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RESEARCH ARTICLE

Functional Analysis

On the construction of unitaries representing minimal inner toral polynomials

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Abstract: A polynomial $p = p(z,w) \in \mathbb{C}[z,w]$ is called an inner toral polynomial if the zero set of p, contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$ where $\mathbb{D}^2$ is the open bidisk, $\mathbb{T}^2$ is the two dimensional torus and $\mathbb{E}^2$ is the exterior of the closed bidisk in $\mathbb{C}^2$. Such a polynomial is called a minimal inner toral polynomial if it divides all the other polynomials with same zero set as itself. The bidegree of p is defined as the ordered pair (n,m) whenever p has degrees n and m in variables z and w respectively. For a minimal inner toral polynomial $p(z,w)$ of bidegree (n,m), there exist block unitary matrices, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{((m+n)\times(m+n))}$, such that $p(z,w)$ is a constant multiple of the determinant of $\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix}$, where, blocks A, B, C and D are matrices with complex entries of sizes $(m \times m), (m \times n), (n \times m)$ and $(n \times n)$ respectively. Because the size of such unitaries solely depends on the bidegree of the considered minimal inner toral polynomial, the process of computing such unitary matrices is manageable when the bidegree is very small but become strenuous exponentially with the bidegree. In this work, we introduce a method of constructing unitary matrices representing reducible minimal inner toral polynomials in terms of the unitaries representing its factors and vice versa.

Keywords: Block matrices, distinguished varieties, inner toral polynomials, unitary matrices.

INTRODUCTION

Inner toral polynomials are polynomials in two complex variables, say in z and w, such that their zero sets lie in the open bidisk and exit the open bidisk to the exterior of the closed bidisk through the two-dimensional torus. These polynomials became interesting, because the variety generated by them have this special form, and most importantly, a given pair of pure commuting isometries that satisfies an algebraic relationship contains an inner toral polynomial that annihilates the same pair of isometries. Therefore, studies related to such polynomials appear in many research articles in functional analysis as well as operator theory (Agler & McCarthy, 2002, 2005, 2006; Agler *et al*., 2006, 2012; Knese, 2010; McCarthy, 2012; Jury *et al*., 2012; Pal & Shalit, 2014; Wijesooriya, 2018; Bhattacharyya *et al*, 2022). An inner toral polynomial is called a minimal inner toral polynomial or simply minimal, if it divides all the polynomial with the same zero set as itself. In 2005, Agler and McCarthy have proved that a minimal inner toral polynomial can be represented with a unitary block matrix in a way that the determinant of a shifted version of the unitary block matrix is a constant multiple of the polynomial.

If the inner toral polynomial we consider has bidegree (n, m), then a unitary representing it, is a square matrix of size $n + m$. Due to the large size of these matrices, and the fact that the number of unknowns in the matrices increases significantly with the bidegree of the polynomial, it is not computationally efficient to come up with examples. In this study we paid attention to reducible minimal inner toral polynomials which are obviously factorizable into simpler irreducible minimal inner toral polynomials and we found a method to compute unitaries representing such reducible minimal inner toral polynomials in terms

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of unitaries representing their factors and vice versa. We define a special type of a direct sum for block matrices, which we call the diagonal wise direct sum, and we construct desired unitary matrices by performing the diagonal wise direct sum on them.

PRELIMINARIES

In this section we discuss necessary background materials to discuss the main result in the preceding section. Throughout this paper we consider $\mathbb{D}$ to be the open unit disk, $\mathbb{T}$ to be the unit circle and $\mathbb{E}$ to be the exterior of the closed unit disk in the complex plane.

Definition 1: Let $p = p(z,w) \in \mathbb{C}[z,w]$ be a polynomial in *z* and *w*. We say that *p* has bidegree (*n*, *m*) if *p* has degree *n* in *z* and *m* in *w*.

Definition 2: A polynomial $\mathfrak{p} = \mathfrak{p}(z,w) \in \mathbb{C}[z,w]$ is called an inner toral polynomial if the zero set of $\mathfrak{p}$, $Z(\mathfrak{p})$, contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.

