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RESEARCH ARTICLE

Applied Statistics

A coupled system of stochastic differential equations for probabilistic wind speed modelling

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Abstract: Wind-based electricity generation plays a vital role in the renewable energy industry. Benefits from its use can only be obtained if it is possible to accommodate its variability and limited predictability. Stochastic differential equation (SDE) based approaches have demonstrated an improved capability of predicting temporal wind speed patterns that have statistical properties that are similar to those observed in reality. However, no standard approach for deriving such models exist due to the wide variations in the temporal statistical properties that one observes in wind speed data measured from location to location. Wind speed data which have been recorded at coastal locations, exhibit non-stationary features, like diurnal effect, seasonal effects, and temporal trends. In this work, such effects are eliminated using standard smoothing techniques. A coupled system of second-order ordinary linear differential equations driven by a white noise forcing term was used for the probabilistic modelling of the residual data. The model was then used to predict the wind speed distribution and the corresponding autocorrelation function of wind speed data, recorded at the wind measurement centre of Kokkilai in northern Sri Lanka, from February 2015 to February 2016. Finally, the results of this novel approach were compared against the probabilistic modelling methods existing in the literature.

Key words: Autocorrelation function, probabilistic modelling, smoothing techniques, stochastic differential equation, white noise.

INTRODUCTION

Environmental concerns and supply uncertainties are forcing many countries to rethink their energy mix and develop diverse sources of clean, renewable energy. Cost-effective energy that can be produced without major negative environmental impact has become the goal worldwide. Wind energy, as a clean and renewable resource, has been under wide consideration around the world in the last decade. The benefits of wind energy are accompanied by several challenges: high variability, limited predictability, and non-storability. Thus wind can be identified as a highly volatile energy source and hence an efficient mechanism for modelling wind speed has become a mandatory requirement in power system regulations, control, and management of wind-based electricity generation (Zhu & Genton, 2012). How to reduce the uncertainty in predicting wind speeds has been the focus of studies over the last two decades. To smoothly integrate large-scale wind power into power systems, wind generation forecasting models have been developed to improve the accuracy of forecasts. Time series analysis-based methods, probability distribution-based approaches and applications of Kalman filters are found as the major statistical and mathematical modelling approaches used in modelling wind speeds. In Zhu & Genton (2012), it is pointed out that the accurate probabilistic modelling of wind speeds is more useful than point-forecasting for the power system operations of wind-based energy industry. Thus, in addition to the traditional mathematical and statistical tools for wind speed modelling, Stochastic Differential Equations (SDE) based models have become popular and have shown much promise (Zhu & Genton, 2012). Most of these SDE-based approaches have employed a single

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equation to produce forecasting trajectories of wind speeds (Bibby *et al.,* 2005; Calif, 2012; Zarate-Minano *et al.,* 2016). In addition, applications of those methods have been confined to modelling wind speed data that exhibit significant stationary features. Yet, from a practical point of view, wind speed data exhibit considerable non-stationary features. Therefore, applications of SDEs that generate stationary Ornstein-Uhlenbeck (O-U) processes are not sufficient to obtain reliable probabilistic forecasting for the wind energy industry. In this regard, the sums of diffusion method (Bibby *et al.,* 2005) has demonstrated successful results for weakly stationary real process that have several time scale effects. However, when the influence of cyclic effects becomes intensive, the method becomes ineffective due to the increased non-stationariness. In addition, there are other sophisticated approaches based on non-linear filtering and particle filtering that can be successfully adapted to probabilistic modelling of wind speed data (Fernando & Hausenblas, 2018; Hausenblas *et al.,* 2021).

In this paper, we propose a novel approach that first eliminates the non-stationary effects using a smoothing method and then uses two coupled oscillators driven by white noise to model the residual process. The approach was then used to successfully model empirical wind speed data, recorded at Kokkilai in the North Eastern coast of Sri Lanka from February 2015 to February 2016 under the Quantum Leap Wind Power Assessment Project of the Sustainable Energy Authority of Sri Lanka.

MATERIALS AND METHODS

In this section, we present a novel method for predicting wind speeds and validate it against the wind speed data measured at Kokkilai, a coastal point located on the North Eastern shore of Sri Lanka. The wind speed data that were originally recorded at 10 min intervals were converted to hourly averaged wind speeds. The empirical autocorrelation function of hourly wind speed data in Figure 1(a) exhibits a complex cyclic behaviour due to the influence of diurnal and seasonal effects (Brett & Tuller, 1991). As a coastal point, Kokkilai also comes under the influence of both North Eastern and South Western monsoonal wind patterns. In addition, due to the inter-monsoonal wind variations, short-scale temporal trends of wind speed variation were crucial factors that contributed to the non-stationariness. Figure 1(b) shows that the empirical wind speed distribution deviates significantly from a normal Gaussian distribution.

