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RESEARCH ARTICLE

Fuzzy Mathematics

Networking model based on fuzzy graphs of almost primary fuzzy ideals of near-rings

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Abstract: Sometimes traditional mathematical methods do not work properly to find the solutions of the problems with uncertainties in different field of sciences. The reason is that there are lot of complexities and ambiguities which occur in real-world problems. Researchers from all across the world have created new theories such as fuzzy set theory, rough set theory, *etc*., to overcome such circumstances. Over the past few decades numerous fuzzy algebraic structures and substructures have been studied, like fuzzy rings, fuzzy near-rings, fuzzy ideals of near-rings, and so on. In this study, we present the notion of almost primary fuzzy ideals (almost primary *f*-ideals) of near-rings and provide their application through the fuzzy graphs associated with them. Firstly, we provide numerous characterizations of almost primary *f*-ideals in near-rings. We observe that every almost primary *f*-ideal is not a primary fuzzy ideal (PFI) in near-rings, and show when a primary fuzzy ideal is an almost primary *f*-ideals in near-rings. Some inter-relationships between a classical almost primary ideal (API) and almost primary *f*-ideals of near-rings are also established. Finally, we introduce the fuzzy graph structure related to APFIs of near-rings and present a model of general networks.

Keywords: Almost primary fuzzy ideal in near-rings, fuzzy graphs, fuzzy ideals of near-rings, fuzzy prime ideal of near-rings.

INTRODUCTION

Zadeh (1965) introduced the notion of fuzzy sets (FSs). In FSs, each element was allocated a value from the interval [0, 1]. Due to the flexible nature of FSs, many generalizations of FSs have been explored in the literature. The very first extension of the FSs was also presented by Zadeh (1975) and termed as the interval-valued FSs. Afterwards, different extensions of FSs such as intuitionistic FSs (Atanassov & Atanassov, 1999), bipolar FSs (Zhang, 1998), Pythagorean FSs (Razaq & Alhamzi, 2023), picture FSs (Cuong, 2014), *etc* have been added in the literature.

FSs were utilized in different fields especially where uncertainties were involved. FSs have resolved many issues in decision-making procedures involving vague information, enabling more flexible and sensitive assessments in contexts such as multi-criteria decision-making. Fuzzy logic controllers, particularly in engineering, are useful for controlling complicated systems like robots and automated temperature control because they approximate human thinking. Due to their ability to handle pixel value discrepancies, FSs are especially useful in image processing, where they aid in tasks like edge recognition and noise reduction. On the other hands, fuzzy graphs (FGs) were first introduced by Rosenfeld (1975). FGs have fuzzy associations between nodes which extend the idea of classical graphs. Features of FGs become helpful in solving problems with uncertainties in different fields, such as networking, medicine, engineering etc. They have also been used in clustering algorithms to enhance the way that overlapping data points and unclear group memberships are involved.

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FSs and FGs become useful tools for controlling ambiguity and imprecision in a variety of areas from data mining to artificial intelligence (Bloch & Ralescu, 2023).

In the last few decades, many concepts of classical rings, near-rings, and near-semirings have been shifted towards FSs theory. In this regard, the concepts of fuzzy ideals (in short, f-ideals) in rings, near-rings, and near-semirings have been introduced in the literature. Liu (1982) introduced f-ideals in rings. If for every $x, y \in R, \mathcal{I}(x \cdot y) \geq \mathcal{I}(x)(resp., \mathcal{I}(x \cdot y) \geq \mathcal{I}(y))$, then we call it a f-left (resp. right) ideal of a ring R. The fuzzy prime ideal in a ring was first introduced by Mukherjee *et al.* (1987). The primary f-ideal in ring was introduced by Malik and Mordeson (1991). Various kinds of prime ideals of near-rings have been introduced by Birkenmeier *et al.* (1993), Elavarasan (2011), and Groenewald (1991). Birkenmeier *et al.* (1993) discussed the link between the prime ideal and type-1 prime ideal. Elavarasan (2011) introduced the almost prime ideal in near-rings. The concepts of $\mathcal{V}$-primary ideals and their graphs have been discussed by Muhammad *et al.* (2023). Recently, Khan *et al.* (2018) introduced almost prime ideals in Γ-near-rings. The concepts of f-ideal and prime f-ideal of $\mathcal{N}$ were introduced by Abou-Zaid (1991). Different types of prime f-ideals in near-rings were introduced by Kedukodi *et al.* (2009).

Recently, Razaq and Alhamzi (2023) examined many algebraic aspects of Pythagorean f-ideals in classical rings. Onar *et al.* (2023) extended the understanding of algebraic structures within G-rings by introducing the concept of non-symmetric 2-absorbing vague weakly complete G-ideals in commutative G-rings. Jebapresitha (2024) explored the hybrid structure of FSs within near-rings, effectively transformed f-ideals into hybrid ideals and fuzzy maximal ideals into hybrid maximal ideals. The transformation of f-ideals into hybrid ideals, such as fuzzy maximal ideals into hybrid maximal ideals, addressed the uncertainty problems more efficiently. Subsequently, Batool *et al.* (2023) discovered intuitionistic multi-fuzzy sub-near-rings and ideals within near-rings, exploring foundational operations and relationships within these structures. Adak *et al.* (2023) studied the concepts of semi-prime ideals within the context of picture FSs. They introduced the methods for constructing picture fuzzy regular and intra-regular ideals along with fundamental insights related to these constructions.

