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RESEARCH ARTICLE

Survey Sampling

Estimation of mean considering the joint influence of measurement errors and non-response in two-phase sampling designs

S Sabir[1], A Sanaullah[1*] and S Gupta[2]
[1] *Department of Statistics, COMSATS University Islamabad, Lahore Campus, Pakistan.*
[2] *Department of Mathematics and Statistics, The University of North Carolina at Greensboro, USA.*

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Abstract: Recently, a few studies have presented the problem of estimation of the mean for a finite homogenous population considering the joint influence of non-response and measurement errors and following the assumption that population mean of auxiliary variable is available. However, in real situations the population under observation may not be homogeneous and population mean of the auxiliary variable may also not be available. Therefore, in this study, we present new estimators which are the generalized form of the difference-cum-exponential type estimator of the finite population mean in the presence of non-response and measurement errors in two-phase simple random sampling and stratified random sampling. The expressions for mean square error and bias of the proposed estimators are shown under the first order approximation in each sampling design. Theoretically, it has been shown that the proposed estimators are more efficient than some existing estimators in both sampling designs. A numerical study has also been carried out using two different simulated populations under simple random sampling designs as well as under stratified random sampling design to support the theoretical results. Numerical results show that the proposed estimators perform more efficiently than some modified versions of the existing estimators in both sampling designs.

Keywords: Auxiliary variable, exponential estimator, mean square error, measurement error, ratio estimator, regression estimator.

INTRODUCTION

Survey errors arise when a difference is found between the true and recorded values. In practice, it is observed that researchers face the problem of measurement error and/or non-response while collecting information from the samples. For example, in surveys regarding agriculture, the households are asked to report a total area of their holdings, which may vary from the cadastral data. Many researchers have studied the problem of mean estimation considering only measurement errors like Shalabh (1997), Wang (2002), Shukla *et al*. (2012) Sharma *et al*. (2013) and Khalil *et al*. (2017). Hansen and Hurwitz (1946) considered the problem of non-response only while estimating the population mean. They recommended that the questionnaires are mailed to all the respondents included in the sample and as the deadline is passed, a list of non-respondents is prepared in order to draw a sub-sample from the list of non-respondents and the necessary information is collected by direct interview. Many researchers have discussed this issue following Hansen and Hurwitz's sub-sampling plan, including Cochran (1977), Rao (1986), Singh and Kumar (2010), Kumar and Bhougal (2011), Kumar (2015), Sanaullah *et al*. (2015; 2018), Riaz *et al*. (2014; 2016), and Saleem *et al*. (2018).

Many authors who have studied non-response have neglected the measurement errors whereas many of those who have studied measurement errors have ignored the presence of non-response. However, Azeem (2014) and Kumar (2015; 2016) considered both the presence of non-response and measurement errors simultaneously. Azeem (2014) and Kumar (2015) considered only the situation where population is homogeneous and the population mean of the auxiliary variable is available, which may not be true for many practical situations.

* Corresponding author (chaamirsanaullah@yahoo.com; https://orcid.org/0000-0001-8114-0678)

Sabir and Sanaullah (2019) examined the mean estimator introduced by Kumar (2016) and provided corrected expressions for the mean square error. Zahid and Shabbir (2018) provided estimators for mean considering the joint presence of non-response and measurement errors in stratified simple random sampling. Irfan *et al*. (2018) studied how to optimize the estimation of mean if non-response and measurement errors are jointly present in a simple random sample.

Deviating from Azeem (2014) and Kumar (2015; 2016), this study is aimed at providing some improved estimators for population mean considering the situation when the population mean of an auxiliary variable may not be readily available. Further, the proposed estimators are compared with the modified ratio, exponential and regression estimators in both simple random sampling and stratified random sampling designs.

Notations and sampling methodology:

Let $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_N\}$ be a finite population of N homogeneous units which is divided into two mutually exclusive groups say the respondent group of size $N_{(1)}$ and the non-respondent group of size $N_{(2)}=N-N_{(1)}$. Some basic notations to be used in Sections 1 and 2 are defined as given below.

n': 1st-phase sample size

n'': sub-sample \ 2nd-phase sample size to be selected from

$n''_{(1)}$: number of respondents in 2nd-phase sample

$n''_{(2)}$: number of non-respondents in 2nd-phase sample

r: size of sub-sample to be selected for interview from $n''_{(2)}$

$Y_i \setminus X_i$: True values on Y and X for the i^{th} unit

$\mu_Y \setminus \mu_X$: Population mean of Y given population mean of X

$\mu_{Y(1)}, \mu_{X(1)}$: Population means of Y and X groups of respondents

$\mu_{Y(2)}, \mu_{X(2)}$: Population means of Y and X groups of non-respondents

σ_Y^2, σ_X^2: Population variances of Y and X, respectively

$\sigma_{Y(1)}^2, \sigma_{X(1)}^2$: Population variances for Y and X groups of respondents

$\sigma_{Y(2)}^2, \sigma_{X(2)}^2$: Population variances for Y and X groups of non-respondents

$C_{Y(1)} = \frac{\sigma_{Y(1)}}{\mu_{Y(1)}}, C_{X(1)} = \frac{\sigma_{X(1)}}{\mu_{X(1)}}$: Coefficient of variations for Y and X groups of respondents

$C_{Y(2)} = \frac{\sigma_{Y(2)}}{\mu_{Y(2)}}, C_{X(2)} = \frac{\sigma_{X(2)}}{\mu_{X(2)}}$: Coefficient of variation for Y and X groups of non-respondents

$y_i \setminus x_i$: reported values on Y and X for the i^{th} unit

$U_i = y_i - Y_i$: measurement error on the study variable associated with i^{th} unit in h^{th} stratum

$V_i = x_i - X_i$: measurement error on the auxiliary variable associated with i^{th} unit in h^{th} stratum

$U_i^* = y_i^* - Y_i^*$: measurement error and non-response on Y associated with i^{th} unit in h^{th} stratum

$V_i^* = x_i^* - X_i^*$: measurement error and non-response on X associated with i^{th} unit in h^{th} stratum

$\sigma_{U(2)}^2$ and $\sigma_{V(2)}^2$:Population variances of U and V groups of non-respondents

$\rho_{YX(1)}$ and $\rho_{YX(2)}$: Coefficients of correlation between the study and auxiliary variables

μ_y' and μ_x': Sample mean estimator for y and x based on units

$\mu''_{y(1)}, \mu''_{x(1)}$: sample mean estimators for y and x based on units

$\mu''_{y(2)r}, \mu''_{x(2)r}$: sample mean estimators for y and x based on units

μ''^*_y, μ''^*_x: sample mean estimators for y and x following Hansen and Hurwitz (1946)

The measurement errors are assumed to have means zero and the variances $\sigma_{u(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(U_i - \bar{U}_{(2)})^2$ and $\sigma_{v(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(V_i - \bar{V}_{(2)})^2$ for the non-respondent group. Let $\mu_{Y(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1} Y_i$ and $\mu_{X(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1} X_i$ be the means and, $\sigma_{Y(1)}^2 = \frac{1}{N_1-1}\sum_{i=1}^{N_1}(Y_i - \mu_{Y(1)})^2$ and $\sigma_{X(1)}^2 = \frac{1}{N_1-1}\sum_{i=1}^{N_1}(X_i - \mu_{X(1)})^2$ be the variances of the study and the auxiliary variables from the respondent group. Similarly, let $\mu_{Y(2)} = \frac{1}{N_2}\sum_{i=1}^{N_2} Y_i$ and $\mu_{X(2)} = \frac{1}{N_2}\sum_{i=1}^{N_2} X_i$ be the means and $\sigma_{Y(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(Y_i - \mu_{Y(2)})^2$ and $\sigma_{X(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(X_i - \mu_{X(2)})^2$ be the variances for the non-respondent group. Let $\rho_{XY(1)} = \frac{\sigma_{XY(1)}}{\sigma_{X(1)}\sigma_{Y(1)}}$ and

$\rho_{XY(2)} = \frac{\sigma_{XY(2)}}{\sigma_{X(2)}\sigma_{Y(2)}}$ be the coefficients of correlation between X and Y in the populations of respondent and non-respondent groups, respectively.

In the presence of non-response and measurement errors, we can define the population mean by

$\mu_Y = W_1\mu_{Y(1)} + W_2\mu_{Y(2)}$, where $W_1 = \frac{N_1}{N}$ and $W_2 = \frac{N_2}{N}$.