In order to eliminate the non-stationary features of wind speed data, the corresponding time plot and correlogram were carefully inspected to identify various cyclic patterns with distinct periods of temporal trends. Accordingly, the dataset was divided into distinct blocks based on the visual inspection of the time plot and correlogram. Regression models were then incorporated to eliminate the non-stationary features that were specific to each block.

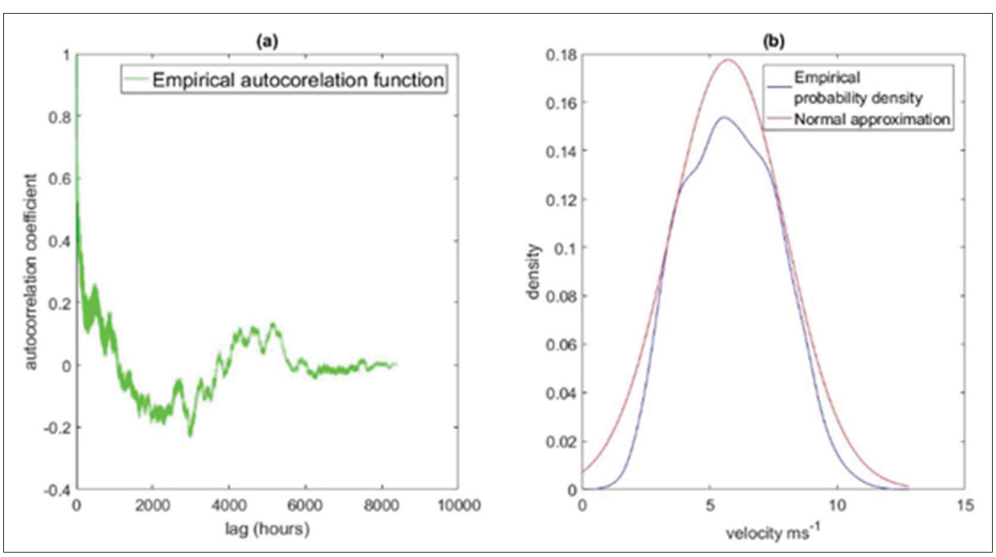

Figure 1: (a): Empirical autocorrelation of hourly wind speeds and (b): empirical wind speed distribution approximated by a normal Gaussian distribution

When a cyclic variation of fixed period is apparent in a wind speed data set, it is eliminated with a regression model of the form,

$$u_r = b_0 + b_1 \cos\left(\frac{2\pi}{T} t\right) + b_2 \sin\left(\frac{2\pi}{T} t\right), \qquad ...(1)$$

where $b_i (i = 0, 1, 2)$ are constants and T is the period of the cyclic variation (Chatfield, 2004; Shumway & Stoffer 2016). Here, b_0 captures the mean value of the data.

In the event where several cyclic effects are present a regression model of the following form can be used (Chatfield, 2004; Shumway & Stoffer, 2016).

$$u_r = b_0 + \sum_{i=1}^{n} \left(a_i \cos\left(\frac{2\pi}{T_i} t\right) + b_i \sin\left(\frac{2\pi}{T_i} t\right) \right). \qquad ...(2)$$

Here, $a_i, b_i (i = 0, 1, 2 \ldots)$ are constants and $T_i (i = 0, 1, 2 \ldots)$ is the period of the i^{th} cycle.

Furthermore, a polynomial regression model of the form shown below can be used to capture the temporal trend patterns included in a wind speed dataset (Chatfield, 2004; Shumway & Stoffer, 2016)

$$u_r = \sum_{i=0}^{m} c_i t^i, \qquad ...(3)$$

where $c_i (i = 0, 1, 2 \ldots)$ are constants.

In the case, when both cyclic effects and temporal trends are present a sum of the regression models given by equation (2) and (3) can be used,

$$u_r = b_0 + \sum_{i=1}^{n} \left(a_i \cos\left(\frac{2\pi}{T_i} t\right) + b_i \sin\left(\frac{2\pi}{T_i} t\right) \right) + \sum_{i=1}^{m} c_i t^i. \qquad ...(4)$$

By visual inspection of the time plot of wind speed data and the corresponding correlogram, the dataset was partitioned into eleven blocks based on the observed different non-stationary features. The non-stationary features and the regression models incorporated are summarized in the following table (Table 1).