Numerous graph structures associated with algebraic structures such as rings, near-rings, ideals of rings and near-rings, etc., have been discussed in the literature. Similarly, fuzzy graphs linked with fuzzy algebraic structures have also been explored. The graphs based on ring structure consist of ring elements as vertices of graphs, while the edges of the graphs represent some operations of the rings, including addition and multiplication. In contrast to conventional graphs, the vertices and edges of the FGs represent a membership value that measures the degree of membership and the strength of the link. FGs of rings become useful in many fields like coding theory, cryptography or network analysis. Algebraic structures usually interact with uncertain data because they enable the modelling of imprecise interactions inside the ring. This method allows for studying uncertainties that may occur in real-world applications, and allows for a more clear understanding of the interactions inside the ring by adding fuzziness to the graphs (Mordeson *et al.*, 2023).

Jagadeesha *et al*. (2023), introduced equiprime fuzzy graphs associated with a near-ring in relation to the level ideal of a f-ideal. They discussed the theoretical properties of both the graphs and ideals, and demonstrated their key characteristics such as vertex cut and connectedness influenced by the properties of the f-ideal. Srinivas and Prasad (2024) introduced the notion of a c-prime f-graph associated with a near-ring. They focused on the relationship between the properties of a f-ideal and their corresponding fuzzy graphs. Additionally, a connection between near-ring homomorphism and graph homomorphism was also described by them. They also considered that the homomorphic image of a c-prime f-ideal remains a c-prime f-ideal.

Research gap

From literature review, we come to know that fuzzy set theory in relation to rings and near-rings is playing a key role in solving the problems lying in different fields of sciences. Within this fuzzy framework, classical rings and ideals have been transformed into fuzzy rings and fuzzy ideals. In this regard, prime and primary f-ideals of near-rings have been defined parallel to rings. However, almost primary f-ideals of near-rings have not been discussed to date. To fill this gap, we initiate the idea of almost primary f-ideal in near-rings. In addition, we explore fuzzy graphs associated with these ideals and present their application in networking.

Novelty

The novelty of this study can be described as follows.

i. Firstly, we introduce the notion of almost primary f-ideals of near-rings.
ii. Numerous characterizations of almost primary f-ideals in near-rings are discussed.

iii. We consider that every almost primary *f*-ideal is not a primary fuzzy ideal (PFI) in near-rings and find a condition when a primary fuzzy ideal is almost primary *f*-ideals in near-rings.
iv. For near-rings, we establish some inter-relationships between the classical almost primary ideal (API) and almost primary *f*-ideals.
v. Finally, we present the fuzzy graph structures related to APFIs of near-rings and their application in networking.

METHODOLOGY

This section consists of basic concepts related to the existing algebraic structures like rings, near-rings, semi-near-rings, and their ideals and fuzzy ideals. These concepts will help to understand the presented work.

An algebraic structure $(\mathcal{N}, +, \cdot)$ is called a (right) near-ring, if $\mathcal{N}$ is a group (not necessarily Abelian) under "+", semigroup under "·" and fulfils (right) distributive law. For the elementary notions of near-rings, we refer Pilz (2011). Right ideal of $\mathcal{N}$ is a subset $\mathcal{J}$ of $\mathcal{N}$, where (i) $(\mathcal{J}, +)$ is normal subgroup of $(\mathcal{N}, +), (ii)$ For all $\mathcal{J}\mathcal{N} \subseteq \mathcal{J}$. Similarly, $\mathcal{J}$ is a left ideal, if $(i)(\mathcal{J}, +) \trianglelefteq (\mathcal{N}, +), (ii) a_1 \cdot (a_2 + i) - a_1 \cdot a_2 \in \mathcal{J}$, for every $a_1, a_2 \in \mathcal{N}$ and $i \in \mathcal{J}$. An ideal $\mathcal{J}$ of $\mathcal{N}$ is called prime, if for every $\mathcal{J}_1$, $\mathcal{J}_2 \subset \mathcal{N}$, $\mathcal{J}_1\mathcal{J}_2 \subseteq \mathcal{J} \Rightarrow \mathcal{J}_1 \subseteq \mathcal{J}$ or $\mathcal{J}_2 \subseteq \mathcal{J}$. Various kinds of prime ideals of near-rings have been established in (Birkenmeier *et al*., 1993; Elavarasan, 2011; Groenewald, 1991).

Now we present some useful terminologies which would be helpful in understanding the forthcoming literature.

