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ORIGINAL ARTICLE

Determination of 3D Transformation Parameters for the Sri Lankan Geodetic Reference Network Using Ordinary and Total Least Squares

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ABSTRACT

Since the Global Navigation Satellite System (GNSS) has been a common and convenient practice for many local geospatial applications, it is required to transform the geodetic coordinates from global to local datums. In the Sri Lankan context, datum transformation is performed from the WGS84 to the Everest (1830) geodetic datums. Usually, transformation parameters are determined using the least squares adjustment technique. This research compares the ordinary and total least squares adjustment techniques in three coordinate transformations: Bursa-Wolf; Molodensky-Badekas; Veis, and determines new transformation parameters for coordinate transformation in Sri Lankan local geodetic reference network. Cross-Validations of hold out and leave-one-out were applied for the validation of the results, in which the latter performed best. The research concludes that though the Bursa-Wolf model is simple, Molodensky-Badekas and Veis models are more suitable for coordinate transformation in Sri Lanka.

1 Introduction

Local geodetic datums play an important role as the reference surface required for surveying and mapping activities of many countries. With an advent of Global Positioning System (GPS), the use of the global datum such as WGS84 has rapidly increased. As a consequence, the transformation of coordinates between global and local geodetic systems is a matter of interest in many research studies. Among the different transformation models, the frequently used approaches for geodetic datum transformations are the seven-parameter transformation models of Bursa-Wolf (Bursa, 1962), Molodensky Badekas and Abridged Molodensky (Molodensky et al., 1962), and Veis (Veis, 1960). These transformation models use a set of parameters determined from independent observations through the least squares adjustment technique. For geodetic coordinate transformation, various studies apply and compare the efficiencies of Total Least Square (TLS) and Ordinary Least Square (OLS) adjustment techniques (Okwuashi and Eyoh, 2012). According to these studies, TLS performed better than the OLS though it was computationally robust. Although Sri Lanka has been using a seven parameter transformation model (Bursa-Wolf) for a long time, a comprehensive analysis of its performance has not yet been carried out. Therefore, this study focuses to analyze the performance of TLS and OLS for the Sri Lankan geodetic reference network. Three seven-parameter transformation models: Bursa-Wolf; Molodensky-Badekas; and Veis were selected for the analysis with least squares techniques (OLS and TLS) since they are the widely used transformation models by researchers globally, due to their simplicity and achievable accuracy (Závoti and Kalmár, 2016).

In 1993, the geodetic control network of Sri Lanka was upgraded by using GNSS technology. The seven-parameter Burse-Wolf transformation model was used to realize the coordinates adopted for Sri Lankan Datum 1999 (SLD99) which was based on the Everest (1830) ellipsoid, from the WGS84 ellipsoid. Both Kandawala, the datum based on old geodetic reference network of Sri Lanka, and SLD99 have been parallelly used in the country, and US National Geospatial Agency (NGA) has provided the transformation parameters from WGS84 to Kandawala using the standard Molodensky transformation model (Abeyratne et al., 2010). It is observed that there is a difference between the false origins of the respective grid coordinates, and yet, NGA has not provided the transformation parameters from WGS84 to SLD99 by using the standard Molodensky transformation model. According to that, there is no direct Geographical coordinate conversion from WGS84 to SLD99. The standard Molodensky transformation is used in this study since it is one of the most widely used method for the

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transformation of geodetic coordinates from one datum to another (Leder and Leder, 2009), and also this method is incorporated within most Geographic Information System (GIS) packages, as well as into most GPS receivers.

2 Study area and Data Used

The horizontal geodetic network of Sri Lanka was initially established by using Triangulation (Jackson, 1936), and upgraded in 1993 with GNSS technology. This initial network has been referred in the literature as Kandawala geodetic network. In this study, 274 common points with known WGS84 geodetic coordinates $(\emptyset, \lambda, h)_W$ were selected, covering the entire country (Fig 1). Abridged Molodensky Transformation model, which is the NGA accepted method of converting WGS84 $(\emptyset, \lambda, h)_W$ to Everest (1830) $(\emptyset, \lambda, h)_E$, was used to make common points for this task.

