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RESEARCH ARTICLE

Mathematical statistics

The odd modified Burr-III exponential distribution: properties, estimation and application

H Rasheed[1], IS Dar[1], M Saqib[1], N Abbas[2], and M Suhail[3*]
[1] *College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan.*
[2] *Department of Statistics, Government Postgraduate College, Jhang, Pakistan.*
[3] *Department of Statistics, The University of Agriculture Peshawar, Amir Muhammad Khan Campus Mardan, Pakistan.*

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Abstract: Modelling waiting times have always been the earnest need of experts. In this paper, we proposed a new lifetime model entitled Odd Modified Burr-III Exponential Distribution which has four parameters. The proposed model has the heterogeneity in shapes of the density and hazard curves. The density curves of the model are unimodal and reversed J-shaped. The study of the various statistical properties such as moments (raw moments, mean moments, negative moments, incomplete moments, and conditional moments), mode, harmonic mean, geometric mean, hazard, survival, quantile and characteristics function, coefficient of skewness, and kurtosis of the proposed model are presented. In the reliability study, we derive the expressions of mean residual life, mean waiting time, and entropies (Renyi and Shannon). Furthermore, order statistics and L- moments are also presented in this study. The maximum likelihood technique is used to estimate the parameters of the model. Monte Carlo simulation is used to evaluate the performance of the maximum likelihood estimates in terms of absolute average biases and mean square errors. The impact of sample size on parametric properties is also studied. The importance of the proposed model has been illustrated through real-life application from the field of engineering and manufacturing material.

Keywords: Exponential distribution, maximum likelihood estimation, modified Burr-III distribution, order statistics.

INTRODUCTION

Burr (1942) defined a framework centred on twelve different types of cumulative distribution functions, which are based on the Pearson differential equations that have several types of density functions. This framework is true for a broad variety of applications. The Burr-XII and Burr-III models are the most widely used in the Burr system. The Burr-III distribution, also called Dagum distribution, is useful in the studies of wages, wealth, and income (Dagum, 1977), and kappa distribution in the meteorological field (Mielke Jr, 1973). The Burr-III is the inverse form of the Burr-XII distribution and is used in several fields for modelling statistical data such as forestry, fracture roughness, options market price, life testing, operational risks, etc. (Wingo, 1983; Lindsay *et al*., 1996; Sherrick *et al*., 1996; Nadarajah & Kotz, 2004; 2006; Chernobai *et al*., 2007; Gove *et al*., 2008; Abbas *et al*., 2019). It was later used (Mielke Jr, 1973) in meteorology (Abdel-Ghaly *et al*., 1997), in reliability analysis, and in low-flow frequency analysis (Shao *et al*., 2008).

A lot of attempts have been made over the last few decades to enhance the chances of modelling datasets of different types. Such efforts led to the development of various methods for the generation of new probability distribution families. Further, various extensions and modifications of distributions have been developed by researchers. For instance, the exponentiated Weibull distribution was suggested by Mudholkar and Srivastava (1993), the Marshall-Olkin generated family by Marshall and Olkin (1997), the Kumaraswamy-G family by

* Corresponding author (msuhail88@aup.edu.pk ; https://orcid.org/0000-0003-1784-5272)

Cordeiro and de Castro (2011), the modified Burr III G family of distributions by Arifa *et al*. (2017) and the Pseudo-gamma Distribution by Abbas and Mohsin (2020). The most recent developments on the Burr III are presented in Modi and Gill (2020), Afify *et al*. (2021), El Shekh Ahmed (2021), Mahmoud *et al*. (2021), and ul Haq *et al*. (2021).

We rarely come across a practical situation in the real world that has a constant hazard function during the lifetime. Hence, assuming hazard rate as a function of time seems practical, and this has led to the development of an alternative model for lifetime data analysis. Hence, this study provides distribution of mixtures using existing baseline distributions, which is useful for modelling real-life datasets whose failure rate is not constant. Many compound distributions work better on versatility than their baseline distributions, when applied to real-life datasets. Many authors have discovered that the analytical properties of the baseline distributions can be improved by increasing the number of parameters, as the new distribution can exhibit stronger structure in terms of better results than the baseline distribution. Recently, several statistical models are used to explain the phenomena of real-world situations, but their use is limited to improper exploration in various fields such as engineering, finance, reliability, and the other similar areas. This paper aims at introducing a new flexible and tractable distribution, which among the other competitive models, offers the best fit for various real-life datasets.

MATERIALS AND METHODS

In this section, the proposed model and the mathematical and statistical properties, such as cumulative distribution function (cdf), probability density function (pdf), hazard function (hf), survival function (sf), reverse hazard function (rhf), and cumulative hazard function (chf) of the proposed distribution are derived, along with their graphical representation. Alzaatreh *et al*. (2013) defined the following generator to propose a new family of distributions, called the T-X family. Let $g_2(x)$ be the probability density function (pdf) of a random variable (r. v) $X \in [u,v]$, for $-\infty \le u < v \le \infty$ and let $G(F(x)) = \dfrac{G_1(x)}{1-G_1(x)}$ be a function of the cdf $F(x)$ of any r.v X so that $G(F(x))$ satisfies the following conditions:

$G(F(x)) \in [u,v]$.

$G(F(x))$ is differentiable and monotonically non-decreasing.

$G(F(x)) \to u$ as $x \to -\infty$ and $G(F(x)) \to b$ as $x \to \infty$.

Then, the cdf of the T-X family of distributions for r.vX is given by:

$$F(x) = \int_0^{\frac{G_1(x)}{1-G_1(x)}} g_2(x)\,dx \qquad ...(1)$$

Here, the odd $\dfrac{G_1(x)}{1-G_1(x)}$ is obtained from an exponential distribution (ED), $g_2(x)$ is the pdf of the Modified Burr-III (MB-III) distribution and the resultant distribution is named as the Odd Modified Burr-III Exponential distribution (OMBED-III). This generator would provide the flexible shapes of the pdf and hazard function (hf). The modified Burr III-G family of distributions was proposed by Arifa *et al* (2017). Ali *et al* (2015) defined the cdf and pdf of MB-III respectively as:

$$G_2(x) = \left(1+\lambda x^{-\beta}\right)^{-\frac{\alpha}{\lambda}}, x \ge 0,$$

and

$$g_2(x) = \alpha\beta x^{-\beta-1}(1+\lambda x^{-\beta})^{-\frac{\alpha}{\lambda}-1}, \; x \ge 0 \qquad ...(2)$$

where $a > 0$, $\beta > 0$, and λ > 0 are shape parameters of MB-III. There are various special cases of MB-III. For example, when $\theta \to 0$, it reduces to generalized inverse Weibull distribution (De Gusmao *et al*., 2011); for $\theta = 1$, it transforms to Burr-III distribution (Burr, 1942); and it becomes the log-logistic distribution having the scale parameter $a = \theta = 1$ (Shoukri *et al*., 1988) .