Definition 1. (Groenewald, 1991) $\mathcal{J}$ is called a 0-prime ideal of $\mathcal{N}$, if $\mathcal{J}_1\mathcal{J}_2 \subseteq \mathcal{J} \Rightarrow \mathcal{J}_1 \subseteq \mathcal{J}$ or $\mathcal{J}_2 \subseteq \mathcal{J}$, where $\mathcal{J}_1$, $\mathcal{J}_2$ are ideals of near-ring.

Definition 2. (Groenewald, 1991) An ideal $\mathcal{J}$ is called a prime ideal, if $\mathcal{J}_1\mathcal{J}_2 \subseteq \mathcal{J} \Rightarrow \mathcal{J}_1 \subseteq \mathcal{J}$ or $\mathcal{J}_2 \subseteq \mathcal{J}$, where $\mathcal{J}_1$ & $\mathcal{J}_2$ are ideals of near-ring.

Definition 3. (Groenewald, 1991) $\mathcal{J}$ is said to be 3-prime ideal of $\mathcal{N}$, if $a\mathcal{N}b \subseteq \mathcal{J}$, then $a \in \mathcal{J}$ or $b \in \mathcal{J}$, for $a, b \in \mathcal{N}$.

Definition 4. (Elavarasan, 2011) $\mathcal{J}$ is called an almost prime ideal if for $\mathcal{J}_1\mathcal{J}_2 \subseteq \mathcal{J}$ and $\mathcal{J}_1\mathcal{J}_2 \nsubseteq \mathcal{J}^2 \Longrightarrow \mathcal{J}_1 \subseteq \mathcal{J}$ or $\mathcal{J}_2 \subseteq \mathcal{J}$ where $\mathcal{J}_1$ & $\mathcal{J}_2$ are ideals of $\mathcal{N}$.

Theorem 1. (Elavarasan, 2011) Let $\mathcal{N}$ be a near-ring and $\mathcal{J}$ be an almost prime ideal of $\mathcal{N}$. Then $\mathcal{J}^2 = \mathcal{J}$, if $\mathcal{J}$ is a prime ideal.

Corollary 1. (Elavarasan, 2011) Let $\mathcal{J}$ be a proper ideal of $\mathcal{N}$ and if $\mathcal{J}$ is an almost prime with $(\mathcal{J}^2 : \mathcal{J}) \subseteq \mathcal{J}$, then $\mathcal{J}$ is a prime ideal.

Definition 5. (Zadeh, 1965) A fuzzy set T on a non-empty set U is described as $= \{(u, r(u)) : r(u) \in [0,1]\}$.

Definition 6. (Rosenfeld, 1975) A fuzzy graph is a pair $G = (C, D)$ where $C = \{\rho_C\}$ and $D = \{\rho_D\}$ such that $\rho_C : V \to [0,1]$ and $\rho_D : V \times V \to [0,1]$. Then, $\rho_D(x, w) \leq \rho_c(x) \wedge \rho_c(w)$.

Definition 7. (Mukherjee *et al.*, 1987) A f-ideal $\mathcal{P}$ of a R is a prime fuzzy ideal (prime f-ideal), if $\mathcal{J}_1 \circ \mathcal{J}_2 \subseteq \mathcal{P}$ implies $\mathcal{J}_1 \subseteq \mathcal{P}$ or $\mathcal{J}_2 \subseteq \mathcal{P}$, where $\mathcal{J}_1, \mathcal{J}_2 \in R$.

Definition 8. (Malik & Mordeson, 1991) An ideal $\mathcal{P}$ is called a primary f-ideal of a ring, if the values of $\mathcal{P}$ is not a constant and $\mathcal{P}_1 \circ \mathcal{P}_2 \subseteq \mathcal{P}$ implies $\mathcal{P}_1 \subseteq \mathcal{P}$ or $\mathcal{P}_2 \subseteq \sqrt{\mathcal{P}}$, where $\mathcal{P}_1, \mathcal{P}_2$ are f-ideals.

Definition 9. (Abou-Zaid, 1991) An f-ideal $\mathcal{J}$ is called a right f-ideal of $\mathcal{N}$, if $\mathcal{J}$ is a normal f-subgroup under addition and $\mathcal{J}(n_1 \cdot n_2) \geq \mathcal{J}(n_1)$, for all $n_1, n_2 \in \mathcal{N}$. Similarly, $\mathcal{J}$ is called a left f-ideal of $\mathcal{N}$, if $\mathcal{J}$ is normal f-subgroup under addition and $\mathcal{J}(a \cdot (b + i) - a \cdot b) \geq \mathcal{J}(i)$, for every $a, b, i \in \mathcal{N}$.

Definition 10. (Kedukodi *et al.* 2009) An ideal $\mathcal{P}$ of $\mathcal{N}$ is said to be a c-prime f-ideal, if for every $a, b \in \mathcal{N}$, $\max\{\mathcal{P}(a), \mathcal{P}(b)\} \geq \mathcal{P}(a \cdot b)$. Similarly, $\mathcal{P}$ is called a 3-prime f-ideal, if $\max\{\mathcal{P}(a), \ \mathcal{P}(b)\} \geq \mathcal{P}(a \cdot n \cdot b)$, for every $a, b, n \in \mathcal{N}$.

