\mu_{r:n,m,k}^{p}-abk\beta^{k}\gamma_{r}\mu_{r-1:n,m,k}^{p}=p\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\dfrac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}\mu_{r-1:n,m,k}^{p-k(j-1)}-p\mu_{r:n,m,k}^{p+k}+p\beta^{k}\mu_{r:n,m,k}^{p}$

or $\mu_{r:n,m,k}^{p+k}=\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\dfrac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}\mu_{r-1:n,m,k}^{p-k(j-1)}+\dfrac{\beta^{k}\left(p-abk\gamma_{r}\right)}{p}\mu_{r:n,m,k}^{p}+\dfrac{abk\beta^{k}\gamma_{r}}{p}\mu_{r-1:n,m,k}^{p}.$ ...(21)

Again, for the log-logistic distribution, we have

$$\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_{1}^{p_{1}}w_{2}^{p_{2}-1}r_{H}^{-1}\left(w_{2}\right)H^{-a}\left(w_{2}\right)=\frac{1}{k\beta^{k}}w_{1}^{p_{1}}w_{2}^{p_{2}+k}\left[1-\left(w_{2}/\beta\right)^{-k}\right]^{-(a-1)}$$
$$=\frac{1}{k\beta^{k}}\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\frac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}w_{1}^{p_{1}}w_{2}^{p_{2}-k(j-1)}$$

and $\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_{1}^{p_{1}}w_{2}^{p_{2}-1}r_{H}^{-1}\left(w_{2}\right)=\dfrac{1}{k\beta^{k}}\left(w_{1}^{p_{1}}w_{2}^{p_{2}+k}-\beta^{k}w_{1}^{p_{1}}w_{2}^{p_{2}}\right).$

Using the above formulae in (11), the relation for the joint moments of GOS for the Kumaraswamy log-logistic distribution is obtained as

$$E\left(W_{r:n,m,k}^{p_{1}}W_{s:n,m,k}^{p_{2}}\right)-E\left(W_{r:n,m,k}^{p_{1}}W_{s-1:n,m,k}^{p_{2}}\right)=\frac{p_{2}}{abk\beta^{k}\gamma_{s}}\left[\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\frac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}E\left(W_{r:n,m,k}^{p_{1}}W_{s:n,m,k}^{p_{2}-k(j-1)}\right)\right.$$
$$\left.-\left\{E\left(W_{r:n,m,k}^{p_{1}}W_{s:n,m,k}^{p_{2}+k}\right)-\beta^{h}E\left(W_{r:n,m,k}^{p_{1}}W_{s:n,m,k}^{p_{2}}\right)\right\}\right]$$

or $abk\beta^{k}\gamma_{s}\mu_{r,s:n,m,k}^{p_{1},p_{2}}-abk\beta^{k}\gamma_{s}\mu_{r,s-1:n,m,k}^{p_{1},p_{2}}=q\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\dfrac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}\mu_{r,s:n,m,k}^{p_{1},p_{2}-k(j-1)}-q\mu_{r,s:n,m,k}^{p_{1},p_{2}+k}+q\beta^{k}\mu_{r,s:n,m,k}^{p_{1},p_{2}}$

or $\mu_{r,s:n,m,k}^{p_{1},p_{2}+k}=\sum_{j=0}^{\infty}\left(-1\right)^{j}\beta^{kj}\dfrac{\Gamma\left(a+r-1\right)}{r!\Gamma\left(a-1\right)}\mu_{r,s:n,m,k}^{p_{1},p_{2}-k(j-1)}+\dfrac{\beta^{k}\left(q-abk\gamma_{s}\right)}{q}\mu_{r,s:n,m,k}^{p_{1},p_{2}}+\dfrac{abk\beta^{k}\gamma_{s}}{q}\mu_{r,s-1:n,m,k}^{p_{1},p_{2}}.$ ...(22)

The relations for single and joint moments of GOS for the log-logistic distribution can be easily obtained from (21) and (22) by setting $a = b = 1$.

CHARACTERIZATION RESULTS

In this section, we have given some characterization results for the *Kum–G* family of distributions based on single and joint moments of GOS. These results are given in the following Theorems.

Theorem 3: For a random variable X to have density and distribution functions as given in (5) and (6), respectively, it is necessary and sufficient that the single moments of its *GOS* are related as

$$E\left(W^{p}_{r:n,m,k}\right)-E\left(W^{p}_{r-1:n,m,k}\right)=\frac{p}{ab\gamma_r}\left[E\left\{\Phi\left(W_{r:n,m,k}\right)\right\}-E\left\{\Delta\left(W_{r:n,m,k}\right)\right\}\right],$$

where $\Phi(x)=x^{p-1}r_G^{-1}(x)G^{-a}(x)$ and $\Delta(x)=x^{p-1}r_G^{-1}(x)$.