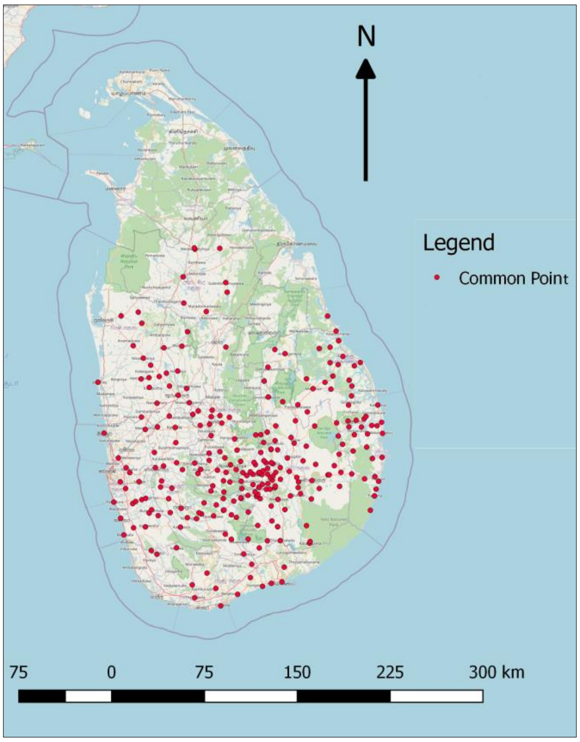

Fig. 1: Study area and control stations

3 Computational Methodology

3.1 The Standard and Abridged Molodensky Transformation Model

Many geodetic practitioners and GIS software have used the standard and abridged Molodensky transformation formulae for geodetic coordinate transformations (Deakin, 2004). The formulae enable the transformation of latitude, longitude and height $(\emptyset, \lambda, h)$ from one ellipsoid to another. It provides $(\nabla\emptyset, \Delta\lambda, \Delta h)$ as the set of corrections of latitude, longitude and ellipsoidal heights respectively to transform $(\emptyset, \lambda, h)_W$ to $(\emptyset, \lambda, h)_E$ as follows:

$$\Delta\emptyset'' = \frac{1}{Msin1''}\begin{pmatrix}-\Delta X sin\emptyset cos\lambda - \Delta Y sin\phi sin\lambda + \Delta Z cos\phi + \\ (a\Delta f + f\Delta a) sin2\phi\end{pmatrix} \quad (1)$$

$$\Delta\lambda'' = \frac{1}{Ncos\phi sin1''}(-\Delta X sin\lambda + \Delta Y cos\phi) \quad (2)$$

$$\Delta h = \Delta X cos\phi cos\lambda + \Delta Y cos\phi sin\lambda + \Delta Z sin\phi + (a\Delta f + f\Delta a)\, sin^2\phi - \Delta a \quad (3)$$

where $\Delta X, \Delta Y$ and ΔZ are difference in X, Y, Z rectangular Cartesian coordinates of WGS84 and Everest (1830) ellipsoids, and Δf, Δa are difference in polar flattening and semi-major axes of ellipsoids. The radii of curvature in the meridian (M) and prime vertical (N) sections are defined in Eq. (4) and (5) as

$$N = \frac{a}{\sqrt{1-e^2 sin^2\phi}} \quad (4)$$

$$M = \frac{a(1-e^2)}{(1-e^2 sin^2\phi)^{\frac{3}{2}}} \quad (5)$$

where a and e are length of semi-major axis and eccentricity of the ellipsoid. The transformed geodetic coordinates are given as follows:

$$\phi_E = \phi_W - \Delta\phi \quad (6)$$

$$\lambda_E = \lambda_W - \Delta\lambda \quad (7)$$

$$h_E = h_W - \Delta h \quad (8)$$

3.2 Computation of Cartesian coordinates

In order to convert the WGS84 and Everest (1830) geodetic coordinates, latitudes, longitudes and ellipsoidal heights (ϕ, λ, h) into rectangular Cartesian coordinates (X, Y, Z), the following relationship was applied;

$$X = \frac{acos\phi cos\lambda}{[1-f(2-f)sin^2\phi]^{1/2}} + hcos\phi cos\lambda \quad (9)$$

$$Y = \frac{acos\phi cos\lambda}{[1-f(2-f)sin^2\phi]^{1/2}} + hcos\phi sin\lambda \quad (10)$$

$$Z = \frac{a(1-f)^2 sin\phi}{[1-f(2-f)sin^2\phi]^{1/2}} + hsin\phi \quad (11)$$

Ellipsoid parameters used in Eq. (9), (10) and (11) for WGS84 are: semi-major axis value of 6,378,137.0 m and a flattening of 1/298.257. The Everest (1830) on the other hand has a semi-major value of 6,377,276.345 m and a flattening of 1/300.802. Eq. (9), (10) and (11) were directly applied to the WGS84 and Everest (1830) geodetic coordinates to obtain its related Cartesian coordinates. The Bursa-Wolf, Mododensky-Badekas and Veis transformation models were used with least square and total least square methods to calculate seven transformation parameters from Everest (1830) to WGS84.