Definition 11. (Mukherjee & Sen, 1987) A f-ideal $\mathcal{P}$ of R is called an primary f-ideal if for $a, b \in R$ such that $\mathcal{P}(\mathrm{ab}) \geq \mathcal{P}(\mathrm{a})$ implies $\mathcal{P}(\mathrm{b}^{\mathrm{n}}) \geq \mathcal{P}(\mathrm{ab})$.

Theorem 2. (Mukherjee & Sen, 1987) Let $\mathcal{P}$ be an ideal of R. Then its characteristic function is a fuzzy primary ideal of R if and only if $\mathcal{P}$ is a primary ideal.

Theorem 3. (Mukherjee & Sen, 1987) Suppose R is a ring, then every f-prime ideal in R is a f-almost prime ideal in R.

RESULTS AND DISCUSSIONS

In this section, we introduce the notion of almost primary f-ideals in near-rings and explore some of the generalizations of the almost primary f-ideals of near-rings. In addition, we also provide several characterizations of these ideals.

Definition 12. An f-ideal $\mathcal{P}$ of $\mathcal{N}$ is said to be almost primary f-ideal, if $\mathcal{P}(a \cdot b) > \mathcal{P}(1_{\mathcal{N}})$ and $(\mathcal{P} \circ \mathcal{P})(a \cdot b) = \mathcal{P}(1_{\mathcal{N}})$, we have $\mathcal{P}(a \cdot b) \geq \mathcal{P}(a^n)$ or $\mathcal{P}(a \cdot b) \geq \mathcal{P}(b^n)$, for all $a, b, c \in \mathcal{N}$ and some $n \in \mathbb{Z}^+$, where

$$(\mathcal{P} \circ \mathcal{P})(c) = \begin{cases} \sup(min\{\mathcal{P}(a), \quad \mathcal{P}(b)\}) & \text{for} \quad a \cdot b = c \\ 0, & a \cdot b \neq c \end{cases}$$

Example 1. Let us consider a near-ring defined in tables set 1.

Tables set. 1

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	3	0	1	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	7	5	6	2	0	1	3
5	5	6	4	7	0	2	3	1
6	6	4	7	5	1	3	0	2
7	7	5	6	4	3	1	2	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	2	2	0	0
3	0	3	2	1	5	4	6	7
4	0	4	2	5	4	5	6	7
5	0	6	2	4	5	4	6	7
6	0	6	0	6	0	0	0	0
7	0	7	0	7	2	2	0	0

Consider a mapping $\mathcal{P}: \mathcal{N} \longrightarrow [0,1]$ such that $\mathcal{P}(0) = \mathcal{P}(1) = \mathcal{P}(3) = \mathcal{P}(5) = \mathcal{P}(1_{\mathcal{N}}) = 0, \mathcal{P}(2) = \mathcal{P}(4) = \mathcal{P}(6) = \mathcal{P}(7) = \alpha$, where $\alpha \in (0,1]$. Here, $\mathcal{P}$ is an almost primary f-ideal. To verify the definition we chose $3, 5 \in \mathcal{N}$ such that $\mathcal{P}(3 \cdot 6) = \mathcal{P}(6) = \alpha > \mathcal{P}(1_{\mathcal{N}}) = 0$ and $(\mathcal{P} \circ \mathcal{P})(6) = \sup_{6=3\cdot 6}(min\{\mathcal{P}(3), \mathcal{P}(6)\}) = \sup_{6=3\cdot 6}(min\{0, \alpha\}) = 0 = \mathcal{P}(1_{\mathcal{N}})$, then $\Rightarrow \mathcal{P}(3 \cdot 6) = \alpha \geq \mathcal{P}(3) = 0$ or $\mathcal{P}(3 \cdot 6) \geq \mathcal{P}(6^2) = \mathcal{P}(0) = 0$. Similarly, $5, 7 \in \mathcal{N}$ such that $\mathcal{P}(5 \cdot 7) = \mathcal{P}(7) = \alpha > \mathcal{P}(1_{\mathcal{N}}) = 0$ $\mathcal{P}(5 \cdot 7) = \mathcal{P}(7) = \alpha > \mathcal{P}(1_{\mathcal{N}}) = 0$ and $(\mathcal{P} \circ \mathcal{P})(7) = \sup_{7=5\cdot 7}(min\{\mathcal{P}(5), \mathcal{P}(7)\}) = \sup_{7=5\cdot 7}(min\{0, \alpha\}) = 0 = \mathcal{P}(1_{\mathcal{N}}), \Rightarrow \mathcal{P}(5 \cdot 7) = \alpha \geq \mathcal{P}(5) = 0$ or $\mathcal{P}(5 \cdot 7) \geq \mathcal{P}(7^2) = \mathcal{P}(0) = 0$. According to the definition of primary f-ideal, we can easily verify that $\mathcal{P}$ is not a primary f-ideal. For this, let $2, 4 \in \mathcal{N}$ such that $\mathcal{P}(2 \cdot 4) = \mathcal{P}(2) = \alpha \leq \mathcal{P}(2) = 0$ but $\mathcal{P}(2 \cdot 4) = \mathcal{P}(2) = \alpha > \mathcal{P}(4^2) = \alpha$, which contradicts the definition of primary f-ideal of $\mathcal{N}$.