A random variable X is said to have an exponential distribution with cdf defined as:

$$G_1(x) = 1 - e^{-\theta x}, x \ge 0, \theta > 0 \qquad ...(3)$$

Then, the cdf of OMBED-III is obtained by putting equation (2) and equation (3) in equation (1) as:

$$F(x;\varphi) = \left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}}, \; x \ge 0, \qquad ...(4)$$

with the pdf

$$f\left(x;\varphi\right)=\alpha\beta\theta\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}-1}\frac{\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}},x\geq 0, \quad ...(5)$$

where $\varphi=\left(\alpha,\beta,\theta,\lambda\right)>0$ is a parameter vector, and $\alpha,\beta,\lambda,\ \theta$ are shape parameters. The corresponding sf, hf, rhf, and chf are respectively given by:

$$S\left(x\right)=1-\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}},$$

$$h\left(x\right)=\left[\alpha\beta\theta\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}-1}\frac{\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}\right]$$

$$\left[1-\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}}\right]^{-1},$$

$$\tau\left(x\right)=\left[\alpha\beta\theta\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}-1}\frac{\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}\right]$$

$$\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{\frac{\alpha}{\lambda}},$$

and

$$H\left(x\right)=-\log\left[1-\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}}\right].$$

The above graphs show some possible shapes of pdf and hf of OMBED-III for different values of parameters. Figure 1 (a) shows that the density of distribution exhibits a reverse J-shaped and unimodal behaviour. Figure 1 (b) provides a bathtub trend of hf of the proposed distribution.

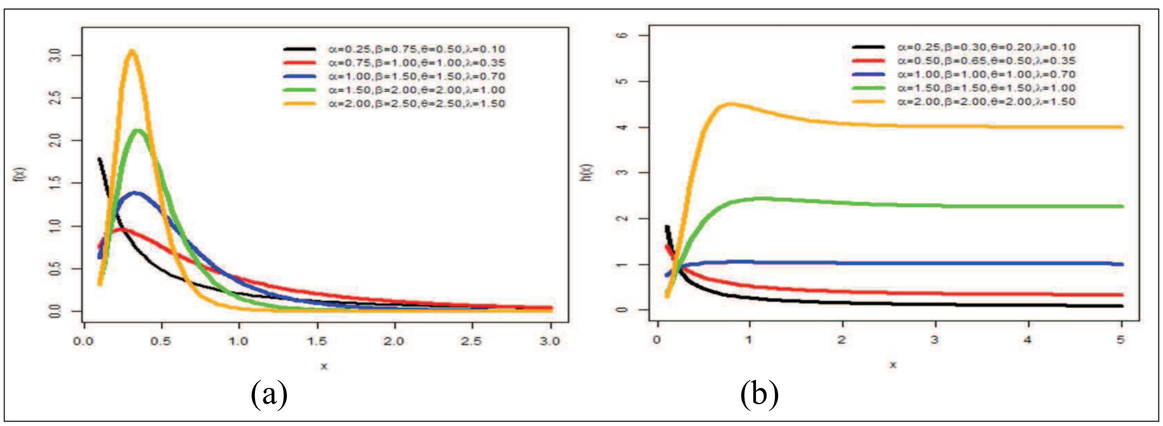

Figure 1: The (a) pdf and (b) hf of OMBED-III at various combinations of parameter values

Expansion of pdf: This section offers a useful linear representation of the density function shown in equation (5).

Using expansion $\left(a+b\right)^{n}=\sum_{i=0}^{\infty}\binom{n}{i}a^{n-i}b^{i}$, we get

$$f(x)=\alpha\beta\theta\sum_{i=0}^{\infty}\binom{-\frac{\alpha}{\lambda}-1}{i}\lambda^{i}\left(e^{-\theta x}\right)^{\beta(i+1)}$$

$$\left(1-e^{-\theta x}\right)^{-(\beta(i+1)+1)}$$

Using $\left(1-z\right)^{-k} = \sum_{j=0}^{\infty}\binom{-k}{j}\left(-1\right)^{j} z^{j}$ now we can write

$$\left(1-e^{-\theta x}\right)^{-(\beta(i+1)+1)} =$$

$$\sum_{j=0}^{\infty}\binom{-\left(\beta\left(i+1\right)+1\right)}{j}\left(-1\right)^{j} e^{-\theta xj} .$$

Then, we get

$$f\left(x\right)=\alpha\beta\theta\sum_{i,j=0}^{\infty}\binom{-\frac{\alpha}{\lambda}-1}{i}\binom{-\left(\beta\left(i+1\right)+1\right)}{j}$$

$$\lambda^{i}\left(-1\right)^{j}\left(e^{-\theta x}\right)^{\beta(i+1)+j}$$

Thus,

$$f\left(x;\varphi\right)=\alpha\beta\theta\sum_{i,j=0}^{\infty}\delta_{ij}\left(e^{-\theta x}\right)^{\beta(i+1)+j} \qquad ...(6)$$

where $\delta_{ij} = \binom{-\frac{\alpha}{\lambda}-1}{i}\binom{-\left(\beta\left(i+1\right)+1\right)}{j}\lambda^{i}\left(-1\right)^{j}$.

Properties

In this section, we derived measures of central tendency (*i.e.*, mean, median, mode, geometric mean, and harmonic mean), the measures of dispersion such as variance, moments (raw moments, mean moments, negative moments, incomplete moments, and conditional moments), coefficients of skewness and kurtosis, coefficient of variation, and moment generating function (mgf), characteristic function (cf), mean residual life (MRL), mean waiting time (MWT), order statistics, L-moments, and entropies of OMBED-III.