3.3 Bursa-Wolf Model

According to the Bursa-Wolf Model (Bursa, 1962), the two rectangular coordinate systems relate as follows:

$$[X_w, Y_w, Z_w] = [T_x, T_y, T_z] + \eta R(\alpha_1, \alpha_2, \alpha_3)[X_E, Y_E, Z_E] \quad (12)$$

where X_w, Y_w and Z_w are the rectangular Cartesian coordinates and their subscripts refer the Everest (1830) and WGS84 reference frames; T_x,T_y and T_z are the translations along X, Y and Z axes; η is the scale-factor and R is the overall rotational matrix.

$$R = \begin{bmatrix} cos\alpha_3 & sin\alpha_3 & 0 \\ -sin\alpha_3 & cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\alpha_2 & 0 & -sin\alpha_2 \\ 0 & 1 & 0 \\ sin\alpha_2 & 0 & cos\alpha_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha_1 & sin\alpha_1 \\ 0 & -sin\alpha_1 & cos\alpha_1 \end{bmatrix} \quad (13)$$

where α_1, α_2 and α_3 are the rotation angles around X, Y and Z axes. The expansion of Eq. (12) is given as:

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \begin{bmatrix} 1+\eta & R_Z & -R_y \\ -R_Z & 1+\eta & R_x \\ R_y & -R_x & 1+\eta \end{bmatrix} \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} \quad (14)$$

where R_x, R_y and R_z denote rotation matrix about *X, Y* and *Z* axes. Eq. (14) can be simplified as Eq. (15) – (17):

$$X_w - X_E = T_x + \eta X_E + R_Z Y_E - R_y Z_E \quad (15)$$

$$Y_w - Y_E = T_Y - R_Z X_E + \eta Y_E + R_X Z_E \quad (16)$$

$$Z_w - Z_E = T_Z + R_Y X_E - R_X Y_E + \eta X_E \quad (17)$$

In matrix form:

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & -Z_E & Y_E & X_E \\ 0 & 1 & 0 & Z_E & 0 & -X_E & Y_E \\ 0 & 0 & 1 & -Y_E & X_E & 0 & Z_E \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \\ R_X \\ R_Y \\ R_Z \\ \eta \end{bmatrix} \quad (18)$$

The above equation can be represented in the form $V = AX - L$ as illustrated in Eq. (19).

$$\begin{bmatrix} V_x \\ V_Y \\ V_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -Z_E & Y_E & X_E \\ 0 & 1 & 0 & Z_E & 0 & -X_E & Y_E \\ 0 & 0 & 1 & -Y_E & X_E & 0 & Z_E \end{bmatrix} \begin{bmatrix} T_Z \\ T_Y \\ T_Z \\ R_X \\ R_Y \\ R_Z \\ \eta \end{bmatrix} - \begin{bmatrix} X_w - X_E \\ Y_w - Y_E \\ Z_w - Z_E \end{bmatrix} \quad (19)$$

where, matrixes *V, A, X* and *L* represents residuals, design, unknown parameters and observation matrixes. The least squares method can be applied to determine the unknown transformation parameters (matrix *X*).

3.4 Molodensky-Badekas Model

According to the Molodensky-Badekas model, the two rectangular coordinate systems are expressed as

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \eta R(\alpha_1, \alpha_2, \alpha_3) \begin{bmatrix} X_E - X_c \\ Y_E - Y_C \\ Z_E - Z_C \end{bmatrix} \quad (20)$$

where, X_C, Y_C and Z_C are the respective centroids of points in the Everest (1830) reference Frame. With n as the number of observation points, the centroids are given by Eq. (21) as:

$$X_C = \frac{1}{n}\sum_{i=1}^{n} X_{E_i}, Y_C = \frac{1}{n}\sum_{i=1}^{n} Y_{E_i} \text{ and} Z_C = \frac{1}{n}\sum_{i=1}^{n} Z_{E_i} \quad (21)$$