Then, the residual process (v) was obtained by eliminating the respective non-stationary component (u_r) from the empirical data (u) as in (5) and the corresponding time plots are given in Figure 2.

$$v = u - u_r \qquad ...(5)$$

Table 1: Periods of cyclic effects and the order of regression polynomial functions corresponding to each wind speed data block

Block numbers	Parameter values of regression models	Regression model
1 and 3	T = 24 hours	$b_0 + b_1 \cos\left(\frac{2\pi}{T} t\right) + b_2 \sin\left(\frac{2\pi}{T} t\right)$
2, 4, 5, 6, 7 and 9	T_1 = 24 hours,m = 2	$b_0 + \sum_{i=1}^{n} \left(a_i \cos\left(\frac{2\pi}{T_i} t\right) + b_i \sin\left(\frac{2\pi}{T_i} t\right) \right) + \sum_{i=1}^{m} c_i t^i$
8	T_1 = 24 hours, T_2 = 350 hours	$b_0 + \sum_{i=1}^{n} \left(a_i \cos\left(\frac{2\pi}{T_i} t\right) + b_i \sin\left(\frac{2\pi}{T_i} t\right) \right)$
10 and 11	T_1 = 24 hours, T_2 = 144 hours, m = 2	$b_0 + \sum_{i=1}^{n} \left(a_i \cos\left(\frac{2\pi}{T_i} t\right) + b_i \sin\left(\frac{2\pi}{T_i} t\right) \right) + \sum_{i=1}^{m} c_i t^i$

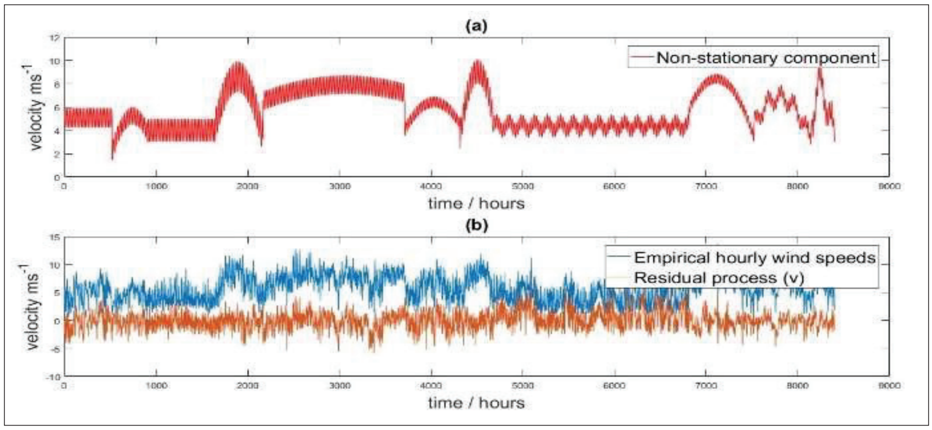

Figure 2: (a): Non-stationary component (u_r) of empirical wind speed data, (b): Time plot of empirical wind speeds and residua l process (v)

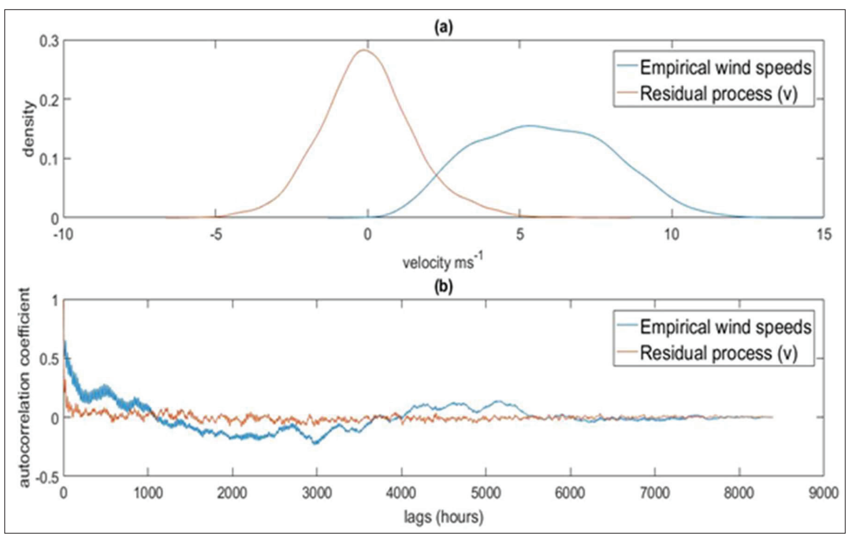

Figure 3: (a): Distributions of empirical wind speeds and residual process (v), (b): Correlograms of empirical wind speeds and residual process (v)

The correlogram of Figure 3(b) indicates that the residual process can be approximately treated as a wide sense stationary process (Chatfield, 2004).

In the rest of this section, we show how this approximately stationary residual process can be modeled by a process that is generated by a coupled pair of second-order linear SDEs.

Consider the general n-dimensional SDE,

$$\dot{\boldsymbol{X}}=\boldsymbol{A}(X,t)+\boldsymbol{B}(X,t)\boldsymbol{\eta}(t) \quad \text{...(6a)}$$

$$y=\boldsymbol{C}X \quad \text{...(6b)}$$

where $\boldsymbol{X}$ is a $n\times 1$ matrix, $\boldsymbol{A}$ is a $n\times n$ matrix, $\boldsymbol{B}$ is a $n\times m$ matrix, and $\boldsymbol{\eta}(.)$ is a $m\times 1$ white noise process.