By using the inverse transformation technique, the simplified form of the quantile function of OMBED-III is given by:

$$x=\frac{1}{\theta}\log\left[\left\{\lambda\left(u^{-\frac{\lambda}{\alpha}}-1\right)^{-1}\right\}^{\frac{1}{\beta}}+1\right]. \qquad ...(7)$$

Hence, the median of the underlying distribution is obtained by substituting $u=0.50$ in equation (7). Similarly, we get the first quartile (Q_1) and third quartile (Q_3) by substituting u = 0.25, 0.75, respectively in equation (7).

Mode

The mode of the proposed distribution can be obtained by taking the logarithm of the pdf given in equation (5) and by taking the derivative with respect to x. Then, we have

$$\frac{d\left[\log f\left(x\right)\right]}{dx}=-\frac{\theta\left[(e^{\theta x}-1)^{\beta}+e^{\theta x}\left\{(e^{\theta x}-1)^{\beta}-\alpha\right\}\beta+\lambda\right]}{(e^{\theta x}-1)\left[(e^{\theta x}-1)^{\beta}+\lambda\right]}.$$

In order to obtain the mode we write $\frac{\left(d\left(logf\left(x\right)\right)\right)}{dx}=0$ and after simplification, we get

$$(e^{\theta x}-1)^{\beta}\left(\beta e^{\theta x}+1\right)=\alpha\beta e^{\theta x}-\lambda$$

The modal equation of OMBED-III is a non-linear one, that can be solved numerically for various parameter values of α,β,θ and λ.

Harmonic mean

The simplified form of the harmonic mean of OMBED-III is

$$H=\frac{1}{\alpha\beta\theta\sum_{i,j=0}^{\infty}\delta_{ij}}.$$

Geometric mean

The simplified form of the geometric mean of OMBED-III is

$$G=antilog\left[\sum_{i,j=0}^{\infty}\delta_{ij}\frac{\alpha\beta\left[-\gamma-\log\left(\theta\left(\beta\left(i+1\right)+j\right)\right)\right]}{\beta\left(i+1\right)+j}\right]$$

Moments

The r^{th} moment of OMBED-III is obtained as

$$E\left(X^r\right)=\int_0^{+\infty} x^r f(x)\,dx.$$

To solve $\mu_r^{'}$, substitute $f(x)$ as given in equation (5) into the above equation as given below:

$$\mu_r^{'}=\int_0^{+\infty} x^r \alpha\beta\theta \sum_{i,j=0}^{\infty} \delta_{ij}\left(e^{-\theta x}\right)^{\beta(i+1)+j} dx$$

Let $y=\theta x\left(\beta(i+1)+j\right);\ \ x=\frac{y}{\theta\left(\beta(i+1)+j\right)};$

$$dx=\frac{dy}{\theta\left(\beta(i+1)+j\right)}, \text{ and}$$

$$limits=\begin{cases} x=0\ , y=0 \\ x=\infty,\ \ y=\infty \end{cases}$$

we get,

$$\mu_r^{'}=\alpha\beta\theta\sum_{i,j=0}^{\infty}\delta_{ij}\int_0^{\infty}\left(\frac{y}{\theta\left(\beta(i+1)+j\right)}\right)^r e^{-y}\frac{dy}{\theta\left(\beta(i+1)+j\right)}$$

$$\mu_r^{'}=\alpha\beta\sum_{i,j=0}^{\infty}\delta_{ij}\frac{1}{\theta^r\left(\beta(i+1)+j\right)^{r+1}}\int_0^{\infty} e^{-y}y^r dy$$

Since $\int_0^{\infty} e^{-y}y^{n-1}dy=\Gamma(n)$, the r[th] ordinary moment of OMBED-III is

$$\mu_r^{'}=\sum_{i,j=0}^{\infty}\delta_{ij}\frac{\alpha\beta r!}{\theta^r\left(\beta(i+1)+j\right)^{r+1}} \qquad ...(8)$$

Now, we will find the central moments of the proposed distribution, using the following relationship:

$$\mu_r=\sum_{i=0}^{r}\binom{r}{i}(-1)^i\left(\mu_1^{'}\right)^i\mu_{r-i}^{'} \qquad ...(9)$$

where, $\mu_r^{'}$ is given in equation (8).

Remarks:

i. We can obtain the mean of OMBED-III by inserting $r=1$ in equation (8).

ii. By inserting $r=2$ in equation (9), we can obtain the variance of OMBED-III.

The coefficient of skewness and kurtosis of OMBED-III can be obtained from equation (9) by putting $r=3$ and $r=2$ as $\beta_1=\frac{\mu_3^{2}}{\mu_2^{3}}$, respectively.

We get the coefficient of variation

$$C.V=\frac{S.D}{Mean}\times 100.$$

by substituting $r=1$ in equation (8), and $r=2$ in equation (9) and taking its square root.

The negative moment of OMBED-III can be obtained by substituting negative r in equation (8) as

$$E\left(X^{-r}\right)=\sum_{i,j=0}^{\infty}\delta_{ij}\frac{\alpha\beta(-r)!}{\theta^{-r}\left(\beta(i+1)+j\right)^{(-r)+1}};$$
$$r=1,2,3,4,\cdots$$

The lower and upper incomplete moments of OMBED-III are respectively obtained by using the following formulae:

$$m_{r(t)}=\int_{-\infty}^{x} x^r f(x)\,dx$$

and

$$M_{r(t)}=\int_x^{\infty} x^r f(x)\,dx$$

After solving the above equations, we get the following simplified results:

$$m_{r(t)}=\sum_{i,j=0}^{\infty}\delta_{ij}\frac{\alpha\beta\theta\gamma\left[r+1,\theta x\left(\beta(i+1)+j\right)\right]}{\left[\theta\left(\beta(i+1)+j\right)\right]^{r+1}}$$

and

$$M_{r(t)}=\sum_{i,j=0}^{\infty}\delta_{ij}\frac{\alpha\beta\theta\Gamma\left[r+1,\theta x\left(\beta(i+1)+j\right)\right]}{\left[\theta\left(\beta(i+1)+j\right)\right]^{r+1}}.$$

The conditional moment of OMBED-III is obtained as

$$E\left(x^d\,\middle|\,X>x\right)=\frac{1}{S(x)}\int_x^{+\infty} x^d f(x)\,dx.$$