At first phase, we select a large sample of size $n^{'}$ $(n^{'} < N)$ by simple random sampling without replacement (*SRSWOR*) to estimate μ_X and a sub-sample of size is taken on 2nd phase by *SRSWOR* to observe the study variable *Y*. It is assumed that $n^{''}_{(1)}$ of these units respond and $n^{''}_{(2)} = (n^{''} - n^{''}_{(1)})$ do not. Let $r\left(= \frac{n^{''}_{(2)}}{k}; k > 1\right)$ be a sub-sample of the individuals who do not respond to the mailed questionnaire but respond when they are contacted again for their personal interviews, where *k* is the inverse sampling ratio. It is further assumed that all units respond while interviewing them for the study variable *y*.

Now, the unbiased sample estimators of μ_Y following the first-phase and second-phase samples, respectively are

$\mu^{'}_y = \frac{1}{n^{'}_1}\sum_{i=1}^{n^{'}_1} y_i$,

and $t_0 = \mu^{*''}_y = \frac{n^{''}_1}{n^{''}}\mu^{''}_{y(1)} + \frac{n^{''}_2}{n^{''}}\mu^{''}_{y(2)r}$, where

$\mu^{''}_{y(1)} = \frac{1}{n^{''}_1}\sum_{i=1}^{n^{''}_1} y_i$ and $\mu^{''}_{y(2)r} = \frac{1}{r}\sum_{i=1}^{r} y^*_i$,

with their variances given respectively by

$\mathrm{Var}(\mu^{'}_y) = \lambda^{'}\mu_Y^2 C_Y^2$, where $\lambda^{'} = \frac{1}{n^{'}} - \frac{1}{N}$ and

$\mathrm{Var}\,(t_0) = \mathrm{Var}(\mu^{*''}_y) = \lambda^{''}\mu_Y^2\left(C_Y^2 + \frac{\sigma_U^2}{\mu_Y^2}\right) + \theta^{''}\mu_Y^2\left(C_{Y(2)}^2 + \frac{\sigma_{U(2)}^2}{\mu_Y^2}\right)$,

where $\lambda^{''} = \frac{1}{n^{''}} - \frac{1}{N}$ and $\theta^{''} = \frac{W_2(r-1)}{n^{''}}$.

Similarly, let $\mu^{''}_{x(1)} = \frac{1}{n^{''}_1}\sum_{i=1}^{n^{''}_1} x_i$ and $\mu^{''}_{x(2)r} = \frac{1}{r}\sum_{i=1}^{r} x^*_i$ denote the means of responding units and the *r* sub-sampled units for the auxiliary variable.

We modify the usual ratio estimator, usual exponential ratio estimator and usual regression estimator to cover the case of non-response and measurement errors in two-phase sampling. The modified estimators are given respectively by,

$$t_1 = \frac{\mu^{*''}_y}{\mu^{*''}_x}\,\mu^{'}_x, \qquad \text{...(1)}$$

$$t_2 = \mu^{*''}_y \exp\left(\frac{\mu^{'}_x - \mu^{*''}_x}{\mu^{'}_x + \mu^{*''}_x}\right), \qquad \text{...(2)}$$

and

$$t_3 = \mu^{*''}_y + b^*(\mu^{'}_x - \mu^{*''}_x), \qquad \text{...(3)}$$

with the Mean Square Errors (*MSE)* of t_1, t_2 and t_3 respectively, upto the first order approximation, given by

$$MSE(t_1) = \lambda^{''}\mu_Y^2\left(\left(C_Y^2 + C_X^2 - 2\rho_{yx}C_YC_X\right)\left(\frac{\sigma_u^2}{\mu_Y^2} + \frac{\sigma_v^2}{\mu_X^2}\right)\right)$$

$$+\theta^{''}\mu_Y^2\left(\left(C_{Y(2)}^2 + C_{X(2)}^2 - 2\rho_{yx(2)}C_{Y(2)}C_{X(2)}\right)\right.$$

$$\left.+\left(\frac{\sigma_{u(2)}^2}{\mu_Y^2} + \frac{\sigma_{v(2)}^2}{\mu_X^2}\right)\right) - \lambda^{'}\mu_Y^2\left(2\rho_{yx}C_YC_X - C_X^2\right),$$

$$\text{...(4)}$$

$$MSE(t_2) = \lambda^{''}\mu_Y^2\left(\left(C_Y^2 + \frac{C_X^2}{4} - \rho_{yx}C_YC_X\right) + \left(\frac{\sigma_u^2}{\mu_Y^2} + \frac{\sigma_v^2}{4\,\mu_X^2}\right)\right)$$

$$+\theta^{''}\mu_Y^2\left(\left(C_{Y(2)}^2 + \frac{C_{X(2)}^2}{4} - \rho_{yx(2)}C_{Y(2)}C_{X(2)}\right)\right.$$

$$\left.+\left(\frac{\sigma_{u(2)}^2}{\mu_Y^2} + \frac{\sigma_{v(2)}^2}{4\,\mu_X^2}\right)\right) - \lambda^{'}\mu_Y^2\left(\rho_{yx}C_YC_X - \frac{C_X^2}{4}\right),$$

$$\text{...(5)}$$

$$MSE(t_3) = \mu_Y^2\left(\lambda^{''}\left(C_Y^2 + \frac{\sigma_u^2}{\mu_Y^2}\right) + \theta^{''}\left(C_{Y(2)}^2 + \frac{\sigma_{u(2)}^2}{\mu_Y^2}\right)\left\{1 - \vartheta^{''}_0\right\}\right),$$

$$\text{...(6)}$$

where

$$\vartheta^{''}_0 = \frac{\left(\lambda^{''}\rho_{YX}\sigma_Y\sigma_X + \theta^{''}\rho_{YX(2)}\sigma_{Y(2)}\sigma_{X(2)} - \lambda^{'}\rho_{yx}C_YC_X\right)^2}{\lambda^{''}\left(C_Y^2 + C_X^2 + \left(\frac{\sigma_u^2}{\mu_Y^2} + \frac{\sigma_v^2}{\mu_X^2}\right)\right) + \theta^{''}\left(C_{Y(2)}^2 + C_{X(2)}^2 + \left(\frac{\sigma_{u(2)}^2}{\mu_Y^2} + \frac{\sigma_{v(2)}^2}{\mu_X^2}\right)\right) - \lambda^{'}C_X^2}$$

The expression of MSE() is obtained iff

$$b^* = \frac{\lambda^{''}\rho_{YX}\sigma_Y\sigma_X + \theta^{''}\rho_{YX(2)}\sigma_{Y(2)}\sigma_{X(2)}}{\mathrm{R}\left(\lambda^{''}\left(C_X^2 + \frac{\sigma_v^2}{\mu_X^2}\right) + \theta^{''}\left(C_{X(2)}^2 + \frac{\sigma_{v(2)}^2}{\mu_X^2}\right)\right)}, \qquad \text{and} \quad R = \frac{\mu_Y}{\mu_X}.$$

The proposed improved generalized class of estimator (*IGCE*):

In survey sampling, the surveys often yield biased estimates because of the non-response and/or the measurement errors. Many estimators have been used to deal with the non-response and the measurement errors independently. In this section, we propose an *IGCE* considering the presence of both the non-response and the measurement errors together on the study variable as well as on the auxiliary variable. We are considering the situation when the population mean of the auxiliary variable is not available.

The proposed *IGCE* in two-phase sampling considering the non-response and measurement errors is given by

$$t_{4(a)} = \left(H + \omega_1\left(\mu_x^{'} - \mu_x^{*''}\right) + \omega_2\mu_y^{*''}\right)\exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right), \quad ...(7)$$

where $H = \frac{\mu_y^{*''}}{2}\left(exp\left(\frac{\mu_x^{'(\frac{1}{a})} - \mu_x^{*''(\frac{1}{a})}}{\mu_x^{'(\frac{1}{a})} + \mu_x^{*''(\frac{1}{a})}}\right) + exp\left(\frac{\mu_x^{*''(\frac{1}{a})} - \mu_x^{'(\frac{1}{a})}}{\mu_x^{*''(\frac{1}{a})} + \mu_x^{'(\frac{1}{a})}}\right)\right)$,

ω_1 and ω_2 are constants which need to be optimized to get minimum mean square error of $t_{4(a)}$, and a is a generalizing constant to get different estimators under the a^{th} root transformation.