In other words, for a two-variable polynomial $p = p(z,w) \in \mathbb{C}[z,w]$, in order to be inner toral, every zero (z_0, w_0) of *p* must satisfy the condition that either $|z_0|, |w_0| < 1$ or $|z_0| = |w_0| = 1$ or $|z_0|, |w_0| > 1$.

For example, $p_1(z,w) = z^n - w^m$ for $n, m \in \mathbb{N}$ is inner toral. However, $p_2(z,w) = z^n - 2w^m$ is not inner toral because $\left(2^{\frac{1}{n}}, 1\right)$ is a zero of p_2 but it is not in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.

Definition 3: An inner toral polynomial $\mathfrak{p}(z,w) \in \mathbb{C}[z,w]$ is called a minimal inner toral polynomial, if it divides any $\mathfrak{q} \in \mathbb{C}[z,w]$ with $Z(\mathfrak{p}) = Z(\mathfrak{q})$.

A minimal inner toral polynomial is minimal in the sense that it has smallest possible bidegree, and it is square free (unique up to constant multiples) with the same zero set as the original one. Therefore, clearly the minimal polynomial is also inner toral. Given an inner toral polynomial $\mathfrak{p}$, its minimal version always exists as it is the product of all the irreducible factors (of algebraic multiplicity one) of $\mathfrak{p}$ (up to constant multiples). For example, $(z-w)^2(z+w)$ is inner toral and $(z-w)(z+w)$ is its minimal version.

Definition 4: For an inner toral polynomial $\mathfrak{p}$ the set, $\mathfrak{V}(\mathfrak{p}) = Z(\mathfrak{p}) \cap \mathbb{D}^2$ is called a distinguished variety.

Definition 5: A square matrix U is called a unitary matrix if its conjugate transpose U^* is its inverse. That is, $UU^* = U^*U = I$ where *I* is the identity matrix.

Our work is based on the unitaries representing minimal inner toral polynomials. The Theorem given below explain the existence of such unitary matrices and how exactly they look like. This Theorem was first proved by Agler and McCarthy in 2005 and an alternative proof was given by Knese in 2010. This result is one of the breakthrough results in this setting as it eventually became a very helpful tool in the studies carried out by many mathematicians related to inner toral polynomials, distinguished varieties and pure algebraic iso-pairs (Knese, 2010; Jury *et al*., 2012; McCarthy, 2012; Pal & Shalit 2014; Wijesooriya, 2018).

Theorem 1: (Knese, 2010) For a minimal inner toral polynomial $\mathfrak{p}(z,w)$ of bidegree (n,m) there exists an unitary matrix $U = \begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{array}{c} m \\ n \end{array} \end{array}$ such that

- $\det\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} = k\,\mathfrak{p}(z,w)$ where k is a constant and
- if $\det(D - \lambda I_n) = 0$, then $|\lambda| \neq 1$.

Here, A, B, C and D are matrices of sizes $(m \times m)$, $(n \times m)$, $(m \times n)$ and $(n \times n)$ respectively. Such unitary matrices, U, are called unitaries representing minimal inner toral polynomial $\mathfrak{p}(z,w)$.

For example, let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$ be a unitary representing the minimal inner toral polynomial $z - w$. That means the determinant of $\begin{pmatrix} a - w & zb \\ c & zd - 1 \end{pmatrix}$ is equal to $k(z-w)$ for some constant $k \in \mathbb{C}$. By comparing the coefficients of the equation $(ad - bc)z - dzw + w - a = k(z - w)$, and considering the fact that U is a unitary matrix, we have $a = d = 0$, and $|b| = |c| = 1$ with $bc = 1$. Therefore, unitaries representing $z - w$ are of the form $\begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}$ where α is a unimodular constant. This computation can be done easily due to the fact that U has a relatively small size and the system of equations involved is simple. However, that is not the case all the time. For example, unitaries representing the minimal inner toral polynomial $z^4 - w^4$ are 8 × 8 matrices and it is not quite easy to compute such unitaries as it involved finding 64 unknowns.

Remark 1: It is evident from the above example that, given an inner toral polynomial one can find more than one unitary matrix representing the considered polynomial.

Before moving to the results and discussions, we prefer to discuss the invertibility of the block matrix $zD - I$ to avoid any ambiguities that could appear in the proof of the main Theorem.