Then, we get the simplified form as

$$E\left(x^{d}\middle|X>x\right)=\left[1-\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}}\right]^{-1}$$

$$\sum_{i,j=0}^{\infty}\delta_{ij}\,\frac{\alpha\beta\theta\Gamma\left[d+1,\theta x\left(\beta\left(i+1\right)+j\right)\right]}{\left[\theta\left[\beta\left(i+1\right)+j\right]\right]^{d+1}}$$

The moment generating function (mgf) of OMBED-III is given by

$$M_{x}\left(t\right)=E\left(e^{tx}\right)=\int_{0}^{+\infty}e^{tx}f\left(x\right)dx=\int_{0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(t\right)^{m}x^{m}}{m!}f\left(x\right)dx$$

$$=\sum_{m=0}^{\infty}\frac{\left(t\right)^{m}}{m!}E\left(x^{m}\right)$$

By substituting equation (8) to the above equation, we get mgf as

$$M_{x}\left(t\right)=\sum_{m=0}^{\infty}\frac{\left(t\right)^{m}}{m!}\left[\sum_{i,j=0}^{\infty}\delta_{ij}\,\frac{\alpha\beta m!}{\theta^{m}\left[\beta\left(i+1\right)+j\right]^{m+1}}\right]$$

Similarly, we derive the characteristic function (cf) of OMBED-III using its well-known definition given by

$$\varphi\left(x\right)=\int_{0}^{+\infty}e^{itx}f\left(x\right)dx=\sum_{p=0}^{\infty}\frac{\left(it\right)^{p}}{p!}\int_{0}^{\infty}x^{p}f\left(x\right)dx$$

$$=\sum_{p=0}^{\infty}\frac{\left(it\right)^{p}}{p!}E\left(x^{p}\right)$$

After substituting equation (8), now we have

$$\varphi\left(x\right)=\sum_{p=0}^{\infty}\frac{\left(it\right)^{p}}{p!}\left[\sum_{i,j=0}^{\infty}\delta_{ij}\,\frac{\alpha\beta p!}{\theta^{p}\left[\beta\left(i+1\right)+j\right]^{p+1}}\right].$$

Mean residual life and mean waiting time

The mean residual life of the random variable X following OMBED-III has the form

$$m\left(t\right)=E\left[X-t\middle|X>t\right]=\left[S\left(x\right)\right]^{-1}\int_{t}^{+\infty}\left(x-t\right)f\left(x\right)dx$$

By solving the above equation, we obtained

$$m\left(t\right)=\left[1-\left\{1+\lambda\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}}\right]^{-1}$$

$$\sum_{i,j=0}^{\infty}\delta_{ij}\,\frac{\alpha\beta\Gamma\left[2,\theta t\left(\beta\left(i+1\right)+j\right)\right]}{\theta\left[\beta\left(i+1\right)+j\right]^{2}}-t$$

and the mean waiting time is given by

$$r\left(t\right)=E\left(t-X\middle|X\le t\right)=\frac{1}{F\left(t\right)}\int_{0}^{t}\left(t-x\right)f\left(x\right)dx.$$

Now, we obtained the following simplified result.

$$r\left(t\right)=t-\left[1+\lambda\left(\frac{e^{-\theta t}}{1-e^{-\theta t}}\right)^{\beta}\right]^{\frac{\alpha}{\lambda}}$$

$$\sum_{i,j=0}^{\infty}\delta_{ij}\,\frac{\alpha\beta\gamma\left[2,\theta t\left(\beta\left(i+1\right)+j\right)\right]}{\theta\left[\beta\left(i+1\right)+j\right]^{2}}$$

Order statistics

The k[th] order statistics of OMBED-III can be obtained from the following equation:

$$f_{k:n}\left(x\right)=\frac{n!}{\left(k-1\right)!\left(n-k\right)!}f\left(x\right)\left[F\left(x\right)\right]^{k-1}\left[1-F\left(x\right)\right]^{n-k}$$

...(10)

Substituting pdf and cdf to the above equation, we have

$$f_{k:n}\left(x\right)=\frac{n!}{\left(k-1\right)!\left(n-k\right)!}\left[\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{k\alpha}{\lambda}-1}\right]$$

$$\left[\frac{\alpha\beta\theta\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}\right]\left[1-\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}}\right]^{n-k}$$

The minimum and maximum order statistics can be obtained by substituting $k=1$ and $k=n$ in equation (10), respectively as

$$f_{1:n}\left(x\right)=n\left[\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}-1}\frac{\alpha\beta\theta\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}\right]$$

$$\left[1-\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{\alpha}{\lambda}}\right]^{n-1}$$

and

$$f_{n:n}(x)=n\left[\left\{1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right\}^{-\frac{n\alpha}{\lambda}-1}\frac{\alpha\beta\theta\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}\right]$$

L-Moments

The L-moments can be computed by using the following expression:

$$\lambda_r=\sum_{i=0}^{r-1}(-1)^{r-1-i}\binom{r-1}{i}\binom{r-1+i}{i}\beta_i,\ \ r=1,2,3,4,\cdots \qquad ...(11)$$

where

$$\beta_i=E\left(x\left[F(x)\right]^i\right)=\int_0^{\infty}x\left[F(x)\right]^i f(x)\,dx$$

$$=\alpha\beta\theta\int_0^{\infty}x\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{i\alpha}{\lambda}}$$

$$\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}-1}\frac{\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}dx$$

$$=\alpha\beta\theta\int_0^{\infty}x\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}(i+1)-1}$$

$$\frac{\left(e^{-\theta x}\right)^{\beta}}{\left(1-e^{-\theta x}\right)^{\beta+1}}dx$$