For example, for a=1, the proposed estimator $t_{4(a)}$ is reduced to the form

$$t_{4(1)} = \left(\frac{\mu_y^{*''}}{2}\left(\exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right) + \exp\left(\frac{\mu_x^{*''} - \mu_x^{'}}{\mu_x^{*''} + \mu_x^{'}}\right)\right) + \omega_1^{opt(1)}\left(\mu_x^{'} - \mu_x^{*''}\right) + \omega_2^{opt(1)}\mu_y^{*''}\right)\exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right),$$

and for a=2, the proposed estimator is reduced to the form

$$t_{4(2)} = \left(\frac{\mu_y^{*''}}{2}\left(\exp\left(\frac{\mu_x^{'(1/2)} - \mu_x^{*''(1/2)}}{\mu_x^{'(1/2)} + \mu_x^{*''(1/2)}}\right) + \exp\left(\frac{\mu_x^{*''(1/2)} - \mu_x^{'(1/2)}}{\mu_x^{*''(1/2)} + \mu_x^{'(1/2)}}\right)\right) + \omega_1^{opt(2)}\left(\mu_x^{'} - \mu_x^{*''}\right) + \omega_2^{opt(2)}\mu_y^{*''}\right)\exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right).$$

The bias and *MSE* of *IGCE* $t_{4(a)}$

In order to obtain the expressions for the bias and *MSE* of $t_{4(a)}$ let us consider,

$W_X^* = \sum_{i=1}^{n''}\left(x_i^{*''} - \mu_X\right)$, $W_U^* = \sum_{i=1}^{n''}U_i^*$, $W_V^* = \sum_{i=1}^{n''}V_i^*$,

and $W_X = \sum_{i=1}^{n'}\left(x_i^{'} - \mu_X\right)$. ...(8)

Now from (8), the errors due to sampling in the presence of non-response and measurement errors may be written as

$e_0^{*''} = \frac{1}{n''\mu_Y}\left(W_Y^* + W_U^*\right)$, $e_1^{*''} = \frac{1}{n''\mu_X}\left(W_X^* + W_V^*\right)$, and $e_1^{'} = \frac{1}{n'\mu_X}\left(x_i^{'} - \mu_X\right)$, or alternatively $\mu_y^{*''} = \mu_Y(1 + e_0^{*''})$, $\mu_x^{*''} = \mu_X(1 + e_1^{*''})$, and $\mu_x^{'} = \mu_X(1 + e_1^{'})$,

such that $E(e_0^{*''}) = E(e_1^{*''}) = E(e_1^{'}) = 0$, and

$E(e_0^{*''})^2 = \left(\lambda''\left(C_Y^2 + \frac{\sigma_u^2}{\mu_Y^2}\right) + \theta''\left(C_{Y(2)}^2 + \frac{\sigma_{u(2)}^2}{\mu_Y^2}\right)\right) = V_{02}^{''}$,

$E(e_1^{*''})^2 = \left(\lambda''\left(C_X^2 + \frac{\sigma_v^2}{\mu_X^2}\right) + \theta''\left(C_{X(2)}^2 + \frac{\sigma_{v(2)}^2}{\mu_X^2}\right)\right) = V_{20}^{''}$,

$E(e_0^{*''}e_1^{*''}) = \left(\lambda''\rho_{YX}C_YC_X + \theta''\rho_{YX(2)}C_{Y(2)}C_{X(2)}\right) = V_{11}^{''}$,

$E(e_1^{'})^2 = E(e_1^{*''}e_1^{'}) = \lambda' C_X^2 = V_{20}^{'}$, and

$E(e_0^{*''}e_1^{'}) = \lambda'\rho_{YX}C_YC_X = V_{11}^{'}$.

Now $t_{4(a)}$ can be rewritten as below:

$$t_{4(a)} = \left(\frac{\mu_Y(1+e_0^{*''})}{2}\left(exp\left(\frac{\mu_X^{(\frac{1}{a})}(1+e_1^{'})^{(\frac{1}{a})} - \mu_X^{(\frac{1}{a})}(1+e_1^{*''})^{(\frac{1}{a})}}{\mu_X^{(\frac{1}{a})}(1+e_1^{'})^{(\frac{1}{a})} + \mu_X^{(\frac{1}{a})}(1+e_1^{*''})^{(\frac{1}{a})}}\right) + exp\left(\frac{\mu_X^{(\frac{1}{a})}(1+e_1^{*''})^{(\frac{1}{a})} - \mu_X^{(\frac{1}{a})}(1+e_1^{'})^{(\frac{1}{a})}}{\mu_X^{(\frac{1}{a})}(1+e_1^{*''})^{(\frac{1}{a})} + \mu_X^{(\frac{1}{a})}(1+e_1^{'})^{(\frac{1}{a})}}\right)\right) + \omega_1\left(\mu_X - \mu_X(1+e_1^{'})\right) + \omega_2\mu_Y\left(1+e_0^{*''}\right)\right)$$

$$exp\left(\frac{\mu_X(1+e_1^{'}) - \mu_X(1+e_1^{*''})}{\mu_X(1+e_1^{'}) + \mu_X(1+e_1^{*''})}\right). \quad ...(9)$$

Simplifying the above expression up to the order $O(n^{-1})$ we can write (9) as

$$t_{4(a)} - \mu_Y \cong \mu_Y \left(e_0^{*''} + \frac{e_1^{'}}{2} - \frac{e_1^{*''}}{2} - \frac{e_1^{'2}}{8} + \frac{3e_1^{*''2}}{8} - \frac{e_1^{'} e_1^{*''}}{4} + \right.$$

$$\frac{e_1^{'} e_0^{*''}}{2} - \frac{e_1^{*''} e_0^{*''}}{2} + \frac{e_1^{*''2}}{8a^2} + \frac{e_1^{'2}}{8a^2} - \frac{e_1^{'} e_1^{*''}}{4a^2} + w_2 e_0^{*''} + \omega_2 \frac{e_1^{'}}{2}$$

$$w_2 \frac{e_1^{*''}}{2} - w_2 \frac{e_1^{'2}}{8} + \frac{3 w_2 e_1^{*''2}}{8} - w_2 \frac{e_1^{'} e_1^{*''}}{4} + w_2 \frac{e_1^{'} e_0^{*''}}{2} -$$

$$\left. w_2 \frac{e_1^{*''} e_0^{*''}}{2} \right) + \omega_1 \mu_X \left(e_1^{'} - e_1^{*''} + \frac{e_1^{'2}}{2} + \frac{e_1^{*''2}}{2} - e_1^{'} e_1^{*''} \right).$$

...(10)

The expressions for the bias and *MSE* of $t_{4(a)}$, up to the order $O(n^{-1})$, are given respectively by

$$\text{Bias}(t_{4(a)}) \cong \mu_Y \left(-\frac{V_{20}^{'}}{8} + \frac{3V_{20}^{''}}{8} - \frac{V_{20}^{'}}{4} + \frac{V_{11}^{'}}{2} - \frac{V_{11}^{''}}{2} + \frac{V_{20}^{''}}{8a^2} \right.$$

$$+ \frac{V_{20}^{'}}{8a^2} - \frac{V_{20}^{'}}{4a^2} + \omega_2 - \omega_2 \frac{V_{20}^{'}}{8} + \frac{3 w_2 V_{20}^{''}}{8} - \omega_2 \frac{V_{20}^{'}}{4}$$

$$\left. + \omega_2 \frac{V_{11}^{'}}{2} - \omega_2 \frac{V_{11}^{''}}{2} \right) + \omega_1 \mu_X \left(\frac{V_{20}^{'}}{2} + \frac{V_{20}^{''}}{2} - V_{20}^{'} \right),$$

...(11)

and

$$MSE(t_{4(a)}) \cong \mu_Y^2 \left(V_{02}^{''} - y_{11} + \frac{1}{4} y_{20} + \omega_2^2 (1 + V_{02}^{''} \right.$$

$$- 2y_{11} + y_{20}) + \omega_2 (2V_{02}^{''} - 3y_{11} +$$

$$\left. \frac{5}{4} y_{20} + \frac{1}{4a^2} y_{20}) \right) + \omega_1^2 \mu_X^2 y_{20}$$

$$+ 2\mu_Y \mu_X \omega_1 \begin{pmatrix} \omega_2 (y_{20} - y_{11}) - y_{11} \\ + \frac{1}{2} y_{20} \end{pmatrix},$$

...(12)

where, $y_{20} = V_{20}^{''} - V_{20}^{'}$, and $y_{11} = V_{11}^{''} - V_{11}^{'}$.