The expansion of Eq. (20) is given as:

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \begin{bmatrix} 1+\eta & R_Z & -R_y \\ -R_Z & 1+\eta & R_x \\ R_y & -R_x & 1+\eta \end{bmatrix} \begin{bmatrix} X_E - X_c \\ Y_E - Y_C \\ Z_E - Z_C \end{bmatrix} \quad (22)$$

Eq. (22) can be simplified as Eq. (23) – (25)

$$X_w - X_E - (X_E - X_C) = T_X + (X_E - X_C)\eta + (Y_E - Y_C)R_Z - (Z_E - Z_C)R_Y \quad (23)$$

$$Y_w - Y_E - (Y_E - Y_C) = T_Y - (X_E - X_C)R_Z + (Y_E - Y_C)\eta + (Z_E - Z_C)R_X \quad (24)$$

$$Z_w - Z_E - (Z_E - Z_C) = T_Z + (X_E - X_C)R_Y - (Y_E - Y_C)R_X + (Z_E - Z_C)\eta \quad (25)$$

In matrix form:

$$\begin{bmatrix} X_w - X_E - (X_E - X_C) \\ Y_w - Y_E - (Y_E - Y_C) \\ Z_w - Z_E - (Z_E - Z_C) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -(Z_E - Z_C) & Y_E - Y_C & X_E - X_C \\ 0 & 1 & 0 & Z_E - Z_C & 0 & -(X_E - X_C) & Y_E - Y_C \\ 0 & 0 & 1 & -(Y_E - Y_C) & X_E - X_C & 0 & Z_E - Z_C \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \\ R_x \\ R_y \\ R_z \\ \eta \end{bmatrix} \quad (26)$$

The above equation can be represented in the form $V = AX - L$ as below, and the method of least-squares can be applied to solve the unknown parameters.

$$\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -(Z_E - Z_C) & Y_E - Y_C & X_E - X_C \\ 0 & 1 & 0 & Z_E - Z_C & 0 & -(X_E - X_C) & Y_E - Y_C \\ 0 & 0 & 1 & -(Y_E - Y_C) & X_E - X_C & 0 & Z_E - Z_C \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \\ R_x \\ R_y \\ R_z \\ \eta \end{bmatrix} - \begin{bmatrix} X_w - X_E - (X_E - X_C) \\ Y_w - Y_E - (Y_E - Y_C) \\ Z_w - Z_E - (Z_E - Z_C) \end{bmatrix} \quad (27)$$

3.5 Veis Transformation Model

This similarity transformation model was proposed by Veis (Veis, 1960). This form was an attempt to introduce rotations that could be associated with some process that took place at the datum origin point when the geodetic datum was originally defined (Rapp.H., 1993).

The Veis transformation is defined as in Rapp (1993),

If $(\eta, \xi, \alpha\,)$ denote positive rotations about the *X, Y*, and *Z* axes, and ϕ_0, λ_0 and $h_0$are the geodetic coordinates of the initial point, then *M* matrix is define as:

$$M = R_Z^T(\lambda_0)R_Y^T(90^0 - \phi_0)R_X(\eta)R_Y(\xi)R_X(\alpha)R_Y(90^0 - \phi_0)R_Z(\lambda_0) \quad (28)$$

The rotation matrix (*M*) is then scaled by factor $(1 + \Delta S_v)$ where ΔS_v is the scale change parameter associated with the Veis transformation. The complete transformation is as follows,

$$X_V = \underline{T_v} + x_0 + (1 + \Delta S_v)M(\varphi_0, \lambda_0, \varepsilon, \eta)(x - x_0) \quad (29)$$

where T_v is the translation vector measured in the new axes.