Using

$$(a+b)^n=\sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k$$

we get

$$\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}(i+1)-1}$$

$$=\sum_{k=0}^{\infty}\binom{-\frac{\alpha}{\lambda}(i+1)-1}{k}\left(\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right)^{k}$$

$$=\alpha\beta\theta\sum_{k=0}^{\infty}\binom{-\frac{\alpha}{\lambda}(i+1)-1}{k}\lambda^k\int_0^{\infty}\frac{\left(e^{-\theta x}\right)^{\beta(k+1)}}{\left(1-e^{-\theta x}\right)^{\beta(k+1)+1}}dx$$

$$=\alpha\beta\theta\sum_{k=0}^{\infty}\binom{-\frac{\alpha}{\lambda}(i+1)-1}{k}\lambda^k\int_0^{\infty}\left(e^{-\theta x}\right)^{\beta(k+1)}$$

$$\left(1-e^{-\theta x}\right)^{-[\beta(k+1)+1]}dx$$

Using binomial theorem we get

$$\left(1-e^{-\theta x}\right)^{-[\beta(k+1)+1]}$$

$$=\sum_{j=0}^{\infty}\binom{-(\beta(k+1)+1)}{j}(-1)^j e^{-\theta xj},$$

and the above equation becomes

$$=\alpha\beta\theta\sum_{k=0}^{\infty}\binom{-\frac{\alpha}{\lambda}(i+1)-1}{k}\sum_{j=0}^{\infty}\binom{-(\beta(k+1)+1)}{j}$$

$$\lambda^k(-1)^j\int_0^{\infty}x\left(e^{-\theta x}\right)^{(\beta(k+1)+j)}dx$$

Substitute

$$y=\theta x(\beta(k+1)+j);$$

$$x=\frac{y}{\theta(\beta(k+1)+j)};$$

$$dx=\frac{dy}{\theta(\beta(k+1)+j)},$$

and

$$limits=\begin{cases}x=0,y=0\\x=\infty,y=\infty\end{cases}$$

into the above equation, and we get

$$= \sum_{k,j=0}^{\infty} \binom{-\frac{\alpha}{\lambda}(i+1)-1}{k} \binom{-\left(\beta(k+1)+1\right)}{j}$$

$$\frac{\alpha\beta\lambda^k (-1)^j}{\theta\left(\beta(k+1)+j\right)^2} \int_0^{\infty} e^{-y}\, y\, dy$$

Now, we have

$$\beta_i = \sum_{k,j=0}^{\infty} \binom{-\frac{\alpha}{\lambda}(i+1)-1}{k} \binom{-\left(\beta(k+1)+1\right)}{j}$$

$$\frac{\alpha\beta\lambda^k (-1)^j}{\theta\left(\beta(k+1)+j\right)^2},\ i = 0,1,2,3,\cdots$$

Substituting the above summation in place of β_i in equation (11) allows us to obtain L-moments. Then, by taking $r = 2,3$ and 4, we can obtain the L-skewness (τ_3) and L-kurtosis (τ_4) by using the following formulae:

$\tau_3 = \frac{\lambda_3}{\lambda_2}$ and $\tau_4 = \frac{\lambda_4}{\lambda_2}$.

Entropies

The entropy is defined as a measure of uncertainty; the larger the value of entropy, the more uncertain the data. Given a random variable X with a density function $f(x)$, the Renyi entropy is defined as

$$I_{R(a)} = \frac{1}{1-a} \log\left(\int_0^{\infty} f^a(x)\, dx \right), a > 0, a \neq 1,$$

By solving the above integral, we get the following simplifying result:

$$I_{R(a)} = \frac{1}{1-a} \log\left[\left(\alpha\beta\theta \sum_{i,j=0}^{\infty} \delta_{ij} \right)^a \frac{1}{\theta a\left[\beta(i+1)+j\right]} \right]$$

The Shannon entropy can be obtained from the following expression.

$$E\left(-\log\left[f(x)\right]\right) = -\int_0^{\infty} f(x) \log\left[f(x)\right] dx.$$

Solving the above integral, the Shannon entropy takes the simplest form

$$E\left(\log\left[f(x)\right]\right) = \left[\alpha\beta\theta \sum_{i,j=0}^{\infty} \delta_{ij}\right] \times \log\left[\alpha\beta\theta \sum_{i,j=0}^{\infty} \delta_{ij}\right] \times \frac{1}{\theta\left[\beta(i+1)+j\right]}.$$

Maximum likelihood estimation

Although several approaches exist for parameter estimation; in the present study, we considered the maximum likelihood (ML) method to estimate parameters of OMBED-III. Let $x_1, x_2, \cdots, x_n$ be a random sample from OMBED-III. The log-likelihood function of equation (5) is given as:

$$\log(f(x)) = n\log(\alpha) + n\log(\beta) + n\log(\theta) - \left(\frac{\alpha}{\lambda}+1\right) \sum_{i=1}^{n} \log\left[1 + \lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\beta}\right] - \beta\theta\sum_{i=1}^{n} x_i - (\beta+1)\sum_{i=1}^{n}\log\left(1-e^{-\theta x_i}\right) \quad ...(12)$$

The ML estimates can be achieved by solving the equations obtained by differentiating the log-likelihood function given in equation (12) with respect to the parameters α, β, θ, and λ and equating them to zero, i.e.,

$$\frac{\partial \log(f(x))}{\partial\alpha} = \frac{n}{\alpha} - \left(\frac{1}{\lambda}\right) \sum_{i=1}^{n} \left[\log\left(1 + \lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\beta}\right)\right] = 0$$

$$\frac{\partial \log(f(x))}{d\beta} = \frac{n}{\beta} - \theta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\left[\log\left(1-e^{-\theta x_i}\right)\right] - (\alpha+\lambda)\sum_{i=1}^{n} \left[\frac{\log\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)}{\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{-\beta} + \lambda} \right] = 0$$

$$\frac{\partial \log(f(x))}{\partial \theta}=\frac{n}{\theta}-\left(\frac{\alpha}{\lambda}+1\right)\sum_{i=1}^{n}\left[\frac{\beta\lambda\left(\frac{-x\left(e^{-\theta x_i}\right)^{\beta}}{\left(1-e^{-\theta x_i}\right)^{\beta+1}}\right)}{1+\lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\beta}}\right]$$

$$-\beta\sum_{i=1}^{n}x_i-(\beta+1)\sum_{i=1}^{n}\left(\frac{x_i e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)=0$$

$$\frac{\partial \log(f(x))}{\partial \lambda}=\frac{\alpha}{\lambda^2}\sum_{i=1}^{n}\left[\log\left\{1+\lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\beta}\right\}\right]$$

$$-\left(\frac{\alpha}{\lambda}+1\right)\sum_{i=1}^{n}\left[\frac{\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\beta}}{1+\lambda\left(\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}\right)^{\beta}}\right]=0.$$

The above system of nonlinear equations cannot be solved analytically. However, they can be solved using numerical iterative techniques. The function (optim) is used for solving non-linear equations in R programming language. The R-code is provided in Appendix A-3.