Remarks. Every almost primary f-ideal is not a primary f-ideal of near-rings.

Proposition 1. Let $\mathcal{N}$ be a near-ring. Then every primary f-ideal of $\mathcal{N}$ is an almost primary f-ideal of $\mathcal{N}$.

Proof. Let $\mathcal{P}$ be a primary f-ideal of $\mathcal{N}$ and $a, b \in \mathcal{N}$ such that $\mathcal{P}(a \cdot b) > \mathcal{P}(1_{\mathcal{N}})$ and $(\mathcal{P} \circ \mathcal{P})(a \cdot b) = \mathcal{P}(1_{\mathcal{N}})$. As $\mathcal{P}$ is primary f-ideal then $\mathcal{P}(a \cdot b) \leq \mathcal{P}(\mathrm{a})$ and $\mathcal{P}(a \cdot b) > \mathcal{P}(\mathrm{b^n})$ for some $n \in \mathbb{Z}^+$, so without lose of generality, we have $\mathcal{P}(b^n) = min\{\mathcal{P}(a), \mathcal{P}(b)\} = \mathcal{P}(1_{\mathcal{N}})$, then $(\mathcal{P})(a \cdot b) > \mathcal{P}(b^n) = \mathcal{P}(1_{\mathcal{N}})$since$(\mathcal{P} \circ \mathcal{P})(a \cdot b) = \mathcal{P}(1_{\mathcal{N}})$, then $\mathcal{P}(a \cdot b) \geq \mathcal{P}(b^n)$. This suggests that $\mathcal{P}$ is an almost primary f-ideal of near-ring.

However, the converse of the above proposition need not be true, in general. In this context, we provide an example as evidence.

Example 2. Let $\mathcal{N} = \{0,1,2,3,4,5,6,7\}$ be a right near-ring described in tables set 2.

Tables set 2

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	7	5	6	2	0	1	3
5	5	6	4	7	0	2	3	1
6	6	4	7	5	1	3	0	2
7	7	5	6	4	3	1	2	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	2	2	0	0
3	0	3	2	1	5	4	6	7
4	0	4	2	5	4	5	6	7
5	0	6	2	4	5	4	6	7
6	0	6	0	6	0	0	0	0
7	0	7	0	7	2	2	0	0

As, $\mathcal{N} = \{0,1,2,3,4,5,6,7\}$ and $\mathcal{J} = \{0,2\}$. Set mapping $\mathcal{P}: \mathcal{N} \longrightarrow [0,1]$ is given as

$$\mathcal{P}(x) = \begin{cases} 1, & \text{if } \ x \in < \mathcal{J}^2 > \\ \alpha, & x \in \mathcal{J} - \mathcal{J}^2, x \in \mathcal{N} \text{ and } \alpha \in (0,1] \\ 0, & \text{else} \end{cases}$$

Now, we must demonstrate that $\mathcal{P}$ is an almost primary f-ideal of $\mathcal{N}$. Consider the following cases.

Case I. Suppose $x, y \in \mathcal{N}$. If $x \in \mathcal{J}$ and $y \notin \mathcal{J}$ then $x - y \notin \mathcal{J}$. Therefor $\mathcal{P}(x-y) = \mathcal{P}(1_{\mathcal{N}}) = 0 = \mathcal{P}(y) = min\{\mathcal{P}(x),\ \mathcal{P}(y)\}$ and

$$\mathcal{P}(x \cdot y) = \begin{cases} 1, & \text{if } \ x \cdot y \in \mathcal{J}^2 \\ \alpha, & x \cdot y \in \mathcal{J} - \mathcal{J}^2, \text{where } x \in \mathcal{N} \text{and } \ \alpha \in (0,\ 1]. \end{cases}$$

If $\mathcal{P}(x \cdot y) = 1$, then $\mathcal{P}(x \cdot y) \geq \mathcal{P}(x^n)$ or $\mathcal{P}(y^n)$, where $n \in \mathbb{Z}^+$ and $\mathcal{P}(x^n) = \mathcal{P}(y^n) = min\{\mathcal{P}(x),\ \mathcal{P}(y)\}$ and if $\mathcal{P}(x \cdot y) = \alpha$, then $x \cdot y \in \mathcal{J} - \mathcal{J}^2$, so $\mathcal{P}(x^n) = \alpha$, then $min\{\mathcal{P}(x), \mathcal{P}(y)\} = \alpha$. As a result, $\mathcal{P}(x \cdot y) \geq \mathcal{P}(x^n)$ or $\mathcal{P}(y^n)$.