Proof: The necessity follows immediately from Theorem 1. Now, to prove the sufficiency, we consider (9) as

$$\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}w^{p-1}\left\{\bar{F}(w)\right\}^{\gamma_r}\tau_m^{r-1}\left[F(w)\right]dw$$

$$=\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{c}^{\infty}w^{p-1}\left\{\bar{F}(w)\right\}^{\gamma_r-1}\tau_m^{r-1}\left\{F(w)\right\}$$

$$\times\left[\frac{1}{ab}r_H^{-1}(w)\left\{H^{-a}(w)-1\right\}f(w)\right]dw$$

or

$$\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}w^{p-1}\left\{\bar{F}(w)\right\}^{\gamma_r-1}\tau_m^{r-1}\left[F(w)\right]$$

$$\left[\bar{F}(w)-\frac{1}{ab}r_H^{-1}(w)\left\{H^{-a}(w)-1\right\}f(w)\right]dw=0$$

Applying the *Müntz–Száz* theorem to the above equation (Hwang and Lin, 1984), we have

$$\bar{F}(w)=\frac{1}{ab}r_H^{-1}(w)\left[H^{-a}(w)-1\right]f(w),$$

which is the relationship between the density and distribution function of the Kumaraswamy family of distributions and the theorem is complete.

Theorem 4: For a random variable X to have density and distribution functions as given in (5) and (6), respectively, it is necessary and sufficient that the joint moments of its GOS are related as

$$E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s:n,m,k}\right)-E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s-1:n,m,k}\right)=$$

$$\frac{p_2}{ab\gamma_s}\left[E\left\{\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)\right\}-E\left\{\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)\right\}\right],$$

where $\Phi\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)H^{-a}(w_2)$

and $\Delta\left(W_{r:n,m,k}W_{s:n,m,k}\right)=w_1^{p_1}w_2^{p_2-1}r_H^{-1}(w_2)$.

Proof: The necessity follows immediately from Theorem 2. Now, to prove the sufficiency, we consider (12) as

$$E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s:n,m,k}\right)-E\left(W^{p_1}_{r:n,m,k}W^{p_2}_{s-1:n,m,k}\right)=$$

$$\frac{p_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{w_1}^{\infty}w_1^{p_1}w_2^{p_2-1}f(w_1)\left[\bar{F}(w_1)\right]^m$$

$$\times\tau_m^{r-1}\left[F(x_1)\right]\left[\varphi_m\left\{F(w_2)\right\}-\varphi_m\left\{F(w_1)\right\}\right]^{s-r-1}$$

$$\bar{F}(w_2)\left[\bar{F}(w_2)\right]^{\gamma_s-1}dw_2dw_1.$$

Using the above expression alongside (7), we have

$$\frac{p_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{c}^{\infty}\int_{w_1}^{\infty}w_1^{p_1}w_2^{p_2-1}f(w_1)\left[\bar{F}(w_1)\right]^m\tau_m^{r-1}\left[F(w)\right]\left[\varphi_m\left\{F(w_2)\right\}-\varphi_m\left\{F(w_1)\right\}\right]^{s-r-1}$$

$$\times\left[\bar{F}(w_2)\right]^{\gamma_s}dw_2dw_1=\frac{p_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{c}^{\infty}\int_{w_1}^{\infty}w_1^{p_1}w_2^{p_2-1}f(w_1)\left[\bar{F}(w_1)\right]^m\tau_m^{r-1}\left[F(w)\right]$$

$$\times\left[\varphi_m\left\{F(w_2)\right\}-\varphi_m\left\{F(w_1)\right\}\right]^{s-r-1}\left[\bar{F}(w_2)\right]^{\gamma_s-1}\left[\frac{1}{ab}r_H^{-1}(w_2)\left\{H^{-a}(w_2)-1\right\}f(w_2)\right]dw_2dw_1$$

or

$$\frac{p_2 C_{s-1}}{\gamma_s (r-1)!(s-r-1)!}\int_c^\infty \int_{w_1}^\infty w_1^{p_1} w_2^{p_2-1} f(w_1)\left[\bar{F}(w_1)\right]^m \tau_m^{r-1}\left[F(w_1)\right]\left[\varphi_m\{F(w_2)\}-\varphi_m\{F(w_1)\}\right]^{s-r-1}$$

$$\times\left[\bar{F}(w_2)\right]^{\gamma_s-1}\left[\bar{F}(w_2)-\frac{1}{ab}r_H^{-1}(w_2)\left\{H^{-a}(w_2)-1\right\}f(w_2)\right]dw_2 dw_1=0.$$

Applying the *Müntz–Száz* theorem to the above equation (Hwang and Lin, 1984), we have

$$\bar{F}(w_2)=\frac{1}{ab}r_H^{-1}(w_2)\left[H^{-a}(w_2)-1\right]f(w_2),$$

which is (7) and hence the theorem.

CONCLUSION

In this paper, we have derived some recurrence relations for the single and joint moments of GOS for the Kumaraswamy (*Kum–G*) family of distributions. The relations derived are useful to obtain the relations for single and joint moments of GOS for any member of the *Kum–G* family of distributions. The obtained relations are also helpful in deriving the relations for single and joint moments of the special cases of GOS. The applicability of the derived relations is illustrated with the help of some example distributions, including Kumaraswamy power function, Kumaraswamy reflected exponential, Kumaraswamy inverse Weibull, Kumaraswamy log-logistic, etc. The recurrence relations are also used to characterize the *Kum–G* family of distributions.

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