RESULTS AND DISCUSSION

In this section, the results from Monte Carlo simulation and a real-life application are discussed in detail.

Simulation study

A simulation study was conducted to examine the performance of ML estimators, the impact of parameters, and sample size on the fit of the model. For this purpose, 1000 samples of different sizes n = 25, 50, 150, 250 and 350 were generated from OMBED-III with different combinations of parameters values $\varphi=\alpha,\beta,\theta,\lambda$. In the simulation, all programmes were written in the R version 3.6.1 using the following procedure.

Step 1: A random number U was generated from a uniform distribution on unit interval (0, 1), *i.e.*, Generate U ~Uniform (0, 1).

Step 2: The cdf was calculated, and used to generate random numbers from a particular distribution by using the inverse transformation technique, as:

$$F(x;\varphi)=\left[1+\lambda\left(\frac{e^{-\theta x}}{1-e^{-\theta x}}\right)^{\beta}\right]^{-\frac{\alpha}{\lambda}},$$

Step 3: x was obtained from the following expression

$$x=\frac{1}{\theta}\log\left[\left\{\lambda\left(U^{-\frac{\lambda}{\alpha}}-1\right)^{-1}\right\}^{\frac{1}{\beta}}+1\right].$$

Step 4: We considered four different cases for parameter estimation as follows: Set I $(\alpha=1.0,1.5,2.0,2.5)$ at fixed values of $(\beta,\theta,\lambda)=(0.75,1.6,1.5)$; Set II $(\beta=0.75,1.0,1.5,2.0)$ at fixed values of $(\alpha,\theta,\lambda)=(0.75,2.0,1.5)$; Set III $(\theta=1.0,1.5,2.0,2.5)$ at fixed values of $(\alpha,\beta,\lambda)=(0.75,1.0,1.5)$; and Set IV $(\lambda=0.75,1.0,1.5,2.0)$ at fixed values of $(\alpha,\beta,\theta)=(1.0,1.5,0.75)$.

Step 5: We repeated steps 1, 2, 3, and 4 for each of the combinations of parameters and sample sizes; 1000 samples were generated using the R software version 3.6.1.

Let $\hat{\theta}$ be the estimator of θ. An average absolute bias (AAB) and mean squared error (MSE) of $\hat{\theta}$ is obtained as:

$$AAB(\hat{\theta})=|\hat{\theta}-\theta|$$

$$MSE(\hat{\theta})=variance(\hat{\theta})+(bias(\hat{\theta}))^2$$

For each of the 1000 samples and combinations of parameters, AAB and MSE of estimates is obtained.

Step 6: We plotted the absolute bias and MSE versus sample size n for four sets of parameters in Figures A-1 and A-2 given in the appendix. We observed from figures that MSE and absolute bias decreases for all parameters by increasing the sample size. Moreover, MSE and absolute bias increase by increasing the particular parameter while keeping the effect of other parameters constant.

Real-life application

This section provides application of the proposed distribution using a real-life dataset to show its importance in the real world. Furthermore, model parameters are estimated using the ML method and different information criteria, such as Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent Akaike information criteria (CAIC), and Hannon-Quinn information criteria (HQIC), and the Kolmogorov-Smirnov (K-S) statistic is given to suggest the best-fit model. Lower values of these criteria provide a well-fitted model. The fitted pdf and cdf plot, total time test (TTT) plot and Q-Q plot of the proposed distribution are also provided for better understanding. All the required results are estimated using R software.

Illustrative example

The dataset comprises 64 observations for the breaking strength of single carbon fibres of length 10 mm. This dataset was studied by (AL-Hussaini & Hussein, 2011) and given in Table 1. For comparison purpose, we choose the following models:

1. Modified Burr-III Distribution (MB_III)

$$G(x)=\left(1+\lambda x^{-\beta}\right)^{-\frac{\alpha}{\lambda}}$$

2. Exponential Distribution (ED)

$$G(x)=\left(1-\ x^{-\theta x}\right)$$

3. Exponential Pareto Distribution (EPD)

$$G(x)=\left(1-x^{-\lambda\left(\frac{x}{\beta}\right)^{\beta}}\right)$$

Table 1: Strengths of single carbon fibres of length 10 mm

1. 901	2. 132	2. 203	2. 228	2. 257	2. 350	2. 361	2. 396	2. 397	2. 445	2. 614	2. 616	2. 618
2. 624	2. 659	2. 675	2. 532	2. 575	2. 454	2. 454	2. 474	2. 518	2. 522	2. 525	2. 738	2. 400
2. 856	2. 917	2. 928	2. 937	2. 937	2. 977	2. 996	3. 030	3. 125	3. 139	3. 145	3. 220	3. 223
3. 235	3. 243	3. 264	3. 272	3. 294	3. 332	3. 346	3. 377	3. 408	3. 435	3. 493	3. 501	3. 537
3. 554	3. 562	3. 628	3. 852	3. 871	3. 886	3. 971	4. 024	4. 027	4. 225	4. 395	5. 020	

Table 2 shows the results of ML estimates and -2 logL.

Table 2: Estimates of parameters and the corresponding log-likelihood

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	-2 ***logL***
OMBED-III	41. 0719	2. 9348	0. 5477	3. 4815	115. 30
MBIIID	150. 831	5. 0435	--	2. 5834	119. 78
ED	--	--	0. 3285	--	270. 51
EPD	--	4. 9944	3. 3907	1. 1430	126. 62

In Table 3 goodness of fit statistics such as AIC, BIC, CAIC, HQIC, K-S statistic and p value are reported. It is observed from the results in Tables 5 that the smaller values of different goodness of fit criteria confirm that OMBED-III is the best-fitted model among other competitive models.

According to Table 3, for the dataset of strengths of single carbon fibres of length 10, Odd Modified Burr-III exponential distribution exhibits a better fit comparative to the other defined distributions, based on different goodness of fit criteria.