Theorem 2. (Bezout's Theorem) If two algebraic curves, say described by and $p(z,w) = 0$ and $q(z,w) = 0$ do not have any common components, then they have only finitely many points in common.

In fact, if $p(z,w)$ is a minimal inner toral polynomial and if zeroes of p and q are common except for finitely many zeroes, then p must be a factor of q.

Remark 2: Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a unitary representing the minimal inner toral polynomial $\mathfrak{p}(z,w)$ of bidegree (n,m). By Theorem 1, $\det\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} = k\,\mathfrak{p}(z,w)$ for some constant k. Note that $\det\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} = \det(zD - I) \cdot \det[(A - wI) - zB(zD - I)^{-1}C]$. It is immediate that this equation is true only if the matrix $zD - I$ is invertible. Since the matrix D is of size $n \times n$, the determinant of $zD - I$ can be vanished only at finitely many values of z. Therefore, $\det\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix}$ vanishes at all except finitely many points in $Z(\mathfrak{p})$. By Theorem 2, and the minimality of $\mathfrak{p}$, $\det\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix}$ vanishes at all the points in $Z(\mathfrak{p})$. Eventually, this leads to the fact that $\det\begin{pmatrix} A - wI_m & zB \\ C & zD - I_n \end{pmatrix} = \det(zD - I) \cdot \det[(A - wI) - zB(zD - I)^{-1}C] = k\,\mathfrak{p}(z,w)$[see page 16 of (Knese, 2010) for more details on this]. Therefore, the non-invertibility of $zD - I$ at finitely many $z \in \mathbb{C}$, does not harm the determinantal representation of the unitary to the polynomial . In the proof of the main theorem presented in this paper, we say the fact that $zD - I$ is invertible, with the fully understanding that $zD - I$ is invertible at all except finitely many $z \in \mathbb{C}$.

Now we present the method we found on computing unitaries representing minimal inner toral polynomials.

RESULTS AND DISCUSSION

Let $m_i, n_i \in \mathbb{N}$ and suppose U_i is a block matrix of the form

$$U_i = \begin{array}{cc} \begin{array}{cc} m_i & n_i \end{array} & \\ \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} & \begin{array}{c} m_i \\ n_i \end{array} \end{array}$$

for $i = 1, 2, \ldots, s$ with sizes given above.

$$U_1 \oplus_D U_2 \oplus_D \ldots \oplus_D U_s =$$

$$\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 & B_1 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 & 0 & B_2 & 0 & \cdots & 0 \\
0 & 0 & A_3 & \cdots & 0 & 0 & 0 & B_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_s & 0 & 0 & 0 & \ldots & B_s \\
C_1 & 0 & 0 & \cdots & 0 & D_1 & 0 & 0 & \cdots & 0 \\
0 & C_2 & 0 & \cdots & 0 & 0 & D_2 & 0 & \cdots & 0 \\
0 & 0 & C_3 & \cdots & 0 & 0 & 0 & D_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & C_s & 0 & 0 & 0 & \ldots & D_s
\end{pmatrix}.$$

We call this sum the diagonalwise direct sum. For the ease of notation, let us denote the block matrix $\begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_s \end{pmatrix}$ by $\oplus A_i$ and likewise $\oplus B_i, \oplus C_i$ and $\oplus D_i$. Likewise, we will use the notation, $\oplus$, to denote the block diagonal matrices where it is necessary. Again, for the ease of using notations, given a block matrix $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we denote $\begin{pmatrix} A - wI & Bz \\ C & zD - I \end{pmatrix}$ by $U(z,w)$.