Case II. Assume, $x \in \mathcal{J}$ and $y \in \mathcal{J} - (\mathcal{J} \circ \mathcal{J})$, then $min\{\mathcal{P}(x), \mathcal{P}(y)\} = \alpha$ & $x - y \in \mathcal{J}$. Therefor, $\mathcal{P}(x-y) = \alpha = \mathcal{P}(y^n) = min\{\mathcal{P}(x), \mathcal{P}(y)\}$ and $x \cdot y \in \mathcal{J}^2$ implies that $\mathcal{P}(x \cdot y) = 1 \geq min\{\mathcal{P}(x), \mathcal{P}(y)\}$ $= \mathcal{P}(x^n)$ or $\mathcal{P}(y^n)$.

Case III. If $x \in \mathcal{J}$ and $y \in \mathcal{J}$, then $min\{\mathcal{P}(x),\ \mathcal{P}(y)\} = 0$ & $\mathcal{P}(x-y) \geq 0$. Further to this, $\mathcal{P}(x-y) \geq min\{\mathcal{P}(x),\ \mathcal{P}(y)\}$. Since $\mathcal{J}$ is a primary ideal in $\mathcal{N}$, at that point $x^n \notin \mathcal{J}$ and $y^n \in \mathcal{J}$ for some $\mathbb{Z}^+$. So, $x \cdot y \notin \mathcal{J}$, this implies $\mathcal{P}(x \cdot y) = min\{\mathcal{P}(x),\ \mathcal{P}(y)\} = 0$.

Hence, $\mathcal{P}$ is f-ideal of $\mathcal{N}$. Now, we have to show that $\mathcal{P}$ is an almost primary f-ideal of $\mathcal{N}$.

Suppose, $x,\ y \in \mathcal{N} | \mathcal{P}(x \cdot y) = 0$ and $(\mathcal{P} \circ \mathcal{P})(x \cdot y) = 0$, where $(\mathcal{P} \circ \mathcal{P})(x \cdot y) = 0$ such x and y existed, so $min\{\mathcal{P}(x),\ \mathcal{P}(y)\} = 0$. So consider that $min\{\mathcal{P}(x),\ \mathcal{P}(y)\} = \mathcal{P}(x)$, then $\mathcal{P}(x) = 0$, implies $x \in \mathcal{J}$. But $\mathcal{P}(x \cdot y) = 0$ leads that $x \cdot y \in \mathcal{J}$. Therefore, $y^n \in \mathcal{J}$ (since $\mathcal{J}$ is a primary ideal in $\mathcal{N}$). Assume $y \in \mathcal{J}^2$ then there exists $a, b \in \mathcal{J}$ with $y = a \cdot b$ and $(\mathcal{P} \circ \mathcal{P})(x \cdot y) = 0$ which contradicts the selection of x and y, so $y \in \mathcal{J} - \mathcal{J}^2$ and $x \cdot y \in \mathcal{J} - \mathcal{J}^2$, implies that $\mathcal{P}(x \cdot y) \geq \mathcal{P}(y^n)$. $\mathcal{P}$ is hence an almost primary f-ideal in $\mathcal{N}$. $\mathcal{P}$ still contains three values, therefore $\mathcal{P}$ is not a primary f-ideal of $\mathcal{N}$.

Proposition 2. Let $\mathcal{N}$ be a near-ring and if $\mathcal{P}$ is a f-ideal of $\mathcal{N}$ satisfying the condition that for $a,\ b \in \mathcal{N}$ and $s,\ t \in [0,\ 1]$ with $a_s \cdot b_t \in \mathcal{P}$ and $a_s \cdot b_t \notin \mathcal{P} \circ \mathcal{P}$, $a_s \cdot b_t \notin \mathcal{P} \circ \mathcal{P}$, where $a_s \in \mathcal{P}$ or $b_t{}^n \in \mathcal{P}$, for some $n \in \mathbb{Z}^+$, then $\mathcal{P}$ is an almost primary f-ideal of near-ring.

Proof. Assume $a \cdot b \in \mathcal{N}$ such that $a \cdot b = 0$, $(\mathcal{P} \circ \mathcal{P})(a \cdot b) = 0$ and $s,\ t \in (0,1]$ with $s = t = \mathcal{P}(a \cdot b)$, then $a_s b_t \in \mathcal{P}$. Further to this, $a_s b_t > 0$, then $b_t \notin \mathcal{P} \circ \mathcal{P}$ so either $a_s \in \mathcal{P}$ or $b_t^n \in \mathcal{P}$ as $(\mathcal{P} \circ \mathcal{P})(a \cdot b) = 0$ implies that $\max\{(\mathcal{P}(a), \mathcal{P}(b)\} = \mathcal{P}(a)$, then $a_s(a) = s < \mathcal{P}(a)$, hence $a_s \in \mathcal{P}$ or $b_y{}^n \in \mathcal{P}$ implies that $b_t^n(b) > \mathcal{P}(b^n)$. Therefore, $t \geq \mathcal{P}(b^n) \Rightarrow \mathcal{P}(a \cdot b) > \mathcal{P}(b^n)$, for some $n \in \mathbb{Z}^+$. Hence $\mathcal{P}$ is an almost primary f-ideal of $\mathcal{N}$.