Or alternatively, the MSE of $t_{4(a)}$ can also be given by

$$MSE(t_{4(a)}) \cong \Psi_0 + \omega_2^2 \Psi_1 + \omega_2 \Psi_2 + \omega_1^2 \Psi_3 + \omega_1 \omega_2 \Psi_4 + \omega_1 \Psi_5,$$

...(13)

where,

$\Psi_0 = \mu_Y^2 \left(V_{02}^{''} + \frac{1}{4} V_{20} - y_{11} \right),$

$\Psi_1 = \mu_Y^2 (1 + V_{02}^{''} + y_{20} - 2y_{11}),$

$\Psi_2 = \mu_Y^2 \left(2V_{02}^{''} - 3y_{11} + \frac{5}{4} y_{20} + \frac{1}{4a^2} y_{20} \right),$

$\Psi_3 = \mu_X^2 y_{20},$

$\Psi_4 = -2\mu_Y \mu_X (y_{11} - y_{20}),$ and

$\Psi_5 = -2\mu_Y \mu_X \left(y_{11} - \frac{1}{2} y_{20} \right).$

Now, to get the optimum values, (13) is differentiated partially with respect to ω_1 and ω_2 and then equating each of the first derivatives with zero. This gives two normal equations which are then solved simultaneously to get the optimum values. Finally, the optimum values are shown by,

$$\omega_1^{opt} = \frac{\Psi_2 \Psi_4 - 2 \Psi_1 \Psi_5}{4 \Psi_1 \Psi_3 - \Psi_4^2} \text{ and } \omega_2^{opt} = \frac{\Psi_4 \Psi_5 - 2 \Psi_2 \Psi_3}{4 \Psi_1 \Psi_3 - \Psi_4^2}. \quad ...(14)$$

Substituting the optimum values ω_1^{opt} and ω_2^{opt} in (13), the expression of the minimum $MSE(t_{4(a)})$ may be obtained as

$$\min MSE\left(t_{4(a)}\right) = \Psi_0 - \frac{(\Psi_2 \Psi_4 \Psi_5 - \Psi_2^2 \Psi_3 - \Psi_1 \Psi_5^2)}{\Psi_4^2 - 4 \Psi_1 \Psi_3}. \quad ...(15)$$

Theoretical comparisons:

In this section, the proposed estimator IGCE is compared for its efficiency based on the *MSE* criterion under two-phase sampling with ratio estimator and exponential estimator.

i) The proposed estimator IGCE is more efficient than ratio estimator if $MSE(t_1) - MSE(t_{4(a)}) > 0$, and this gives

$$\frac{1}{V_{02}^{''} y_{20}} - 2 \frac{y_{11}}{V_{02}^{''} y_{20}} > \frac{1}{y_{20}} - \frac{MSE(t_{4(a)})}{y_{20} Var\, t_0},$$

or alternatively, $\varphi_0 > \frac{\varphi_2}{2} + \varphi_2 \frac{MSE(t_{4(a)})}{2Var\, t_0} - \frac{1}{2},$

where $\varphi_0 = \frac{y_{11}}{V_{02}^{''} y_{20}},$ $\varphi_1 = \frac{1}{V_{02}^{''} y_{20}},$ and $\varphi_2 = \frac{1}{y_{20}}.$

ii) The proposed estimator IGCE is more efficient than exponential ratio estimator if
$MSE(t_2) - MSE(t_{4(a)}) > 0$, and this gives

$$\frac{1}{4V_{02}^{''}y_{20}} - \frac{y_{11}}{V_{02}^{''}y_{20}} > \frac{1}{y_{20}} - \frac{MSE(t_{4(a)})}{y_{20}Var\ t_0},$$

or alternatively,

$$\varphi_0 > \frac{\varphi_3}{4} + \varphi_2 \frac{MSE(t_{4(a)})}{2Var\ t_0} - \varphi_2, \quad \text{where} \quad \varphi_3 = \frac{1}{V_{02}^{''}y_{20}}.$$

iii) The proposed estimator IGCE is more efficient than regression estimator if $MSE(t_3) - MSE(t_{4(a)}) > 0$, and this gives

$$\varphi_0 > 1 - \frac{MSE(t_{4(a)})}{Var\ t_0}.$$

RESULTS AND DISCUSSION

In this section, we examine the performance of the proposed estimators in comparison to the modified estimators. We have generated two populations by using normal, and lognormal distributions with different parameter values using R statistical software. The description of each population is given below.

Population I: [normal population]

X=N(5000, 5, 15); *z*=N(5000, 0, 1); *Y*=50*X*+15*z*; *y*=*Y*+N(1, 3); *x*=*X*+N(1, 3); μ_Y = 248.67; μ_X = 4.97; σ_y^2=583685.70; σ_x^2=242.30;σ_U^2=9.25; σ_V^2= 9.25; ρ_{yx}= 0.98; *N*=5000; N_1=3500; N_2=1500;n'=500; $\sigma_{y(2)}^2$=599476.60; $\sigma_{x(2)}^2$ =239.94; $\sigma_{U(2)}^2$ =9.14; $\sigma_{V(2)}^2$ =8.81; $\rho_{yx(2)}$=0.99;

Population II: [lognormal population]

X=logN(5000, log(4), log(6)); *z*=N(5000, 0, 1); *Y*=50*X*+15*z*; *y* =*Y*+N(1, 3); *x*= *X*(1, 3); μ_Y= 703.27; μ_X= 14.07; σ_y^2= 5871164.00; σ_x^2= 2363.44; σ_U^2= 9.03; σ_V^2= 8.69; ρ_{yx}= 0.99; *N*=5000; N_1=3500; N_2=1500; n'=500; $\sigma_{y(2)}^2$= 9035398.00; $\sigma_{x(2)}^2$ = 3615.41; $\sigma_{U(2)}^2$= 8.72; $\sigma_{V(2)}^2$= 8.85; $\rho_{yx(2)}$ = 0.99;

The *MSEs* and the bias of proposed and some available estimators are presented in Tables A1-A4 in appendix A.

The bold numbers in Tables A1 and A2, represents the lowest two $MSE\ t_{4(a)}$ as compared to the *MSE*,s of t_0, t_1, t_2 and t_3. From Tables A1 and A2, one can observe that the proposed class of estimators provides estimators which have less *MSE* values than the *MSE's* of all of the modified estimators. We observe that the best performance is for *a*=1. However, the performance of the modified estimator for *a*=2 still shows smaller *MSE* than the *MSE*s of all other estimators. Tables A3 and A4 display the absolute relative bias. It can be observed that the proposed estimators are slightly more biased than the existing estimators, but the biases are not notable. Therefore, still the proposed estimators can be preferred over other existing estimators.

The proposed improved generalized class of estimator in stratified two-phase sampling (*IGCEStr*)

Let a population be divided into *L* homogenous strata with N_h units (*h*=1,2, …, *L*) such that $\sum_{h=1}^{L} N_h$=*N.* At first-phase, a large sample of size $n_h^{'}$ is selected from *h*th stratum such that $\sum_{h=1}^{L} n_h^{'} = n^{'}$ and a sub-sample of size $n_h^{''}(n_h^{''} < n_h^{'})$ from each stratum is taken on 2nd phase such that $\sum_{h=1}^{L} n_h^{''} = n^{''}$ by *SRSWOR*. Let (y_{hi}, x_{hi}) be the observed values instead of true values (Y_{hi}, X_{hi}) of the two characteristic (Y, X), respectively associated with *i*th unit of *h*th stratum, where *i*=1,2, …, N_h and *h*=1,2, …, *L*. It is assumed that only $n_{h(1)}^{''}$ units respond and $n_{h(2)}^{''}$=$(n_h^{''} - n_{h(1)}^{''})$ do not. Following Hansen and Hurwitz (1946), another sub-sample of size r_h $\left(r_h = \frac{n_{h(2)}^{''}}{k_h}; k_h > 1\right)$ is taken from $n_{h(2)}^{''}$ $(r_h < n_{h(2)}^{''})$ and the subjects are interviewed. It is further assumed that all r_h units respond while interviewing them for the study variable *Y*.