Definition: White noise

A white noise process $\{\boldsymbol{\eta}(t)|\ t\in[0, T]\}$ is a random process with the following properties (Gardiner, 1985; Särkkä & Solin, 2019):

1. For $t_1\neq t_2$, $\eta(t_1)$ and $\eta(t_2)$ are independent.
2. $t\rightarrow\eta(t)$ is a Gaussian process with zero mean and

Dirac delta correlation:
Mean: $m(t) = E[\eta(t)] = 0$
Correlation: $C_\eta(t,s) = E[\eta(t)\eta^T(s)] = \delta(t-s)\boldsymbol{Q}$

where $\boldsymbol{Q}$ is the spectral density of the process.

From the above properties we can also deduce the following peculiar properties of white noise:

a. The sample path $t \rightarrow \eta(t)$ is discontinuous almost everywhere.
b. White noise is unbounded and it takes arbitrarily large positive and negative values at any finite interval.

White noise can be considered a weak derivative of the standard Wiener process (Särkkä & Solin, 2019). Therefore, the system (6a) - (6b) can be expressed as the following state-space model.

$$d\boldsymbol{X} = \boldsymbol{A}(X,t)dt + \boldsymbol{B}(X,t)d\boldsymbol{w} \qquad \text{...(7a)}$$

$$y = \boldsymbol{C}X \qquad \text{...(7b)}$$

Definition: Wiener process

The standard Wiener process $\{\boldsymbol{w}(t) |\, t \in [0, T]\}$ is a continuous stochastic process that satisfies the following conditions (Gardiner, 1985; Särkkä & Solin, 2019).

1. Any increment $\Delta\boldsymbol{w}_k = \boldsymbol{w}(t_{k+1}) - \boldsymbol{w}(t_k)$ is a zero mean multivariate Gaussian random variable with covariance $\boldsymbol{T}\Delta t_k$ where $\boldsymbol{T}$ is the diffusion matrix of the Wiener process and $\Delta t_k = t_{k+1} - t_k$.
2. The increments are independent where the time spans of increments are not overlapped.
3. $w(0) = 0$ with probability one.

The solution of state-space model (7a) - (7b) is a random process (Gardiner, 1985; Allen, 2007; Särkkä & Solin, 2019). Since with a different realization of the noise process we get a different solution, the particular solutions of the equations are not often of interest, but instead, the aim is to determine the statistics of the solutions over all realizations. In the context of SDEs, the term $\boldsymbol{A}(X,t)$ of (7a) is called the drift function which determines the nominal dynamics of the system, and $\boldsymbol{B}(X,t)$ is the dispersion matrix which determines how the noise enters the system.

In our work, we consider a coupled system of differential equations which has a white noise driven forcing term of the following form (Särkkä & Solin, 2019).

$$\ddot{x}(t) + 2\xi_1\omega_1\dot{x}(t) + \omega_1^2 x(t) = \eta(t) \qquad \text{...(8)}$$

$$\ddot{y}(t) + 2\xi_2\omega_2\dot{y}(t) + \omega_2^2 y(t) = \sigma\dot{x}(t) \qquad \text{...(9)}$$

Here, σ is a constant and standard white noise is denoted by and represents the variation of wind speed. Then, following the relation between the white noise and the Wiener process in the context of SDEs, the above system of (8) - (9) can be expressed in the state space form of (7a) - (7b),

where

$$X = \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix};\ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\xi_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sigma & -\omega_2^2 & -2\xi_2\omega_2 \end{pmatrix};$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix};\ C = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

The transfer function of the above state space model is obtained by applying the Laplace transform with zero initial conditions.

$$H(s) = \frac{\sigma s^2}{(s^2 + 2\xi_1\omega_1 s + \omega_1^2)(s^2 + 2\xi_2\omega_2 s + \omega_2^2)} \qquad \text{...(10)}$$

By substituting $s = i\omega$ in the equation (10), the power spectral density function of the system is given by

$$S(\omega) = H(i\omega)H(-i\omega) =$$

$$\frac{\sigma^2\omega^4}{\left((\omega_1^2 - \omega^2)^2 + 4\xi_1^2\omega_1^2\omega^2\right)\left((\omega_2^2 - \omega^2)^2 + 4\xi_2^2\omega_2^2\omega^2\right)} \qquad \text{...(11)}$$

According to the Wiener-Khinchin theorem stated as theorem 1 in Appendix B, the autocorrelation function is the Fourier inversion of the spectral density function. The Fourier inversion was obtained by substituting $p = i\omega$ (Lee,1970).

$$S(p) = \frac{\sigma^2 p^4}{\left[(\omega_1^2 + p^2)^2 - 4\xi_1^2\omega_1^2 p^2\right]\left[(\omega_2^2 + p^2)^2 - 4\xi_2^2\omega_2^2 p^2\right]}$$