Theorem 4. Let $\mathcal{P}$ be an ideal of $\mathcal{N}$. Then its characteristic function $\lambda_{\mathcal{P}}$ is an almost primary f-ideal of $\mathcal{N}$ if and only if $\mathcal{P}$ is an almost primary ideal of $\mathcal{N}$.

Proof. Let $\mathcal{P}$ be an almost primary ideal of $\mathcal{N}$. Let $a, b \in \mathcal{N} \mid \mathcal{P}(a \cdot b) > \mathcal{P}(1_{\mathcal{N}})$ and $\mathcal{P} \circ \mathcal{P}(ab) = \mathcal{P}(1_{\mathcal{N}})$. As we know that $\lambda_{\mathcal{P}}(a)$ is either 0 or 1, so for $\lambda_{\mathcal{P}}(a \cdot b) > \lambda_{\mathcal{P}}(a)$ that $\lambda_{\mathcal{P}}(a) = 0$ implies $\lambda_{\mathcal{P}}(a \cdot b) = 1$. Since for $a \cdot b \in \mathcal{P}$ and $a \notin \mathcal{P}$, according to almost primary ideal $b^n \in \mathcal{P}$ for some positive integer n. Hence $\lambda_{\mathcal{P}}(b^n) = 1 = \lambda_{\mathcal{P}}(a \cdot b)$, so $\lambda_{\mathcal{P}}$ is an almost primary f-ideal of $\mathcal{N}$. Conversely suppose that $\lambda_{\mathcal{P}}$ is an almost primary f-ideal of $\mathcal{N}$, then for $a, b \in \mathcal{N} \mid \mathcal{P}(a \cdot b) > \mathcal{P}(1_{\mathcal{N}})$ and $\mathcal{P} \circ \mathcal{P}(a \cdot b) = \mathcal{P}(1_{\mathcal{N}})$ implies $\mathcal{P}(a \cdot b) > \mathcal{P}(a)$ or $\mathcal{P}(a \cdot b) > \mathcal{P}(b^n)$. To prove $\mathcal{P}$ is an almost primary ideal, let $a, b \in \mathcal{P}$ and $a \notin \mathcal{P}$ then $\lambda_{\mathcal{P}}(a \cdot b) = 1$ and $\lambda_{\mathcal{P}}(a) = 0$. From definition of almost primary f-ideal there is a positive integer n such that $\lambda_{\mathcal{P}}(a \cdot b) > \mathcal{P}(b^n)$. Hence $\lambda_{\mathcal{P}}(b^n) = 1$ implies that $b^n \in \mathcal{P}$. This prove that $\mathcal{P}$ is an almost primary ideal of $\mathcal{N}$.

Proposition 3. Let $\mathcal{N}$ and M be two near-rings and $g: \mathcal{N} \to M$ be a homomorphism. If $h: M \to [0, 1]$ is an almost primary f-ideal, then $g^{-1}(h)$ is an almost primary f-ideal of $\mathcal{N}$.

Proof. We start with proving that $g^{-1}(h)$ is a f-ideal of $\mathcal{N}$. Assume $a, b \in \mathcal{N}$, then

$g^{-1}(h)(a - b) = h(g(a - b)) = h(g(a) - g(b)) \geq min\{h(g(a)), h(g(b))\} = min\{g^{-1}(h)(a), g^{-1}(h)(b)\}$

and

$g^{-1}(h(a \cdot b)) = h(g(a \cdot b)) = h(g(a), g(b)) \geq max\{h(g(a), h(g(b)\} = max\{g^{-1}(h)(a), g^{-1}(h)(b)\}$.

Now, we suppose that h is an almost primary f-ideal of $\mathcal{N}$. Suppose $a, b \in \mathcal{N}$ such that $g^{-1}(h(a \cdot b) > g^{-1}(h)(1_{\mathcal{N}})$ and $(g^{-1}(h) \cdot g^{-1}(h))(a \cdot b) = h(1_{\mathcal{N}})$, so $g^{-1}(h(a \cdot b) = max\ \{min\{g^{-1}(a), \quad g^{-1}(b)\}\} = g^{-1}(h(1_{\mathcal{N}})$, then $(h \circ h)(g(a \cdot b)) = (h \circ h)(g(a)g(b)) = h(1)$. But, since $g^{-1}(h(a \cdot b)) > g^{-1}(h)(1_{\mathcal{N}})$, then $h(g(a) \cdot g(b)) = max\{h(g(a)), h(g(b))\} = max\{g^{-1}(h)(a), g^{-1}(h)(b)\}$.

Fuzzy graph of almost primary fuzzy ideal of near-ring

In this section, we explore some fuzzy graphs associated with almost primary f-ideals of near-rings and provide their application in networking. We give a clue to model any network through our provided application. Moreover, we also discuss the properties of fuzzy graphs of almost primary f-ideal in near-rings.