In addition, let $\mu_Y = \sum_{i=1}^{L} P_h \mu_{Yh}$ and $\mu_X = \sum_{i=1}^{L} P_h \mu_{Xh}$, where $\mu_{Yh} = \frac{1}{N_h}\sum_{i=1}^{N_h} y_{hi}$, $\mu_{Xh} = \frac{1}{N_h}\sum_{i=1}^{N_h} x_{hi}$, are the population means of the study and the auxiliary variables respectively and $P_h = N_h/N$ is the weight of *h*th stratum in the presence of non-response on the study variable as well as the auxiliary variable in *h*th stratum. Let $(x_{hi}^*,\ y_{hi}^*)$ be the observed values and $(X_{hi}^*,\ Y_{hi}^*)$ be the true values for the two characteristics (x_{hi}, y_{hi}) respectively associated with *i*th (*i*=1,2, …,) sample unit of *h*th stratum. The measurement error in the presence of non-response for *h*th stratum may be taken as $U^*_{hi} = y_{hi} - Y_{hi}$ and $V^*_{hi} = x_{hi} - X_{hi}$ with mean zero and variances σ^2_{Uh} and σ^2_{Vh}, respectively. Let $(\mu_{Yh(1)}, \mu_{Xh(1)})$ and $(\sigma^2_{Yh(1)}, \sigma^2_{Xh(1)})$ respectively be the means and variances of the study and the auxiliary variables in h^{th} stratum for respondent group. Similarly, let $(\mu_{Yh(2)}, \mu_{Xh(2)})$ and $(\sigma^2_{Yh(2)}, \sigma^2_{Xh(2)})$ respectively be the means and variances for non-respondent group. Further, let $(C_{Yh(1)}, C_{Xh(1)})$ and $(C_{Yh(2)}, C_{Xh(2)})$ respectively be the coefficient of variations for the respondent and the non-respondent group respectively. Let $\rho_{xyh(1)}$ and $\rho_{xyh(2)}$ be the correlations from the populations of respondents and non-respondents, respectively.

Now, usual unbiased mean estimators assuming non-response and measurement error in stratified first-phase and stratified two-phase sampling are defined respectively by

$$\mu'_{y(st)} = \sum_{h=1}^{L} P_h \mu'_{yh}, \text{ and } t_{01} = \mu^{*''}_{y(st)} = \frac{n''_{h(1)}}{n''_h}\mu''_{yh(1)} + \frac{n''_{h(2)}}{n''_h}\mu''_{y(2)k_h},$$

where $\mu'_{yh} = \frac{1}{n'_h}\sum_{i=1}^{n'_h} y_{hi}$, $\mu''_{yh(1)} = \frac{1}{n''_{h(1)}}\sum_{i=1}^{n''_{h(1)}} y_{hi}$ and

$$\mu''_{y(2)r_h} = \frac{1}{r_h}\sum_{i=1}^{r_h} y''^{*}_{hi}.$$

The expressions for the variances of $\mu^{*'}_{y(st)}$ and $\mu^{*''}_{y(st)}$ are given respectively by

$$\text{var}(\mu^{*'}_{y(st)}) = \mu_Y^2 \sum_{h=1}^{L} P_h^2 \lambda'_h C^2_{Yh},$$

and

$$\text{var}(t_{0(st)}) = \text{var}(\mu^{*''}_{y(st)}) = \mu_Y^2 \sum_{h=1}^{L} P_h^2 \lambda''_h \left(C^2_{Yh} + \frac{\sigma^2_{uh}}{\mu_Y^2}\right) + \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(C^2_{Yh(2)} + \frac{\sigma^2_{uh(2)}}{\mu_Y^2}\right),$$

where $\lambda''_h = \left(\frac{1}{n''_h} - \frac{1}{N_h}\right)$, $\lambda'_h = \left(\frac{1}{n'_h} - \frac{1}{N_h}\right)$,

$$\theta''_h = \frac{W_{h(2)}(k_h - 1)}{n''_h}, \text{ and } W_{h(2)} = \frac{N_{h(2)}}{N_h}.$$

Similarly, let $\mu''_{xh(1)} = \frac{1}{n''_{h(1)}}\sum_{i=1}^{n''_{h(1)}} x_{ih}$ and $\mu''_{x(2)r_h} = \frac{1}{r_h}\sum_{i=1}^{r_h} x^{*''}_{ih}$ denote the means of the responding units, and r_h sub-sampled units for the auxiliary variable.

We modify the usual ratio, exponential ratio and regression estimators to the case of non-response and measurement error in two-phase sampling respectively as

$$t_5 = \frac{\mu^{*''}_{y(st)}}{\mu^{*''}_{x(st)}}\mu'_{x(st)}, \qquad \text{...(16)}$$

$$t_6 = \mu^{*''}_{y(st)} \exp\left(\frac{\mu'_{x(st)} - \mu^{*''}_{x(st)}}{\mu'_{x(st)} + \mu^{*''}_{x(st)}}\right), \qquad \text{...(17)}$$

and

$$t_7 = \left(\mu^{*''}_{y(st)} + b^{**}(\mu'_{x(st)} - \mu^{*''}_{x(st)})\right). \qquad \text{...(18)}$$

The *MSEs* of t_5, t_6 and t_7, respectively are given by,

$$MSE(t_5) = \lambda''_h \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\left(C^2_{Yh} + C^2_{Xh} - 2\rho_{yxh}C_{Yh}C_{Xh}\right) + \left(\frac{\sigma^2_{uh}}{\mu_Y^2} + \frac{\sigma^2_{vh}}{\mu_X^2}\right)\right) + \theta''_h \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\left(C^2_{Yh(2)} + C^2_{Xh(2)} - 2\rho_{yxh(2)}C_{Yh(2)}C_{Xh(2)}\right) + \left(\frac{\sigma^2_{uh(2)}}{\mu_Y^2} + \frac{\sigma^2_{vh(2)}}{\mu_X^2}\right)\right) - \lambda'_h \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(2\rho_{yxh}C_{Yh}C_{Xh} - C^2_{Xh}\right), \qquad \text{...(19)}$$

$$MSE(t_6) = \lambda''_h \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\left(C^2_{Yh} + \frac{C^2_{Xh}}{4} - \rho_{yxh}C_{Yh}C_{Xh}\right) + \left(\frac{\sigma^2_{uh}}{\mu_Y^2} + \frac{\sigma^2_{vh}}{4\mu_X^2}\right)\right) + \theta''_h \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\left(C^2_{Yh(2)} + \frac{C^2_{Xh(2)}}{4} - \rho_{yxh(2)}C_{Yh(2)}C_{Xh(2)}\right) + \left(\frac{\sigma^2_{uh(2)}}{\mu_Y^2} + \frac{\sigma^2_{vh(2)}}{4\mu_X^2}\right)\right) - \lambda'_h \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\rho_{yxh}C_{Yh}C_{Xh} - \frac{C^2_{Xh}}{4}\right), \qquad \text{...(20)}$$

and

$$MSE(t_7) = \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\lambda''_h \left(C^2_{Yh} + \frac{\sigma^2_{uh}}{\mu_Y^2}\right) + \theta''_h \left(C^2_{Yh(2)} + \frac{\sigma^2_{uh(2)}}{\mu_Y^2}\right)\{1 - \vartheta''_0\}\right), \qquad \text{...(21)}$$

where

$$\vartheta_0^{''}=\frac{\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}\rho_{YXh}\sigma_{Yh}\sigma_{Xh}+\theta_h^{''}\rho_{YXh(2)}\sigma_{Yh(2)}\sigma_{Xh(2)}-\lambda_h^{'}\rho_{yxh}C_{Yh}C_{Xh}\right)^2}{\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}\left(C_{Yh}^2+C_{Xh}^2+\left(\frac{\sigma_{uh(2)}^2}{\mu_Y^2}+\frac{\sigma_{vh(2)}^2}{\mu_X^2}\right)\right)+\theta_h^{''}\left(C_{Yh(2)}^2+C_{Xh(2)}^2+\left(\frac{\sigma_{uh(2)}^2}{\mu_Y^2}+\frac{\sigma_{vh(2)}^2}{\mu_X^2}\right)\right)\right)-\lambda_h^{'}\sum_{h=1}^{L}P_h^2C_{Xh}^2},$$

The expression of $MSE(t_7)$ is obtained if

$$b^{**}=\frac{\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}\rho_{YXh}\sigma_{Yh}\sigma_{Xh}+\theta_h^{''}\rho_{YXh(2)}\sigma_{Yh(2)}\sigma_{Xh(2)}\right)}{R\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}(C_{Xh}^2+\frac{\sigma_{vh}^2}{\mu_X^2})+\theta_h^{''}(C_{Xh(2)}^2+\frac{\sigma_{vh(2)}^2}{\mu_X^2})\right)},$$

and $R=\frac{\mu_Y}{\mu_X}$.