By decomposing $S(p)$ into partial fractions and taking the Fourier inversion, we have

$$\mathcal{F}^{-1}\left\{\frac{p}{p^2+2\xi_1\omega_1 p+\omega_1^2}\right\} = u(t)e^{-\xi_1\omega_1 t}\left(\cos\left(\omega_1\sqrt{1-\xi_1^2}\right) - \frac{\xi_1}{\sqrt{1-\xi_1^2}}\sin\left(\omega_1\sqrt{1-\xi_1^2}\right)\right) \quad ...(12)$$

$$\mathcal{F}^{-1}\left\{\frac{1}{p^2-2\xi_1\omega_1 p+\omega_1^2}\right\} = - \frac{1}{\omega_1\sqrt{1-\xi_1^2}}u(t)e^{-\xi_1\omega_1 t}\sin\left(\omega_1\sqrt{1-\xi_1^2}\right) \quad ...(13)$$

$$\mathcal{F}^{-1}\left\{\frac{p}{p^2-2\xi_2\omega_2 p+\omega_2^2}\right\} = -u(-t)e^{-\xi_2\omega_2 t}\left(\cos\left(\omega_2\sqrt{1-\xi_2^2}\right) - \frac{\xi_2}{\sqrt{1-\xi_2^2}}\sin\left(\omega_2\sqrt{1-\xi_2^2}\right)\right) \quad ...(14)$$

$$\mathcal{F}^{-1}\left\{\frac{1}{p^2+2\xi_2\omega_2 p+\omega_2^2}\right\} = \frac{1}{\omega_2\sqrt{1-\xi_2^2}}u(t)e^{-\xi_2\omega_2 t}\sin\left(\omega_2\sqrt{1-\xi_2^2}\right) \quad ...(15)$$

where $u(t)$ is the Heaviside step function. Hence, the autocorrelation function of y, given by equation (7b), is found to be,

$$\kappa(t) = \frac{\sigma^2}{2\pi}\left(A_1 e^{-\xi_1\omega_1 t}\cos(t\omega_{d_1}+\phi_1) + A_2 e^{-\xi_2\omega_2 t}\cos(t\omega_{d_2}+\phi_2)\right) \quad ...(16)$$

Each term A_1, A_2, ϕ_1 and ϕ_2 is a function of the system parameters ξ_1, ω_1, ξ_2 and ω_2. In order to estimate the system parameters, the normalized autocorrelation given below was used.

$$g(t) = \frac{\left(A_1 e^{-\xi_1\omega_1 t}\cos(t\omega_{d_1}+\phi_1) + A_2 e^{-\xi_2\omega_2 t}\cos(t\omega_{d_2}+\phi_2)\right)}{(A_1\cos\phi_1 + A_2\cos\phi_2)} \quad ...(17)$$

The system parameters were estimated by comparing the normalized autocorrelation function $g(t)$ with the empirical autocorrelation function of the residual process v.

RESULTS AND DISCUSSION

To illustrate the validity of the developed model, the wind speed dataset measured at Kokkilai in the North Eastern coast of Sri Lanka was used. As described in section 2, the wind speed dataset was first filtered using (4) and (5) and then the residual was modelled using the coupled second-order SDEs (8) and (9). The theoretically constructed normalized autocorrelation function (17) that corresponds to (8) and (9) and the output was fitted to the normalized autocorrelation function of the residual data in order to estimate the model parameters. The empirical correlogram of Figure 3(b) exhibits rapid decay in the first 100 hours and this decides the time scale of the underlying process (Brett & Tuller, 1991; Zarate-Minano, Mele & Milano, 2016). Thus, the parameters of the normalized theoretical correlogram of equation (17) were estimated using the empirical autocorrelation values of first 100 hours. The least squares errors method was used to fit the parameters. Moreover, the variance of a stationary process is equal to the initial value of the corresponding autocorrelation function (Chatfield, 2004). Therefore, the value of the parameter σ, can be fitted using the following relationship.

$$Var(v) \approx \kappa(0)$$

$$Var(v) \approx \frac{\sigma^2}{2\pi}(A_1\cos\phi_1 + A_2\cos\phi_2) \quad ...(18)$$

where $Var(v)$ is the variance of the residual process.

Using the estimated parameter values, the linear system of SDEs of equation (8) and (9) was simulated using the Euler-Maruyama numerical scheme. Then, the model-generated output was compared with the residual process using 30 sample realizations. Figure 5 shows that the probability density of the model matches that of the residual process v very closely.