Let $\mathfrak{p}(z,w) \in \mathbb{C}[z,w]$ be a reducible minimal inner toral polynomial. Write $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \ldots \mathfrak{p}_s$ for some $s \in \mathbb{N}$. Note that each factor $\mathfrak{p}_i$ is also a minimal inner toral polynomial. Let (n,m) and (n_i, m_i) be the bi-degrees of the polynomial $\mathfrak{p}$ and $\mathfrak{p}_i$ for $i = 1, 2, \ldots, s$ respectively. Note that $n = \sum_{i=1}^{s} n_i$ and $m = \sum_{i=1}^{s} m_i$. Further, let U and U_i be square matrices of order $(m+n)$ and $(m_i + n_i)$ for $i = 1, 2, \ldots, s$ respectively. Now, we present our main result:

Theorem 3

If U_i is a unitary matrix representing the minimal inner toral polynomial $\mathfrak{p}_i$ for each, $i = 1,2, \ldots, s,$ then the matrix $U = U_1 \oplus_D U_2 \oplus_D \dots \oplus_D U_s$ is a unitary matrix representing the polynomial $\mathfrak{p}$. Conversely, suppose $U = \begin{pmatrix} \oplus A_i & \oplus B_i \\ \oplus C_i & \oplus D_i \end{pmatrix}$ is a unitary representing the minimal inner toral polynomial $\mathfrak{p}$. where $i = 1,2,3,\dots,s$. Each matrix given by $U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ is a unitary matrix representing the minimal inner toral polynomial $\mathfrak{p}_i$ (up to re-labelling) for $i = 1,2,\dots,s$.

Proof: Let $U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ for $i = 1,2,3,\dots,s$ with the sizes given as above. First, we must show that U is unitary. Since each $U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ is a unitary, $U_iU_i^* = U_i^*U_i = I_{n_i+m_i}$ for $i = 1,2,\dots,s$. In fact,

$$\begin{pmatrix} A_iA_i^* + B_iB_i^* & A_iC_i^* + B_iD_i^* \\ C_iA_i^* + D_iB_i^* & C_iC_i^* + D_iD_i^* \end{pmatrix} =$$

$$\begin{pmatrix} A_i^*A_i + B_i^*B_i & C_i^*A_i + D_i^*B_i \\ A_i^*C_i + B_i^*D_i & C_i^*C_i + D_i^*D_i \end{pmatrix} = I_{n_i+m_i}$$

for $i = 1,2,\dots,s$.

Clearly $U = \begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ & \begin{pmatrix} \oplus A_i & \oplus B_i \\ \oplus C_i & \oplus D_i \end{pmatrix} \begin{array}{c} m, \\ n \end{array} \end{array}$

$$UU^* = \begin{pmatrix} \oplus(A_iA_i^* + B_iB_i^*) & \oplus(A_iC_i^* + B_iD_i^*) \\ \oplus(C_iA_i^* + D_iB_i^*) & \oplus(C_iC_i^* + D_iD_i^*) \end{pmatrix} \text{ and}$$

$$U^*U = \begin{pmatrix} \oplus(A_i^*A_i + B_i^*B_i) & \oplus(C_i^*A_i + D_i^*B_i) \\ \oplus(A_i^*C_i + B_i^*D_i) & \oplus(C_i^*C_i + D_i^*D_i) \end{pmatrix}.$$

It easily follows that $UU^* = U^*U = I_{n+m}$. Therefore U is also a unitary matrix. It is left to prove that the unitary matrix U represents the polynomial $\mathfrak{p}$. Since U_i represent $\mathfrak{p}_i$, there exist $c_i \in \mathbb{C}$ such that $\det U_i(z,w) = c_i\, \mathfrak{p}_i(z,w)$ for all $i = 1,2,3,\dots,s$. In fact, for $i = 1,2,3,\dots,s$, $\det\begin{pmatrix} A_i - wI_{m_i} & zB_i \\ C_i & zD_i - I_{n_i} \end{pmatrix} = c_i\, \mathfrak{p}_i(z,w)$, and hence

$$\det(zD_i - I_{n_i}) \cdot \det\left[(A_i - wI_{m_i}) - zB_i(zD_i - I_{n_i})^{-1}C_i\right] = c_i\, \mathfrak{p}_i(z,w).$$

Note that

$$U(z,w) = \begin{pmatrix} \oplus(A_i - wI_{m_i}) & \oplus zB_i \\ \oplus C_i & \oplus(zD_i - I_{n_i}) \end{pmatrix}.$$