Now following (16) to (18), we propose another generalized estimator in two-phase sampling considering non-response and measurement error on 2nd phase only, as given by

$$t_{8(a)}=\left(H^{\bullet}+\alpha_1\left(\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}\right)+\alpha_2\mu_{y(st)}^{*''}\right)$$

$$\exp\left(\frac{\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}}{\mu_{x(st)}^{'}+\mu_{x(st)}^{*''}}\right), \quad ...(22)$$

where

$$H^{\bullet}=\frac{\mu_{y(st)}^{*''}}{2}\left(exp\left(\frac{\mu_{x(st)}^{'(\frac{1}{a})}-\mu_{x(st)}^{*''(\frac{1}{a})}}{\mu_{x(st)}^{'(\frac{1}{a})}+\mu_{x(st)}^{*''(\frac{1}{a})}}\right)+exp\left(\frac{\mu_{x(st)}^{*''(\frac{1}{a})}-\mu_{x(st)}^{'(\frac{1}{a})}}{\mu_{x(st)}^{*''(\frac{1}{a})}+\mu_{x(st)}^{'(\frac{1}{a})}}\right)\right),$$

and α_1 and α_2 are the optimizing constants and a is a generalizing constant. For different choices of a, different estimators are obtained. For example, for a=1, the proposed estimator is reduced to the form

$$t_{8(1)}=\left(\frac{\mu_{y(st)}^{*''}}{2}\left(\exp\left(\frac{\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}}{\mu_{x(st)}^{'}+\mu_{x(st)}^{*''}}\right)+\exp\left(\frac{\mu_{x(st)}^{*''}-\mu_{x(st)}^{'}}{\mu_{x(st)}^{*''}+\mu_{x(st)}^{'}}\right)\right)\right.$$

$$+\alpha_1^{opt(1)}\left(\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}\right)+$$

$$\left.\alpha_2^{opt(1)}\mu_{y(st)}^{*''}\right)\exp\left(\frac{\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}}{\mu_{x(st)}^{'}+\mu_{x(st)}^{*''}}\right),$$

and for a=2, the proposed estimator $t_{4(a)}$ is reduced to the form

$$t_{8(2)}=\left(\frac{\mu_{y(st)}^{*''}}{2}\left(\exp\left(\frac{\mu_{x(st)}^{'(1/2)}-\mu_{x(st)}^{*''(1/2)}}{\mu_{x(st)}^{'(1/2)}+\mu_{x(st)}^{*''(1/2)}}\right)\right.\right.$$

$$\left.+\exp\left(\frac{\mu_{x(st)}^{*''(1/2)}-\mu_{x(st)}^{'(1/2)}}{\mu_{x(st)}^{*''(1/2)}+\mu_{x(st)}^{'(1/2)}}\right)\right)+\alpha_1^{opt(2)}\left(\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}\right)$$

$$\left.+\alpha_2^{opt(2)}\mu_{y(st)}^{*''}\right)\exp\left(\frac{\mu_{x(st)}^{'}-\mu_{x(st)}^{*''}}{\mu_{x(st)}^{'}+\mu_{x(st)}^{*''}}\right).$$

The bias and *MSE* of $t_{8(a)}$

In order to obtain the expressions for the bias and *MSE* of $t_{8(a)}$ let us consider

$$W_{Yh}^{*}=\sum_{i=1}^{n_h^{''}}(y_{hi}^{*''}-\mu_{Yh}),\ W_{Xh}^{*}=\sum_{i=1}^{n_h^{''}}(x_{hi}^{*''}-\mu_{Xh}),$$

$$W_{Uh}^{*}=\sum_{i=1}^{n_h^{''}}U_{hi}^{*},\ W_{Vh}^{*}=\sum_{i=1}^{n_h^{''}}V_{hi}^{*},$$

and $W_{Xh}=\sum_{i=1}^{n_h^{'}}(x_{hi}^{'}-\mu_{Xh})$. ...(23)

Now the sampling errors, assuming the presence of non-response and measurement errors in stratified two-phase random sampling, may be given by

$$e_{0(st)}^{*''}=\frac{1}{\mu_Y}\sum_{h=1}^{L}\frac{P_h}{n_h^{''}}(W_{Yh}^{*}+W_{Uh}^{*}),$$

$$e_{1(st)}^{*''}=\frac{1}{\mu_X}\sum_{h=1}^{L}\frac{P_h}{n_h^{''}}(W_{Xh}^{*}+W_{Vh}^{*}),$$ and

$$e_{1(st)}^{'}=\frac{1}{\mu_X}\sum_{h=1}^{L}\frac{P_h}{n_h^{'}}\left(x_{hi}^{'}-\mu_{Xh}\right),$$ or alternatively by,

$\mu_{y(st)}^{*''}=\mu_Y\left(1+e_{0(st)}^{*''}\right)$, $\mu_{x(st)}^{*''}=\mu_X\left(1+e_{1(st)}^{''}\right)$, and $\mu_{x(st)}^{'}=\mu_X\left(1+e_{1(st)}^{'}\right)$,

such that $E\left(e_{0(st)}^{*''}\right)=E(e_{1st}^{*''})=E\left(e_{1st}^{'}\right)=0$, and

$$E(e_{0(st)}^{*''})^2=\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}(C_{Yh}^2+\frac{\sigma_{uh}^2}{\mu_Y^2})+\theta_h^{''}(C_{Yh(2)}^2+\frac{\sigma_{uh(2)}^2}{\mu_Y^2})\right)$$

$$=V_{02(st)}^{''},$$

$$E(e_{1(st)}^{*''})^2=\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}(C_{Xh}^2+\frac{\sigma_{vh}^2}{\mu_X^2})+\theta_h^{''}(C_{Xh(2)}^2+\frac{\sigma_{vh(2)}^2}{\mu_X^2})\right)$$

$$=V_{20(st)}^{''},$$

$$E\left(e_{0(st)}^{*''}e_{1(st)}^{*''}\right)=\sum_{h=1}^{L}P_h^2\left(\lambda_h^{''}\rho_{YXh}C_{Yh}C_{Xh}\right.$$

$$\left.+\theta_h^{''}\rho_{YXh(2)}C_{Yh(2)}C_{Xh(2)}\right)=V_{11(st)}^{''},$$

$$E\left(e_{1(st)}^{'}\right)^2 = E\left(e_{1(st)}^{*''} e_{1(st)}^{'}\right) = \sum_{h=1}^{L} P_h^2 \lambda_h^{'} C_{Xh}^2 = V_{20(st)}^{'},$$

$$E\left(e_{0(st)}^{*''} e_{1(st)}^{'}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_h^{'} \rho_{YXh} C_{Yh} C_{Xh}\right) = V_{11(st)}^{'}.$$

Now we may express $t_{8(a)}$ in terms of e,s by

$$t_{8(a)} = \left(\frac{\mu_Y\left(1+e_{0(st)}^{*''}\right)}{2}\left(\exp\left(\frac{\mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{'}\right)^{\left(\frac{1}{a}\right)} - \mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{*''}\right)^{\left(\frac{1}{a}\right)}}{\mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{'}\right)^{\left(\frac{1}{a}\right)} + \mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{*''}\right)^{\left(\frac{1}{a}\right)}}\right) + \right.\right.$$

$$\left.\left.\exp\left(\frac{\mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{*''}\right)^{\left(\frac{1}{a}\right)} - \mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{'}\right)^{\left(\frac{1}{a}\right)}}{\mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{*''}\right)^{\left(\frac{1}{a}\right)} + \mu_X^{\left(\frac{1}{a}\right)}\left(1+e_{1(st)}^{'}\right)^{\left(\frac{1}{a}\right)}}\right)\right) + \right.$$

$$\left.\alpha_1\left(\mu_X - \mu_X\left(1+e_{1(st)}^{'}\right)\right) + \alpha_2 \mu_Y\left(1+e_{0(st)}^{*''}\right)\right)$$

$$\exp\left(\frac{\mu_X(1+e_{1(st)}^{'}) - \mu_X(1+e_{1(st)}^{*''})}{\mu_X\left(1+e_{1(st)}^{'}\right) + \mu_X(1+e_{1(st)}^{*''})}\right). \qquad \text{...(24)}$$

Simplifying the above expression up to the order $O(n^{-1})$, we get

$$t_{8(a)} - \mu_Y \cong \mu_Y\left(e_{0(st)}^{*''} + \frac{e_{1(st)}^{'}}{2} - \frac{e_{1(st)}^{*''}}{2} - \frac{e_{1(st)}^{'2}}{8} + \frac{3e_{1(st)}^{*''2}}{8}\right.$$

$$- \frac{e_{1st}^{'} e_{1(st)}^{*''}}{4} + \frac{e_{1(st)}^{'} e_{0(st)}^{*''}}{2} - \frac{e_{1(st)}^{*''} e_{0(st)}^{*''}}{2} + \frac{e_{1(st)}^{*''2}}{8a^2}$$

$$+ \frac{e_{1(st)}^{'2}}{8a^2} - \frac{e_{1(st)}^{'} e_{1(st)}^{*''}}{4a^2} + \alpha_2 + \alpha_2 e_{0(st)}^{*''} + \alpha_2 \frac{e_{1(st)}^{'}}{2}$$

$$- \alpha_2 \frac{e_{1(st)}^{*''}}{2} - \alpha_2 \frac{e_{1(st)}^{'2}}{8} + \frac{3\alpha_2\, e_{1(st)}^{*''}}{8} - \alpha_2 \frac{e_{1(st)}^{'} e_{1(st)}^{*''}}{4}$$

$$\left. + \alpha_2 \frac{e_{1(st)}^{'} e_{0(st)}^{*''}}{2} - \alpha_2 \frac{e_{1(st)}^{*''} e_{0(st)}^{*''}}{2}\right)$$

$$+\alpha_1 \mu_X\left(e_{1(st)}^{'} - e_{1(st)}^{*''} + \frac{e_{1(st)}^{'2}}{2} + \frac{e_{1(st)}^{*''2}}{2} - e_{1(st)}^{'} e_{1(st)}^{*''}\right). \qquad \text{...(25)}$$