This can also be represented as;

$$I_2 = T_2 + G_2 + (1 + \Delta s)R_V(\varphi_0, \lambda_0, \varepsilon, \eta)\bar{I_1} \quad (30)$$

Alternatively, Eq. (30) can be written using vector equations as

$$I_2 = t_2 + (1 + \Delta s)R_V(\varphi_0, \lambda_0, \varepsilon, \eta)\bar{I_1} \quad (31)$$

The rotation matrix R_V used in Eq. (31) is given by Eq. (32).

$$R_V = \begin{bmatrix} 1 & \alpha sin\varphi_0 - \eta cos\varphi_0 & -\alpha cos\varphi_0 sin\lambda_0 - \xi cos\lambda_0 - \eta sin\varphi_0 sin\lambda_0 \\ -\alpha sin\varphi_0 + \eta cos\varphi_0 & 1 & \alpha cos\varphi_0 cos\lambda_0 - \xi sin\lambda_0 + \eta sin\varphi_0 cos\lambda_0 \\ \alpha cos\varphi_0 sin\lambda_0 + \xi cos\lambda_0 + \eta sin\varphi_0 sin\lambda_0 & -\alpha cos\varphi_0 cos\lambda_0 + \xi sin\lambda_0 - \eta sin\varphi_0 cos\lambda_0 & 1 \end{bmatrix} \quad (32)$$

$$\bar{I_1} = \begin{bmatrix} X_E - X_G \\ Y_E - Y_G \\ Z_E - Z_G \end{bmatrix}_1 = \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix}_1 \quad (33)$$

X_G,Y_G,Z_G are coordinates of the centroid given as $X_G=\frac{1}{n}\sum_{i=1}^{n} X_i$, $Y_G=\frac{1}{n}\sum_{i=1}^{n} Y_i$, $Z_G=\frac{1}{n}\sum_{i=1}^{n} Z_i$ and (X_i, Y_i, Z_i) are coordinates in the source datum (Everest 1830). R_V Matrix could be expressed as R_V = I+U, where I is an identity matrix

$$R_V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \alpha sin\varphi_0 - \eta cos\varphi_0 & -\alpha cos\varphi_0 sin\lambda_0 - \xi cos\lambda_0 - \eta sin\varphi\_0\ sin\lambda\_0 \\ -\alpha sin\varphi_0 + \eta cos\varphi_0 & 0 & \alpha cos\varphi_0 cos\lambda_0 - \xi sin\lambda_0 + \eta sin\varphi_0 cos\lambda_0 \\ \alpha cos\varphi_0 sin\lambda_0 + \xi cos\lambda_0 + \eta sin\varphi_0 sin\lambda_0 & -\alpha cos\varphi_0 cos\lambda_0 + \xi sin\lambda_0 - \eta sin\varphi_0 cos\lambda_0 & 0 \end{bmatrix} \quad (34)$$

Hence, from Eq. (31), the Veis formula can be written as

$$I_2 = (1 + \Delta s)(I + U)\bar{I_1} + t_2 \quad (35)$$

where $I_2 = [X, Y, Z]_2^T$ are the coordinates in the target datum (WGS84), $\bar{I_1}$=$[\bar{X}, \bar{Y}, \bar{Z}]_1^T$ are the coordinates in the source datum (Everest1830). t_2 is the vector of translation and ΔS is the scale. Expanding I_2 gives

$$I_2 = (I + U)\underline{I_1} + \Delta S(I + U)\underline{I_1} + t_2 \quad (36)$$

$$I_2 = (I + U)\underline{I_1} + \Delta SI\underline{I_1} + t_2 \quad (37)$$

In view of the fact that $R_V = I + U$ and a vector pre-multiplied by the identity matrix is equal to the vector (that is $\Delta SI\bar{I_1} = \Delta S\bar{I_1}$) and $\Delta SU\bar{I_1} \approx 0$ because ΔS is small (usually < 1 ppm) and the off-diagonal elements of U (the small rotations R_X, R_Y, R_Z) are usually less than 1 second arc ($\approx 4.8e^{-6}$), the product will be exceedingly small and may be neglected. Hence, for practical purposes, Eq. (37) can be written as:

$$I_2 = R_V \underline{I_1} + \Delta S \underline{I}_1 + t_2 \tag{38}$$

Substituting the variable terms in equation (38) for a single control point is given as Eq. (39).