Table 3: Goodness of fit measures

Model	AIC	BIC	CAIC	HQIC	KS	p value
OMBED-III	123. 30	131. 94	123. 98	126. 70	0. 0945	0. 6169
MBIIID	125. 78	132. 25	126. 18	128. 33	0. 0876	0. 7079
ED	272. 51	274. 67	272. 57	273. 36	0. 4879	0
EPD	132. 62	139. 09	133. 02	135. 17	0. 0954	0. 6043

Figure 2 displays the TTT plot, Q-Q plot, estimated density plot, and empirical cumulative density plot (ECDF) on the strengths of single carbon fibres of length 10 for OMBED-III. The fitted pdf and cdf plot reveal that the proposed distribution gives a good fit to this dataset; the scaled TTT plot indicates empirical hazard function is an increasing function, and based on the Q-Q plot, we conclude that the proposed distribution provides a good fit for the given dataset.

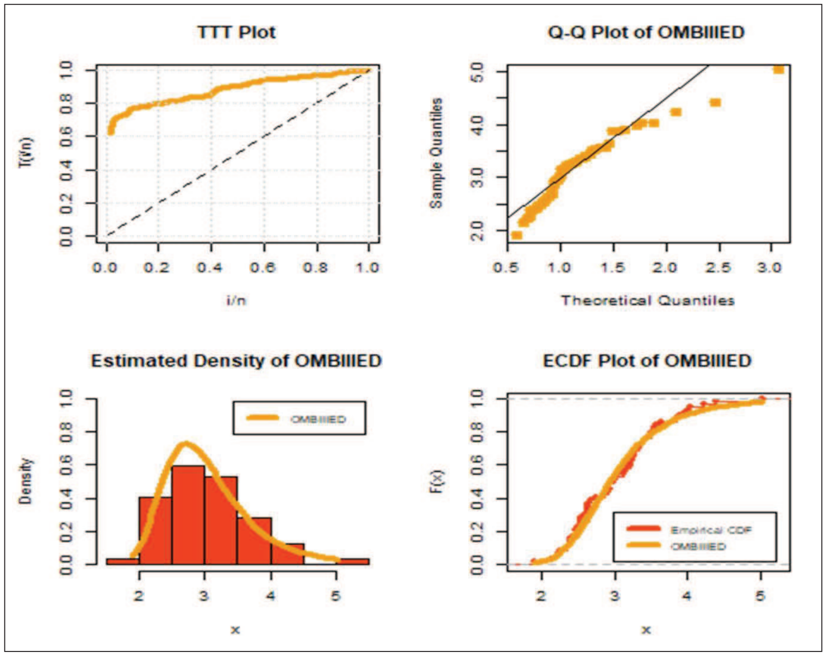

Figure 2: TTT Plot, Q-Q Plot, Estimated Density (EPDF) and ECDF plot of OMBED-III.

CONCLUSIONS

In this article, we proposed a new lifetime distribution named odd modified Burr-III exponential distribution. Statistical properties of the distribution are derived, such as moments, measure of central tendency, and measure of dispersions. The tools of reliability including hazard function, mean residual life, and Shannon and Renyi entropies are derived. Order statistics and L-moments are also included in this study. The ML method is used to estimate the parameters of the distribution. It is found that bias and MSE decrease as sample size increases for all parameters of the distribution. We empirically demonstrate the significance and versatility of the proposed model by using a real-life dataset. The OMBED-III distribution may be considered as the best fitted model on the basis of information criteria. Therefore, we recommend the use of the proposed model which can be preferred to its existing competitors. It is hoped that the proposed model will attract the practitioners.

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Appendix

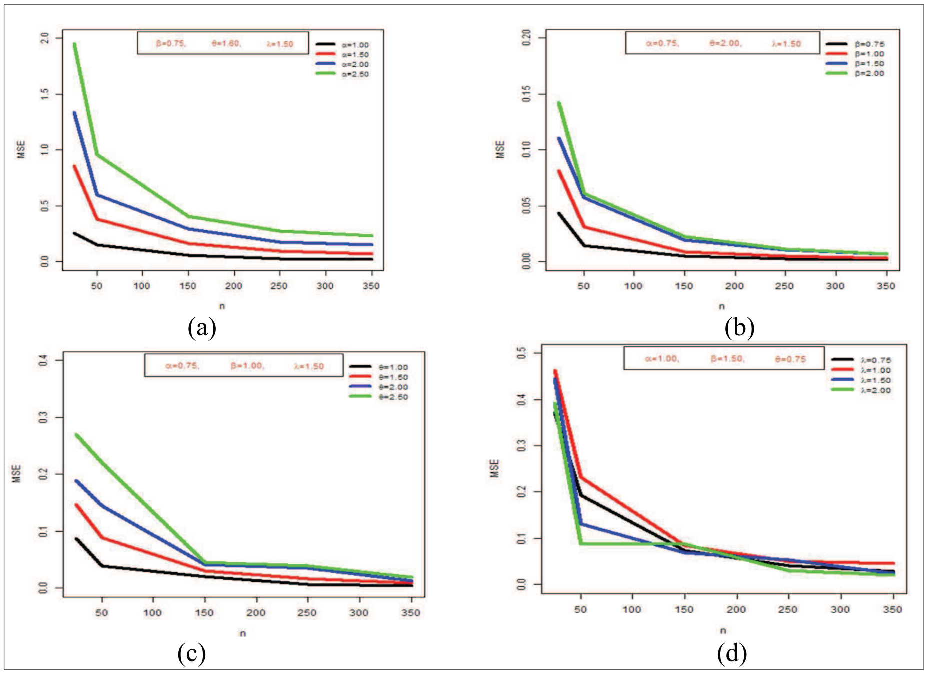

Figure A-1: MSE plot for different values of α, β, θ, λ in (a), (b), (c), (d) respectively against sample size.