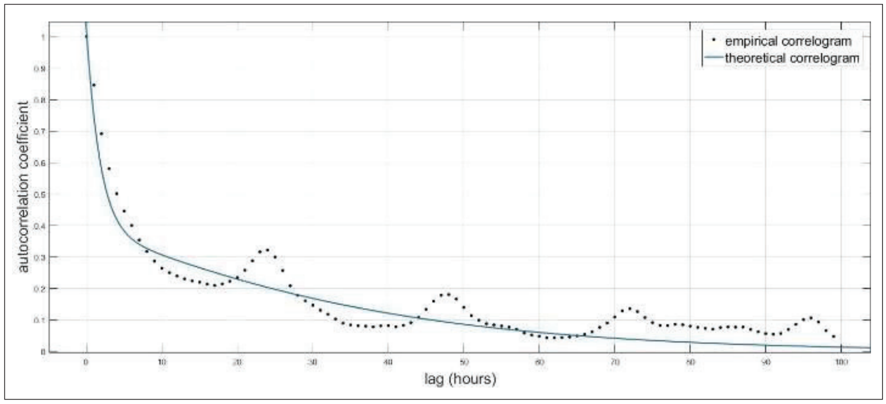

Figure 4: Approximation of empirical correlogram of residual data: ξ_1=0.9812, ω_1=0.3321, ξ_2=0.9907, ω_2= 0.02845 and σ = 0.4818.

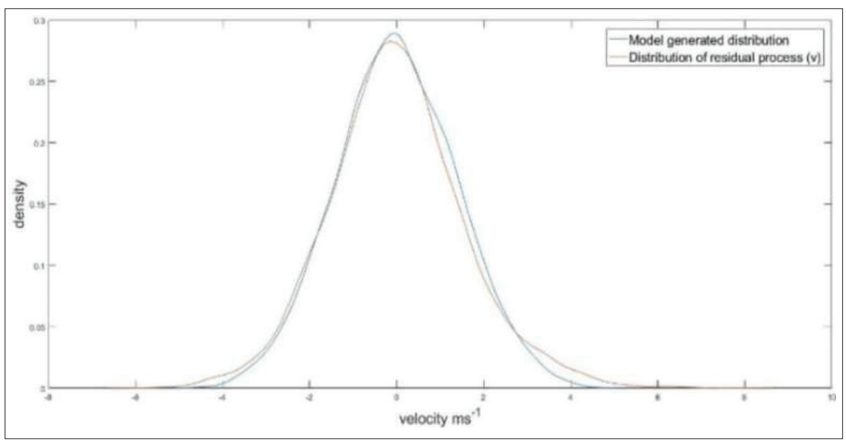

Figure 5: Comparison of probability distributions of SDE model generated stationary process and residual process

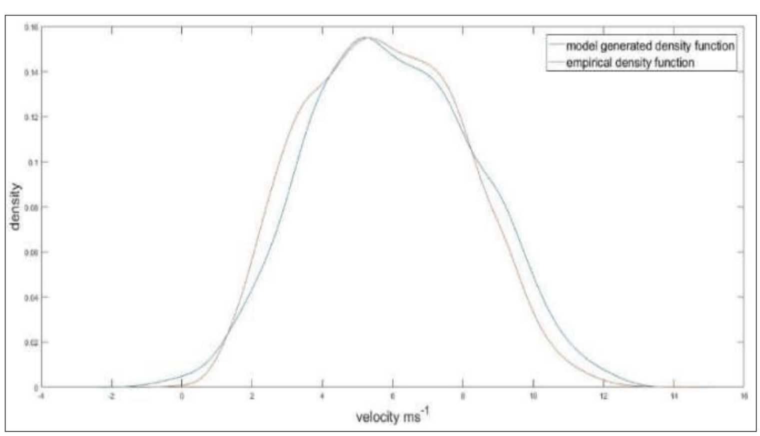

Figure 6: Comparison of final output of SDE model generated probability distribution of hourly wind speeds and that of empirical data

Next, the identified non-stationary component given by (4) was added to the model-generated outcome of the residual process by applying the inverse data transform corresponding to (5). Then the resulting wind speed distribution was compared with the empirical distribution of hourly wind speed distribution in Figure 6. It can be seen that our novel approach is capable of generating a wind speed distribution that is very close to those of the distribution of empirical data.

In addition to the comparison of the probability distributions, the comparison of the autocorrelation functions can be used to verify the level of accuracy. The comparison of autocorrelation functions is given in Figure 7 and shows excellent agreement between the model and empirical data.

Figure 8 shows the comparison of the probability distributions based on the proposed method with a method based on a single one-dimensional SDE. Probabilistic models based on a single one-dimensional SDE have been considered in many situations (Bibby *et al.,* 2005; Calif, 2012; Zarate-Minano *et al.,* 2016). Figure 8 demonstrates that the model proposed in this paper outperforms the models that use a single one-dimensional SDE.