Since $(zD_i - I_{n_i})$ is invertible, $(\oplus(zD_i - I_{n_i}))$ is also invertible and for $i = 1,2,\dots,s$. Therefore,

$$\det[U(z,w)] =$$

$$\det(\oplus(zD_i - I_{n_i})) \cdot \det\left[\oplus(A_i - wI_{m_i}) - \oplus zB_i\left(\oplus(zD_i - I_{n_i})\right)^{-1}\oplus C_i\right]$$

$$= \det\left(\oplus(zD_i - I_{n_i})\right) \cdot \det\left[\oplus\left((A_i - wI_{m_i}) - zB_i(zD_i - I_{n_i})^{-1}C_i\right)\right]$$

$$= \prod_{i=1}^{s} \det(zD_i - I_{n_i}) \cdot \det\left[(A_i - wI_{m_i}) - zB_i(zD_i - I_{n_i})^{-1}C_i\right]$$

$$= \prod_{i=1}^{s} c_i\, \mathfrak{p}_i(z,w)$$

$= c\, \mathfrak{p}(z,w)$ where $c = c_1c_2c_3\dots c_s$.

Therefore, U represents $\mathfrak{p}$.
Conversely, since U represent $\mathfrak{p}$, $\det(U(z,w)) = c\, \mathfrak{p}(z,w)$ for some constant $c \in \mathbb{C}$. In other words,

$$\det(\oplus(zD_i - I_{n_i})) \cdot$$

$$\det[\oplus(A_i - wI_{m_i}) - \oplus zB_i(\oplus(zD_i - I_{n_i}))^{-1}\oplus C_i] = c\, \mathfrak{p}(z,w).$$

Since $(\oplus(zD_i - I_{n_i}))$ is invertible, $(zD_i - I_{n_i})$ is also invertible for $i = 1,2,3,\dots,s$ and

$$\left(\oplus(zD_i - I_{n_i})\right)^{-1} = \oplus(zD_i - I_{n_i})^{-1}.$$

Therefore,

$$\det\left(\oplus(zD_i - I_{n_i})\right) \cdot \det\left[\oplus\left((A_i - wI_{m_i}) - zB_i(zD_i - I_{n_i})^{-1}C_i\right)\right]$$
$$= c\, \mathfrak{p}(z,w).$$

Consequently, $\prod_{i=1}^{s} \det(zD_i - I_{n_i}) \cdot$

$$\det\left[(A_i - wI_{m_i}) - zB_i(zD_i - I_{n_i})^{-1}C_i\right] = c\, \mathfrak{p}_1\, \mathfrak{p}_2 \dots \mathfrak{p}_s.$$

That is, $\prod_{i=1}^{s} \det(U_i(z,w)) = c\, \mathfrak{p}(z,w) = c\, \mathfrak{p}_1\, \mathfrak{p}_2 \dots \mathfrak{p}_s$. Note that each $\mathfrak{p}_i$ is irreducible and distinct. Moreover, bidegree of $\det(U_i(z,w))$ is at most (n_i, m_i) and bidegree of $\mathfrak{p}_i(z,w)$ is (n_i, m_i). Therefore, $\det(U_i(z,w))$ divides $\mathfrak{p}_i(z,w)$ up to relabeling of $\mathfrak{p}_i$'s. Therefore, unitary matrix U_i represents $\mathfrak{p}_i$, upto relabeling of $\mathfrak{p}_i$'s for $i = 1,2,\dots,s$. Since $\mathfrak{p}_i$ is a minimal inner toral polynomial, $\det(U_i(z,w)) = c_i\, \mathfrak{p}_i(z,w)$ up to relabeling of $\mathfrak{p}_i$'s. Therefore each U_i is a unitary representing $\mathfrak{p}_i$, up to relabeling.

For examples, consider the reducible minimal inner toral polynomial $p = z^4 - w^4 = (z - w)(z + w)(z - iw)(z + iw)$. Unitaries representing p are of the size 8 × 8 and obviously, it is difficult to evaluate all the 64 unknowns in such a matrix. However, we can easily compute the unitaries representing each factor of p. Unitaries representing $z-w,\ z+w,\ z-iw$ and $z+iw$ are of the form $\begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \alpha \\ i\bar{\alpha} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \alpha \\ -i\bar{\alpha} & 0 \end{pmatrix}$ respectively where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Now using the diagonal wise direct sum, and the above Theorem, we can easily obtain a collection of unitary matrices representing p, which are of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ \bar{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\bar{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\bar{\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\bar{\alpha} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Likewise, for the minimal inner toral polynomial $q(z,w) = (z^2 + zw) + i(zw + w^2) = (z+w)(z+iw)$, a collection of unitary matrices representing q can be obtained as:

$$\begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ -\bar{\alpha} & 0 & 0 & 0 \\ 0 & -i\bar{\alpha} & 0 & 0 \end{pmatrix}.$$