The expressions for the bias and *MSE* of $t_{8(a)}$, upto the order $O(n^{-1})$, may be obtained from (25) as

$$Bias\ \left(t_{8(a)}\right) \cong \mu_Y\left(-\frac{V_{20(st)}^{'}}{8} + \frac{3V_{20(st)}^{''}}{8} - \frac{V_{20(st)}^{'}}{4} + \frac{V_{11(st)}^{'}}{2}\right.$$

$$- \frac{V_{11(st)}^{''}}{2} + \frac{V_{20(st)}^{''}}{8a^2} + \frac{V_{20(st)}^{'}}{8a^2} - \frac{V_{20(st)}^{'}}{4a^2} + \alpha_2 -$$

$$\left.\alpha_2 \frac{V_{20(st)}^{'}}{8} + \frac{3\,\alpha_2 V_{20(st)}^{''}}{8} - \alpha_2 \frac{V_{20(st)}^{'}}{4} + \alpha_2 \frac{V_{11(st)}^{'}}{2} - \alpha_2 \frac{V_{11(st)}^{''}}{2}\right)$$

$$+ \alpha_1 \mu_X\left(\frac{V_{20(st)}^{'}}{2} + \frac{V_{20(st)}^{''}}{2} - V_{20(st)}^{'}\right), \qquad \text{...(26)}$$

and

$$MSE\ \left(t_{8(a)}\right) \cong \mu_Y^2\left(V_{02(st)}^{''} - ¥_{11} + \frac{1}{4}¥_{20} + \alpha_2^2\right.$$

$$+ \alpha_2^2\left(1 + V_{02(st)}^{''} - 2¥_{11} + ¥_{20}\right) + \alpha_2\left(2V_{02(st)}^{''} -\right.$$

$$\left.\left.3¥_{11} + \frac{5}{4}¥_{20} + \frac{1}{4a^2}¥_{20}\right)\right) + \alpha_1^2 \mu_X^2 ¥_{20}$$

$$+2\mu_Y \mu_X \alpha_1\left(\alpha_2(¥_{20} - ¥_{11}) - ¥_{11} + \frac{1}{2}¥_{20}\right), \qquad \text{...(27)}$$

where, $¥_{20} = V_{20(st)}^{''} - V_{20(st)}^{'}$, and

$¥_{11} = V_{11(st)}^{''} - V_{11(st)}^{'}$.

Or alternatively,

$$MSE\left(t_{8(a)}\right) \cong \nabla_o + \alpha_2^2 \nabla_1 + \alpha_2 \nabla_2 + \alpha_1^2 \nabla_3 + \alpha_1 \alpha_2 \nabla_4 + \alpha_1 \nabla_5, \qquad \text{...(28)}$$

where,

$\nabla_o = \mu_Y^2\left(V_{02(st)}^{''} + \frac{1}{4}¥_{20} - ¥_{11}\right)$,

$\nabla_1 = \mu_Y^2\left(1 + V_{02(st)}^{''} + ¥_{20} - 2¥_{11}\right)$,

$\nabla_2 = \mu_Y^2\left(2V_{02(st)}^{''} - 3¥_{11} + \frac{5}{4}¥_{20} + \frac{1}{4a^2}¥_{20}\right)$,

$\nabla_3 = \mu_X^2 ¥_{20}$,

$\nabla_4 = -2\mu_Y \mu_X\ (¥_{11} - ¥_{20})$,

and $\nabla_5 = -2\mu_Y \mu_X\left(¥_{11} - \frac{1}{2}¥_{20}\right)$.

Now, to get the optimum values, equation (28) is differentiated partially with respect to α_1, and α_2, and then equating each of the first derivatives with zero. This gives two normal equations which are then solved simultaneously to get the optimum values. Finally, the optimum values are shown by,

$$\alpha_1^{(opt)} = \frac{\nabla_2\nabla_4 - 2\nabla_1\nabla_5}{4\nabla_1\nabla_3 - \nabla_4^2} \text{ and } \alpha_2^{(opt)} = \frac{\nabla_4\nabla_5 - 2\nabla_2\nabla_3}{4\nabla_1\nabla_3 - \nabla_4^2}. \qquad ...(29)$$

Substituting the optimum values of α_1 and α_2 in (28), the expression for the minimum $MSE(t_{8(a)})$ may be obtained as

$$\min MSE\left(t_{8(a)}\right) = \nabla_o - \frac{\left(\nabla_2\nabla_4\nabla_5 - \nabla_2^2\nabla_3 - \nabla_1\nabla_5^2\right)}{\nabla_4^2 - 4\nabla_1\nabla_3}. \qquad ...(30)$$

Theoretical comparisons

In this section, the proposed estimator IGCEStr is compared for its efficiency based on the *MSE* criterion under stratified two-phase sampling with ratio estimator and exponential estimator.

i) The proposed estimator IGCEStr is more efficient than ratio estimator under stratified two-phase sampling if $MSE(t_5) - MSE(t_{8(a)}) > 0$, and this gives

$$\frac{1}{V''_{02(st)}\text{¥}_{20}} - 2\frac{\text{¥}_{11}}{V''_{02(st)}\text{¥}_{20}} > \frac{1}{\text{¥}_{20}} - \frac{MSE(t_{8(a)})}{\text{¥}_{20} Var\, t_{0(st)}},$$

or alternatively, $\tau_0 > \frac{\tau_2}{2} + \tau_2 \frac{MSE(t_{8(a)})}{2Var\, t_{0(st)}} - \frac{1}{2}$,

where $\tau_0 = \frac{\text{¥}_{11}}{V''_{02(st)}\text{¥}_{20}}$, $\tau_1 = \frac{1}{V''_{02(st)}\text{¥}_{20}}$, and $\tau_2 = \frac{1}{\text{¥}_{20}}$.

ii) The proposed estimator IGCEStr is more efficient than exponential ratio estimator under stratified two-phase sampling if $MSE(t_6) - MSE(t_{8(a)}) > 0$, and this gives

$$\frac{1}{4V''_{02(st)}\text{¥}_{20}} - \frac{\text{¥}_{11}}{V''_{02(st)}\text{¥}_{20}} > \frac{1}{\text{¥}_{20}} - \frac{MSE(t_{8(a)})}{\text{¥}_{20} var\, t_{0(st)}},$$

or alternatively,

$$\tau_0 > \frac{\tau_3}{4} + \tau_2 \frac{MSE(t_{8(a)})}{2var\, t_{0(st)}} - \tau_2, \text{ where } \tau_3 = \frac{1}{V''_{02(st)}\text{¥}_{20}}.$$

iii) The proposed estimator IGCEStr is more efficient than regression estimator under stratified two-phase sampling if $MSE(t_3) - MSE(t_{4(a)}) > 0$, and this gives

$$\tau_0 > 1 - \frac{MSE(t_{8(a)})}{Var\, t_{0(st)}}.$$

Empirical results and discussion using stratified populations:

In order to compare performance of the proposed estimators $t_{8(a)}$ for a=1,2, from the proposed class of $IGCEStr$ with $t_{0(st)}$ t_5, t_6, and t_7, we consider two different simulated stratified populations. The description on parameters of each stratified population is given respectively as below;

Stratified population I: [Normal Population]