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} 1 & \begin{matrix} \alpha sin\varphi_0 \\ -\eta cos\varphi_0 \end{matrix} & \begin{matrix} -\alpha cos\varphi_0 sin\lambda_0 \\ -\xi cos\lambda_0 \\ -\eta sin\varphi_0 \ sin\varphi_0 \end{matrix} \\ \begin{matrix} -\alpha sin\varphi_0 \\ +\eta cos\varphi_0 \end{matrix} & 1 & \begin{matrix} \alpha cos\varphi_0 cos\lambda_0 \\ -\xi sin\lambda_0 \\ +\eta sin\varphi_0 cos\lambda_0 \end{matrix} \\ \begin{matrix} \alpha cos\varphi_0 sin\lambda_0 \\ +\xi cos\lambda_0 \\ +\eta sin\varphi_0 sin\lambda_0 \end{matrix} & \begin{matrix} -\alpha cos\varphi_0 cos\lambda_0 \\ +\xi sin\lambda_0 \\ -\eta sin\varphi_0 cos\lambda_0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} + \Delta S \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix} \tag{39}$$

Eq. (40) was obtained by expanding Eq. (39)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} \bar{X} + \bar{Y}(\alpha sin\varphi_0 - \eta cos\varphi_0) + \bar{Z}(-\alpha cos\varphi_0 sin\lambda_0 - \xi cos\lambda_0 - \eta sin\varphi_0 \ sin\varphi_0) \\ \bar{X}(-\alpha sin\varphi_0 + \eta cos\varphi_0) + \bar{Y} + \bar{Z}(\alpha cos\varphi_0 cos\lambda_0 - \xi sin\lambda_0 + \eta sin\varphi_0 cos\lambda_0) \\ \bar{X}(\alpha cos\varphi_0 sin\lambda_0 + \xi cos\lambda_0 + \eta sin\varphi_0 sin\lambda_0) + \bar{Y}(-\alpha cos\varphi_0 cos\lambda_0 + \xi sin\lambda_0 - \eta sin\varphi_0 cos\lambda_0) + \bar{Z} \end{bmatrix} + \Delta S \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix} \tag{40}$$

Eq. (41) has been developed by expanding Eq. (40) and applying the matrix form.

$$\begin{bmatrix} 1 & 0 & 0 & \bar{Y}_1 sin\varphi_0 - \overline{Z_1} cos\varphi_0 sin\lambda_0 & \bar{Z} cos\lambda_0 & -\bar{Y}_1 cos\varphi_0 - \overline{Z_1} sin\varphi_0 sin\lambda_0 & \bar{X}_1 \\ 0 & 1 & 0 & -\bar{X}_1 sin\varphi_0 + \overline{Z_1} cos\varphi_0 cos\lambda_0 & -\bar{Z} sin\lambda_0 & \overline{X_1} cos\varphi_0 + \overline{Z_1} sin\varphi_0 cos\lambda_0 & \bar{Y}_1 \\ 0 & 0 & 1 & \bar{X}_1 cos\varphi_0 sin\lambda_0 - \bar{Y}_1 cos\varphi_0 cos\lambda_0 & \bar{X}_1 cos\lambda_0 + \bar{Y}_1 sin\lambda_0 & \bar{X}_1 sin\varphi_0 sin\lambda_0 - \bar{Y}_1 sin\varphi_0 cos\lambda_0 & \overline{Z_1} \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \\ \alpha \\ \varepsilon \\ \eta \\ \Delta S \end{bmatrix} = \begin{bmatrix} X_2 & - & X_1 \\ Y_2 & - & Y_1 \\ Z_2 & - & Z_1 \end{bmatrix} \tag{41}$$

The above equation [Eq. (41)] can be represented in the form $V = AX - L$ as below, and the method of least-squares can be applied to solve the unknown parameters.

$$\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \bar{Y}_1 sin\varphi_0 - \overline{Z_1} cos\varphi_0 sin\lambda_0 & \bar{Z} cos\lambda_0 & -\bar{Y}_1 cos\varphi_0 - \overline{Z_1} sin\varphi_0 sin\lambda_0 & \bar{X}_1 \\ 0 & 1 & 0 & -\bar{X}_1 sin\varphi_0 + \overline{Z_1} cos\varphi_0 cos\lambda_0 & -\bar{Z} sin\lambda_0 & \overline{X_1} cos\varphi_0 + \overline{Z_1} sin\varphi_0 cos\lambda_0 & \bar{Y}_1 \\ 0 & 0 & 1 & \bar{X}_1 cos\varphi_0 sin\lambda_0 - \bar{Y}_1 cos\varphi_0 cos\lambda_0 & \bar{X}_1 cos\lambda_0 + \bar{Y}_1 sin\lambda_0 & \bar{X}_1 sin\varphi_0 sin\lambda_0 - \bar{Y}_1 sin\varphi_0 cos\lambda_0 & \overline{Z_1} \end{bmatrix} - \begin{bmatrix} X_2 & -X_1 \\ Y_2 & -Y_1 \\ Z_2 & -Z_1 \end{bmatrix} \tag{42}$$