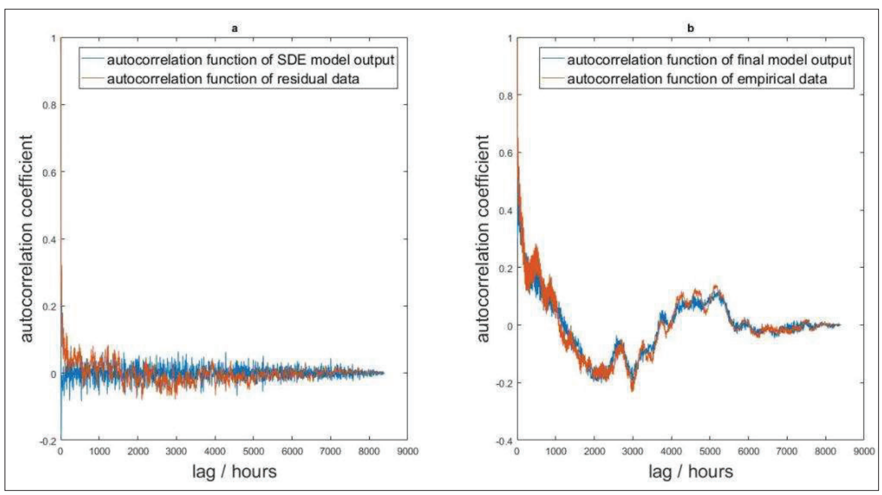

Figure 7: Comparison of correlograms

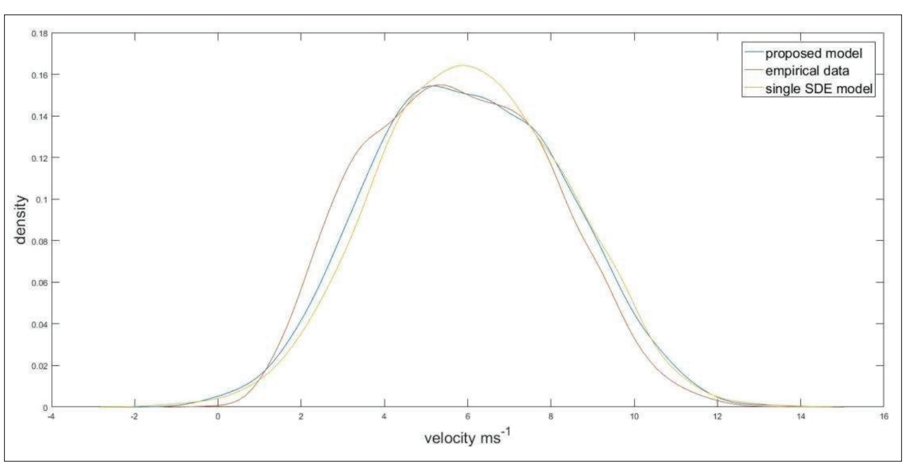

Figure 8: Comparison of distributions of single SDE model and empirical distribution and the proposed model

CONCLUSION

SDE-based approaches for modelling statistical properties of wind speed data are comparatively new. In existing work reported in the literature, probabilistic models are developed based on a single SDE of the standard form. In contrast, in this work, we have developed a novel approach that uses two coupled linear oscillators to model the statistical properties of appropriately filtered hourly wind speed data. The coupled system of SDEs generates a strictly stationary Gaussian process. Since wind speed data usually exhibits complex non-stationary features, we propose the use of a filter to remove these effects from the original wind speed data. The correlogram of the residual process is then fitted to the normalized theoretical autocorrelation function of the coupled SDE system.

The coupled SDE model consists of five unknown parameters. Four of them were estimated by curve fitting the theoretical normalized autocorrelation function to the empirical autocorrelation values, using the least squares method, while the other parameter was computed using these estimated parameter values.

The developed model was used to predict the statistical properties of wind speed data measured at Kokkilai, a coastal point located in the North Eastern shore of Sri Lanka. The simulation results show that the statistical properties of the simulated wind sequence are very close to those of the measured wind sequence. In particular, the model-generated correlogram of our method successfully captures almost all the features of the empirical correlogram. This is a significant improvement in comparison to those predicted by existing methods.

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Appendix A

Linear time invariant stochastic differential equations

A time invariant SDE can be expressed in the standard form as follows,

$$dx = Ax(t)dt + Bdw(t) \quad \text{...(A.1)}$$

Then the formal solution of the above equation (A.1) is written as,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B(\tau)dW(\tau) \quad \text{...(A.2)}$$

where e^{At} is the matrix exponential function. Since the noise process is Gaussian forcing, a linear differential equation can be considered as a linear operator acting on the noise process (and the initial conditions) and the solution is also Gaussian (Gardiner, 1985; Allen, 2007; Särkkä & Solin, 2019).