Definition 13. A fuzzy graph of almost primary f-ideal $\mathcal{P}: \mathcal{N} \longrightarrow [0,1]$ is denoted by $G = (V, \mu, \rho)$, where V is the set of vertices, $\mu: \mathcal{N} \longrightarrow [0,1]$ is the fuzzy vertex set of G and $\rho: \mathcal{N} \cdot \mathcal{N} \longrightarrow [0,1]$ is the fuzzy edge set of G. In a fuzzy graph of almost primary ideal there is an edge between two vertices a and b if $\rho(a, b) = \rho(a \cdot b) \leq \min\{\mu(a), \mu(b)\}$ and $\rho(a, b) = \rho(b \cdot a) \leq \min\{\mu(b), \mu(a)\}$, where $a, b \in \mathcal{N}$.

Example 3. Consider that $\mathcal{P}$ is an almost primary f-ideal of $\mathcal{N}$ in example 2. To construct the fuzzy graph of $\mathcal{P}$, we set the values for fuzzy vertices as $\mu(1) = \rho(1) = 0.1, \mu(2) = \rho(2) = 0.2, \mu(3) = \rho(3) = 0.3, \mu(4) = \rho(4) = 0.4, \mu(5) = \rho(5) = 0.5, \mu(6) = \rho(6) = 0$ and $\mu(7) = \rho(7) = 0$. We can easily find that $\rho(3, 4) = \rho(3 \cdot 4) = \rho(5) = 0.5 \nleq \min\{\mu(3), \mu(4)\} = 0.4$, so there is no fuzzy edge between fuzzy vertices 0.3 and 0.4. Similarly, a fuzzy edge does not exist between fuzzy vertices $0.5, 0.1$ and $0.5, 0.2$. This fuzzy graph of an almost primary f-ideal is depicted in Fig. 1.

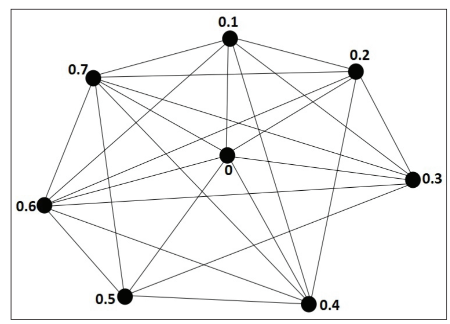

Figure 1: Fuzzy graph of almost primary f-ideal

There is no loop in Fig. 1, so a fuzzy graph of an almost primary f-ideal is a simple graph. Also, the graph is not a star graph because some fuzzy vertices are not connected. Subsequently, it is not a regular as well as complete graph because all fuzzy vertices are not having the same degree. The fact which prevents a fuzzy graph from becoming regular, complete, and a star graph is the operation of a near-ring denoted by “"”. Because in a near-ring structure for all $a, b \in \mathcal{N}$, it is not necessary that $a \cdot b = b \cdot a$. Furthermore, a fuzzy graph of almost primary f-ideal is not a strong fuzzy graph.

Important Note:

Fuzzy graphs are used in networking models to handle uncertainty and vagueness in network connections. In such models, edges between nodes are assigned a membership value (ranging from 0 to 1) instead of a binary presence or absence. This allows for more flexible and realistic modelling of networks where connections may not be entirely certain or strong, such as in social networks, communication networks, or reliability analysis. Fuzzy graphs help in optimizing routes, improving fault tolerance and enhancing the overall efficiency of network systems.

Refer to the above example 3, we can fix any issue related to social networking, computer networking, interconnected networks, etc., by exploiting the above given fuzzy structure in near-rings.

CONCLUSION

Recently, many classical ideals in different types of rings have been shifted in the paradigm of fuzzy sets. We have observed that the notion of almost primary f-ideal has been defined in rings but not extended for the near-rings. To fill this gap, we have initiated the notion of almost primary f-ideal in near-rings. In this article, we have also provided several characterizations of almost primary f-ideal of a near-ring. We have provided several examples and counter-examples in our study. Furthermore, we have explored the relations of almost primary f-ideal with primary f-ideal of a near-ring and concluded that every primary f-ideal is an almost primary f-ideal, but the converse is not true, in general. We have also discussed the homomorphism of two near-rings $\mathcal{N}$ and M denoted by $g: \mathcal{N} \rightarrow M$ along with almost primary f-ideal $h: M \rightarrow [0, 1]$. Moreover, we have also identified the fuzzy graphs associated with almost primary f-ideal of near-rings. We have also provided some properties of these newly established graphs. We discussed how the operation of near-ring affected fuzzy graph of almost primary f-ideal of near-ring. No doubt, almost primary f-ideal and their relations with classical ideals will help us to understand the linkage of classical ideal theory and fuzzy ideal theory. Finally, fuzzy graph of almost primary ideal will also help us to understand the theoretical fuzzy structure and modelling of different networks. One could extend this work towards other generalized structures like semi-near-rings, hemi-rings etc.

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