In another example, a collection of unitaries representing the polynomial $z^2 - w$ is given as $\begin{pmatrix} 0 & 0 & \lambda \\ \beta & 0 & 0 \\ 0 & \overline{\lambda\beta} & 0 \end{pmatrix}$ where $\lambda, \beta \in \mathbb{C}$ with $|\lambda| = |\beta| = 1$. Using the diagonalwise direct sum, we can construct a collection of unitary matrices representing the minimal inner toral polynomial $r(z,w) = z^3 - z^2w - zw + w^2 = (z^2 - w)\,(z - w)$ as

$$\begin{pmatrix} 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{\lambda\beta} & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 & 0 \end{pmatrix}$$

for $\alpha, \lambda, \beta \in \mathbb{C}$ with $|\alpha| = |\lambda| = |\beta| = 1$.

CONCLUSIONS

In this work, we have been able to introduce a method to construct unitaries representing reducible minimal inner toral polynomials in terms of unitaries representing its factors. This method of construction unitaries is useful as it reduces and simplifies the number of computations involve in computing unitaries representing reducible minimal inner toral polynomials of higher bidegree.

Conflict of interest

Authors hereby declare that there is no conflict of interest to disclose.

REFERENCES

Agler J. & McCarthy J.E. (2002). *Pick Interpolation and Hilbert Function Spaces.* American Mathematical Society, Providence , Rhode Island, USA.
DOI: https://doi.org/10.1090/gsm/044

Agler J. & McCarthy J.E. (2005). Distinguished varieties. *Acta Mathematica* **194**(2005): 133–153.
DOI: https://doi.org/10.1007/bf02393219

Agler J. & McCarthy J.E. (2006). Parametrizing distinguished varieties: Recent advances in operator-related function theory. In: *Contemporary Mathematics*, Volume 393. American Mathematical Society, Providence , Rhode Island, USA.
DOI: https://doi.org/10.1090/conm/393/07368

Agler J., Knese G. & McCarthy J.E. (2012). Algebraic pairs of isometries. *Journal of Operatory Theory* **67**: 215–236.

Agler J., McCarthy J.E. & Mark S. (2006). Toral algebraic sets and function theory on polydisks. *Journal of Geometric Analysis* **16**: 551–562.
DOI: https://doi.org/10.1007/bf02922130

Bhattacharyya T., Kumar P. & Sau D. (2022). Distinguished varieties through the Berger--Coburn--Lebow theorem. *Analysis and PDE* **15**(2): 477–506
DOI: https://doi.org/10.2140/apde.2022.15.477

Jury M.T., Knese G. & McCullough S. (2012). Nevanlinna-Pick interpolation on distinguished varieties in the bidisk. *Journal of Functional Analysis* **262**: 3812–3838.
DOI: https://doi.org/10.1016/j.jfa.2012.01.028

Knese G. (2010). Polynomials defining distinguished varities. *Transactions of the American Mathematical Society* **362**: 5635–5655.
DOI: https://doi.org/10.1090/s0002-9947-2010-05275-4

McCarthy J.E. (2012). Shining a Hilbertian lamp on the bidisk: Topics in complex analysis and operator theory In: *Contemporary Mathematics*, Volume 561. American Mathematical Society, Providence , Rhode Island, USA.
DOI: https://doi.org/10.1090/conm/561/11109

Pal S. & Shalit O. (2014). Spectral sets and distinguished varieties in the symmetrized bidisc. *Journal of Functional Analysis* **266**: 5779–5800.

Wijesooriya U. (2018). Algebraic pairs of pure commuting isometries with finite multiplicity. *Journal of Operator Theory* **79(**2): 507–527.
DOI: http://dx.doi.org/10.7900/jot.2017may12.2171