- **Expectation and correlation**

Since the white noise process has zero mean, taking expectations from both sides of equation (A.2) gives,

$$m(t) = E\big(x(t)\big) = exp(At)E\big(x(t_0)\big) \quad \text{...(A.3)}$$

which is the expected value of the SDE solutions over all realizations of noise. The corresponding covariance function is calculated by using the delta-correlation property of white noise as follows,

$$P(t) = E[\big(x(t)\text{-}m(t)\big)\big(x(t)\text{-}m(t)\big)^T]$$

$$P(t) = exp(At)P_0\,exp\big(A^Tt\big) + \int_{t_0}^{t} exp\big(A(t\text{-}\tau)\big)BQB^Tw(\tau)\,exp\big(A^T(t\text{-}\tau)\big)d\tau \quad \text{...(A.4)}$$

where $P_0=E[x(0)x(0)^T]$. By differentiating the above expressions for mean and covariance, the governing differential equations for the mean and covariance are given by

$$\frac{dm(t)}{dt} = Am(t) \quad \text{...(A.5)}$$

$$\frac{dP(t)}{dt} = AP(t)+P(t)A^T+BQB^T \quad \text{...(A.6)}$$

Moreover, it can be seen that the solutions of equation (A.1) generate a multivariate Gaussian distribution (Gardiner, 1985; Särkkä & Solin, 2019).

It should be noted when $P(t)$ is a constant, (A.6) is reduced to a Lyapunov equation. Then, the steady covariance function can be obtained by following the context of a matrix Lyapunov equation (Mao, 2007; Särkkä & Solin, 2019).

Appendix B

- **Fourier analysis of linear time invariant stochastic differential equations**

One way to study linear time invariant SDEs is the Fourier domain (Lindgren *et al.*, 2002; Särkkä & Solin, 2019). In this case, the power spectral density which is the squared absolute value of the Fourier transform of the process is defined as follows,

$$\boldsymbol{S_X}(\omega) = \boldsymbol{X}(i\omega)\boldsymbol{X}^T(\text{-}i\omega) \quad \text{...(B.1)}$$

The power spectral density $\boldsymbol{Q}$ of a multi-dimensional white noise process is also defined accordingly.

$$\boldsymbol{Q} = \boldsymbol{W}(i\omega)\boldsymbol{W}^T(\text{-}i\omega)$$

The spectral density is an efficient tool for estimating the covariance function generated as a result of SDE. Suppose $\boldsymbol{x}(t)$ is a stationary multi-dimensional stochastic process with zero mean. Then the corresponding covariance function is defined as,

$$\boldsymbol{C_x}(\tau) = E[\boldsymbol{x}(t)\boldsymbol{x}^T(t+\tau)]$$

The covariance function of a wide sense stationary process is only dependent on the time lag (τ). The Wiener-Khinchin theorem (Lindgren *et al.*, 2002; Särkkä & Solin, 2019) states the Fourier inversion of the power spectral density function of a given wide sense stationary process provides the corresponding covariance function as follows,

Theorem 1 (Wiener-Khinchin theorem):

The covariance of a wide sense stationary random process has a spectral decomposition given by the power spectrum of that process, i.e., $\boldsymbol{C_x}(\tau)=\mathcal{F}^{-1}\{\boldsymbol{S_X}(\omega)\}$.

Therefore, we have the autocorrelation function for a purely white noise process as follows,

$$\boldsymbol{C_W}(\tau)=\mathcal{F}^{-1}[\boldsymbol{Q}]=Q\mathcal{F}^{-1}[1]=\boldsymbol{Q}\delta(\tau)$$

- **Covariance of the solution of linear time invariant stochastic differential equations**

The spectral density function of resultant process $\boldsymbol{x(t)}$ of equation (A.1) can be given in the matrix version with the assumption of zero initial conditions. Note that the stationary stage can only exist if the matrix corresponds to a stable system, which means that all its eigenvalues have negative real parts. The Fourier transform of $\boldsymbol{x(t)}$ is,

$$\boldsymbol{X(i\omega)} = \left((i\omega)\boldsymbol{I}\text{-}\boldsymbol{A}\right)^{-1}\boldsymbol{BW}(i\omega)$$

where $\boldsymbol{W(i\omega)}$ is the formal Fourier transform of the white noise $w(t)$. It should be noted that the above Fourier transform does not strictly exist, as white noise is not square integrable. Then the spectral density of $\boldsymbol{x(t)}$ is given by the matrix,

$$\boldsymbol{S_X}(\omega)=(\boldsymbol{A}\text{-}(i\omega)\boldsymbol{I})^{-1}\boldsymbol{BQB}^T(\boldsymbol{A}+(i\omega)\boldsymbol{I})^{-T} \qquad \text{...(B.2)}$$

Hence the covariance function is,

$$\boldsymbol{C_x}(\tau)=\mathcal{F}^{-1}\left\{(\boldsymbol{A}\text{-}(i\omega)\boldsymbol{I})^{-1}\boldsymbol{BQB}^T(\boldsymbol{A}+(i\omega)\boldsymbol{I})^{-T}\right\} \qquad \text{...(B3)}$$

This provides useful means to estimate the autocovariance function of a solution to a SDE without explicitly solving the equation.