Stratum-1X_1=N(5000,4,15); z_1=N(5000,0,1); Y_1=50X+15z; y_1 =Y+N(1,3); x_1= X+N(1,3);
Stratum-2X_2=N(5000,5,15); z_2=N(5000,0,1); Y_2=50X+15z; y_2 =Y+N(1,3); x_2= X+N(1,3);
Stratum-3X_3=N(5000,6,15); z_3=N(5000,0,1); Y_3=50X+15z; y_3 =Y+N(1,3); x_3= X+N(1,3);

stratum	N_h	ρ_{yxh}	μ_{Yh}	μ_{Xh}	σ^2_{Uh}	σ^2_{Vh}	σ^2_{yh}	σ^2_{xh}
1	5000	0.98	183.62	3.67	9.19	9.32	543610.50	227.16
2	5000	0.98	248.67	4.97	9.19	9.25	583685.70	242.30
3	5000	0.98	301.68	6.04	9.10	9.18	578114.50	240.50

stratum	n'_h	n''_h	$\sigma^2_{yh(2)}$	$\sigma^2_{xh(2)}$	$\sigma^2_{Uh(2)}$	$\sigma^2_{Vh(2)}$	$\rho_{yxh(2)}$
1	500	300	575428.90	230.66	9.15	9.75	0.99
2	500	300	599476.60	239.94	9.14	8.80	0.99
3	500	300	577044.10	230.86	9.32	8.63	0.99

Stratified population II: [Lognormal Population]

Stratum-1X_1=log N(5000,log(2),log(6)); z_1=N(5000,0,1); Y_1=50X+15z; y_1 =Y+N(1,3); x_1= X+N(1,3);
Stratum-2X_2=log N(5000,log(3),log(6)); z_2=N(5000,0,1); Y_2=50X+15z; y_2 =Y+N(1,3); x_2= X+N(1,3);
Stratum-3X_3=log N(5000,log(4),log(6)); z_3=N(5000,0,1); Y_3=50X+15z; y_3 =Y+N(1,3); x_3= X+N(1,3);

stratum	N_h	ρ_{yxh}	μ_{Yh}	μ_{Xh}	σ^2_{Uh}	σ^2_{Vh}	σ^2_{yh}	σ^2_{xh}
1	5000	0.99	703.27	14.07	9.03	8.69	5871164.00	2363.44
2	5000	0.99	915.35	18.31	8.73	9.16	8522645.00	3424.65
3	5000	0.99	1221.84	24.44	8.94	9.03	20392993.00	8165.68
stratum	n'_h	n''_h	$\sigma^2_{yh(2)}$	$\sigma^2_{xh(2)}$	$\sigma^2_{Uh(2)}$	$\sigma^2_{Vh(2)}$	$\rho_{yxh(2)}$	
1	500	300	9035398.00	3615.41	8.72	8.85	0.99	
2	500	300	5158451.00	2063.84	8.65	9.15	0.99	
3	500	300	17392184.00	6956.91	8.85	8.88	0.9	

We have computed the mean squared error (*MSEs*) and bias of the proposed and some available estimators and the results are presented in Tables 5 – 8.

In this section the relative performance of the proposed $IGCEStr$ with some modified estimators is observed taking different values of k_h (=2,3,4,&5). It can be observed from Tables 5 – 6 that the proposed performs better than $t_{0(\mathrm{st})}, t_5, t_6$ and t_7. We note that performance of the estimator $t_{8(a)}$ is the best for a=1, however for a=2 the estimator is showing relatively large MSE. Therefore, we note that the square root transformation is not working better than the case a=1 for . Further, we can note that *MSE*s of for a=2 are smaller than the *MSE*s of the all the other modified estimators and it is therefore $t_{8(a)}$ may be considered as the best estimator for both a=2 and a=1. Tables 7 – 8 display the absolute relative bias. It can be observed that the proposed estimators are slightly more biased than the existing estimators, but the biases are not notable. Therefore, still the proposed estimators can be preferred over other existing estimators.

CONCLUSION

In this study, we have proposed $IGCE\ t_{4(\mathrm{a})}$ for estimating population mean in the presence of joint influence of non-response and measurement error using simple random sampling assuming that the population mean of the auxiliary variable is not available. The performance of $IGCE\ t_{4(\mathrm{a})}$ is compared with some modified estimators t_1, t_2 and t_3 and it is observed that $t_{4(a)}$ provides the more efficient class of estimators $t_{4(1)}$, and $t_{4(2)}$ as these estimators have shown less *MSE* values than the *MSE* values of t_0 t_1, t_2 and t_3. We have also proposed another $IGCEStr\ t_{8(a)}$ estimator in stratified two-phase sampling for estimating population mean in the joint presence of non-response and measurement error. The performance of the $IGCEStr\ t_{8(a)}$ is compared with some modified estimators and it is observed that $t_{8(a)}$ is the most efficient class of estimators as some estimators from this class such as $t_{8(1)}$, and $t_{8(2)}$ have shown less *MSE* values than the *MSE* values of $t_{0(\mathrm{st})}$, t_5, t_6 and t_7.Finally, the study is supported by three simulated populations in both, simple random sampling and stratified random sampling. Based on numerical results presented in Tables 2 – 3 and Tables 5 – 6, we conclude that $IGCE\ t_{4(a)}$ and $IGCEStr\ t_{8(a)}$ to be more efficient for estimating the population mean if non-response and measurement error are jointly present. We observe that in both simple random and stratified sampling, a reasonable choice for a is 1 or 2 although a=1 may be preferred.

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Appendix A

Table A1: *MSE* at different values of *k* using Normal Distribution

	1/*k*			
Estimators	½	1/3	1/4	1/5
t_0	2428.39	3027.88	3627.37	4226.85
t_1	1176.62	1198.90	1221.17	1243.44
t_2	1426.71	1581.99	1737.29	1892.58
t_3	1166.11	1188.55	1210.54	1232.33
$t_{4(1)}$	1128.37	1141.62	1153.43	1164.07
$t_{4(2)}$	1134.96	1151.86	1167.89	1183.28

Table A2: *MSE* at different values of using Log Normal Distribution

	1/*k*			
Estimators	½	1/3	1/4	1/5
t_0	27431.75	36467.16	45502.56	54537.97
t_1	10687.62	10709.97	10732.32	10754.67
t_2	14807.86	17071.51	19335.17	21598.82
t_3	10686.60	10709.04	10731.40	10753.73
$t_{4(1)}$	10244.96	10121.80	9977.86	9813.21
$t_{4(2)}$	10334.20	10276.60	10210.75	10136.69

Table A3: *Bias* at different values of *k* using Normal Distribution

	1/*k*			
Estimators	1/2	1/3	1/4	1/5
t_1	0.1446	0.1748	0.2004	0.2232
t_2	0.1293	0.1554	0.1776	0.1974
$t_{4(1)}$	0.1452	0.1752	0.2009	0.2236
$t_{4(2)}$	0.1473	0.1775	0.2032	0.2262

Table A4: *Bias* at different values of *k* using Log Normal Distribution

	1/*k*			
Estimators	1/2	1/3	1/4	1/5
t_1	0.1366	0.1695	0.1969	0.2209
t_2	0.1186	0.1470	0.1708	0.1916
$t_{4(1)}$	0.1366	0.1695	0.1969	0.2210
$t_{4(2)}$	0.1384	0.1714	0.1990	0.2233

Table A5: *MSE* at different values of using Normal Distribution

	$1/k_h$=1/k			
Estimators	½	1/3	¼	1/5
	888.44	1109.37	1330.31	1551.24
t_5	559.45	599.27	639.08	678.90
t_6	554.61	620.15	685.70	751.25
t_7	514.66	550.71	585.83	620.53
$t_{8(1)}$	506.91	540.51	572.88	604.51
$t_{8(2)}$	508.28	542.66	575.95	608.64

Table A6: *MSE* at different values of using Lognormal Distribution

	$1/k_h$=1/k			
Estimators	½	1/3	¼	1/5
	14748.44	18344.68	21940.93	25537.17
t_5	7037.73	7189.29	7340.86	7492.42
t_6	8651.65	9589.94	10528.23	11466.52
t_7	6993.81	7141.86	7288.81	7435.28
$t_{8(1)}$	6899.41	7024.02	7144.98	7262.88
$t_{8(2)}$	6915.81	7049.66	7181.26	7311.20

Table 7: *Bias* at different values of using Normal Distribution

	$1/k_h$=1/k			
Estimators	½	1/3	1/4	1/5
t_5	0.07294	0.090825	0.105727	0.118773
t_6	0.073475	0.088946	0.1021	0.113741
$t_{8(1)}$	0.077747	0.09505	0.109724	0.122683
$t_{8(2)}$	0.078549	0.095913	0.110675	0.123734

Table 8: *Bias* at different values of using Log Normal Distribution

	$1/k_h$=1/k			
Estimators	½	1/3	1/4	1/5
t_5	0.124861	0.150183	0.171813	0.191009
t_6	0.111027	0.133045	0.151905	0.168669
$t_{8(1)}$	0.125216	0.150502	0.172119	0.191311
$t_{8(2)}$	0.125975	0.151291	0.172961	0.19222