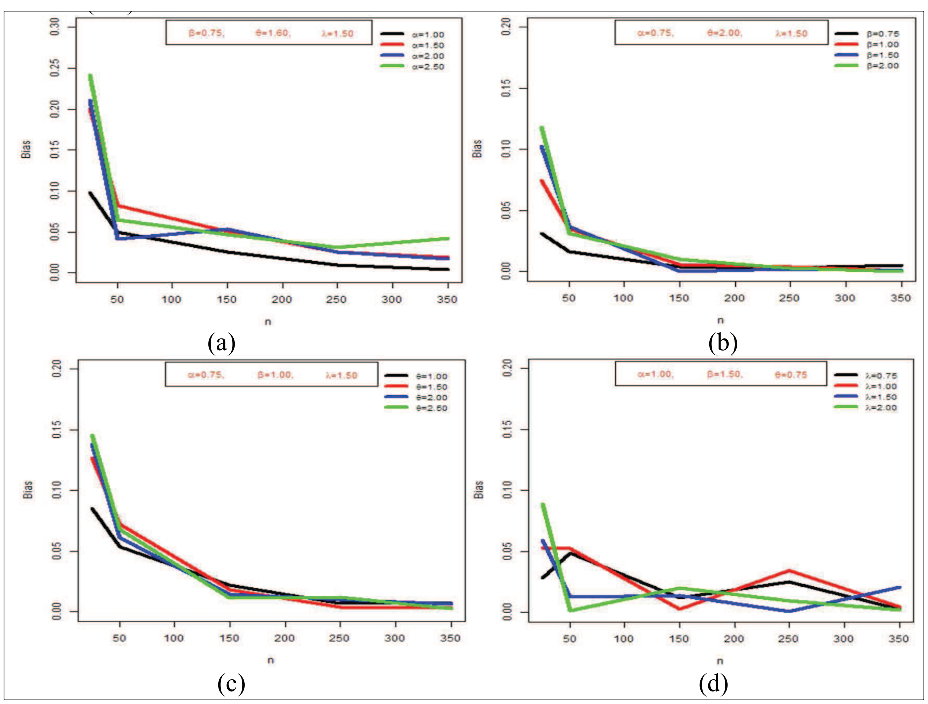

Figure A-2: Absolute bias plot for different values of α, β, θ, λ in (a), (b), (c), (d) respectively against sample size

A-3: R-code

```
==========================================================================
        Maximum Likelihood Estimation (MLE) of Parameters of OMBED-III distribution
==========================================================================
 rm(list=ls())
  library(lars)
  library(lamW)
  library(data.table)
   m = 1000       #Monte Carlo Runs
   n = 350        #sample size 25, 50, 150, 250, 350
   true.values = c(2.50,0.75,1.60,1.50)
   initial    = c(2.49,0.74,1.59,1.49)
   m11    = rep(0,m)
   m12    = rep(0,m)
   m13    = rep(0,m)
   m14    = rep(0,m)
 set.seed(10)
 for(i in 1:m){
              y = rep(0,n)
             a = true.values[1]
             b = true.values[2]
             C = true.values[3]
             d = true.values[4]
             u = runif(n)
        for(j in 1:n){        #Quantile Function
                 y[j] = (1/C)*( log(( (d/ (((u[j])^(-d/a))-1))^(1/b))+1))
                 }
        f2 = function(x){                    #Log Likelihood Function
          x1 = x[1]
          x2 = x[2]
          x3 = x[3]
          x4 = x[4]
    return( n*(log(x1)) + n*(log(x2)) + n*(log(x3)) + ((-(x1/x4)-1)*(sum(log(1+x4*(((exp(-x3*y))/(1-(exp(-
x3*y))))^x2)))))
       - (x2*x3*(sum(y)))-((x2+1)*(sum(log(1-(exp(-x3*y)))))))
              }
         g2 = function(x){   #Differential Equations of Parameter
           x1 = x[1]        #a
           x2 = x[2]        #b
           x3 = x[3]        #C
           x4 = x[4]        #d
          da = (n/x1) - ((1/x4)*(sum(log(1+x4*(((exp(-x3*y))/(1-(exp(-x3*y))))^x2))))) # eq(1) for alpha.


        db = (n/x2) - (C*(sum(y))) - (sum(log(1-(exp(-x3*y))))) +
              ((-x1/x4)-1)*(sum( ((x4*(((exp(-x3*y))/(1-(exp(-x3*y))))^x2))*(log((exp(-x3*y))/
          (1-(exp(-x3*y)))))) / (1+x4*(((exp(-x3*y))/(1-(exp(-x3*y))))^x2))))  # eq(2) for beta.
          dC = (n/x3) - ( (x2*x4*((x1/x4)+1)) * (sum(   ((((exp(-x3*y))/(1-(exp(-x3*y))))^(x2-1)))*((-y*(exp(-
x3*y)))/
          ((1-(exp(-x3*y)))^2)) / (1+x4*(exp(-x3*y) /(1-(exp(-x3*y))))^x2))))  -(x2*(sum(y)))-
          ((b+1)*(sum((y*(exp(-y*x3)))/(1-(exp(-y*x3)))))) #eq(3) for theta.
         dd = ((x1*(sum(log(1+x4*(((exp(-x3*y))/(1-(exp(-x3*y))))^x2)))))/(x4^2))+((-(x1/x4)-1)*
         (sum((((exp(-x3*y))/(1-(exp(-x3*y))))^x2)/(1+x4*(((exp(-x3*y))/(1-(exp(-x3*y))))^x2))))) #eq(4) for
lemda.
```

```
    return(c(da,db,dC,dd))
    }
  result = optim(initial,f2,g2,method="BFGS",control=list(trace=2,fnscale=-1))   #optim command
    estimates = result$par
    m11[i]  = estimates[1]
    m12[i]  = estimates[2]
    m13[i]  = estimates[3]
    m14[i]  = estimates[4]
}
  length(m11)
  length(m12)
  length(m13)
  length(m14)
              ##============= Average MLE =============##
                    round(mean(m11),4)
                    round(mean(m12),4)
                    round(mean(m13),4)
                    round(mean(m14),4)
              ##============== Average Bias =============##
                  m1  = m11- true.values[1]
                  AB1 = (1/m)*sum(m1); round(AB1,4)
                  m2  = m12- true.values[2]
                  AB2 = (1/m)*sum(m2); round(AB2,4)
                  m3  = m13- true.values[3]
                  AB3 = (1/m)*sum(m3); round(AB3,4)
                  m4  = m14- true.values[4]
                  AB4 = (1/m)*sum(m4); round(AB4,4)
              ##============== Average MSE =============##
                MSE1 = (1/m)*sum(m1^2); round(MSE1,4)
                MSE2 = (1/m)*sum(m2^2); round(MSE2,4)
                MSE3 = (1/m)*sum(m3^2); round(MSE3,4)
                MSE4 = (1/m)*sum(m4^2); round(MSE4,4)
```