3.6 Ordinary Least Square (OLS) and Total Least Square (TLS) techniques

Least-squares adjustment is a model for the solution of an over determined system of equations based on the principle of least squares of observation residuals. This method is extensively used in regression analysis and estimation (Miller, 2006). Considering a system of equations to be solved by least squares:

$$AX \approx L \tag{43}$$

Where, A is the design matrix, X is the matrix of the unknown parameters, and L is the observation matrix

3.6.1 Ordinary Least Squares Method (OLS)

The objective of OLS is the estimation of unknown parameters, given a set of observations or measurements, satisfying the minimization of squared errors. Considering a system of equations shown by Eq. (43), where, $A \in R^{m \times n}, L \in R^{m \times d}$, $X \in R^{n \times d}$ and m≥n , the OLS takes the form of Eq. (44):

$$AX = L + V \tag{44}$$

Assuming the unit weight, the unknown parameter in matrix form is estimated from Eq. (45)

$$x = (A^T A)^{-1} (A^T L) \tag{45}$$

The residual vector V is associated with the OLS is given by Eq. (46) as:

$$V = AX - L \tag{46}$$

3.6.2 Total Least Squares Method (TLS)

The total least squares method was introduced by Golub and van Loan (1980). Consider Eq. (43), where$A \in R^{m \times n}$, $L \in R^{m \times d}$ and$X \in R^{n \times d}$. With m>n typically, there is no exact solution for X, so that an approximation method is applied when data in both A and L is perturbed. To find X that minimizes error matrices E and F for A and L respectively,

$$argmin_{E,F}\|[EF]\|_F, \text{ such that } (A+E)X = L + F \tag{47}$$

where $\|[EF]\|_F$ is the Frobenius norm of the augmented matrix of $[EF]$. This matrix can be rewritten as;

$$[(A+E)(L+F)][B - I_k] = 0 \tag{48}$$

where I_k is the identity matrix of dimension k

The Singular Value Decomposition (SVD) of matrix [A, L] was used in solving the TLS through Eq. (48) as

$$[A, L] = USV^T \tag{49}$$

$U = [U_1, U_2], U_1 = [U_1 \dots \dots \dots . U_m],\ U_2 = [U_{m+1} \dots \dots U_n],$
$U^T U = I_n$ and $u_i \in R^n, i = 1,2 \dots m$
where $V = [V_1 \dots \dots \dots V_m, V_{m+1}],\ V^T V = I_{m+1}$ and $V_I \in R^{m+1}, I = 1,2, .. m+1$

The final solution for the TLS problem is given by Eq. (50) as:

$$\left[\hat{X}^T, -1\right]^T = \frac{-1}{V_{m+1,m+1}} . V_{m+1} \tag{50}$$

If $V_{m+1,m+1} \neq 0,$ then $\hat{L} = A\hat{X} = -\frac{1}{V_{m+1,m+1}\hat{A}\left[V_{1,m+1} \dots\dots\dots V_{m,m+1}\right]^T}$ which belongs to the column space of $\hat{A}$, so $\hat{X}$ solves the basic TLS problem (Okwuashi and Eyoh, 2012).

The TLS correction is given in Eq. (51) as:

$$\left[\Delta\hat{A}, \Delta\hat{L}\right] = [A, L] - \left[\hat{A} - \hat{L}\right] \tag{51}$$

3.7 Model validation

Usually, when validating the results, a separate dataset which has not been used for computation of the model is used. However, this is questionable for some regions where a limited known dataset is available. Therefore, in this study, the holdout and leave-one-out cross-validation methods were used for validating the result in which the latter is more applicable for the regions with limited known data coverage. The final model was selected by using the minimum average residual model.

3.7.1 Holdout Cross-Validation

In holdout or test sample estimation method, the whole dataset is divided into two parts called a training set and a test set, or holdout set. It is common practice to designate 2/3 of the data as the training set and the remaining 1/3 as the test set (Mahobia et al., 2010). In this process, the primary data of WGS84 $(\phi, \lambda, h)_w$ coordinates were divided into two subsets as 75% (205 points) as the training set (Fig 2) and 25% (69 points) as test set (Fig 3). The training dataset were used to determine the parameters and the test dataset were used to calculate the residuals.

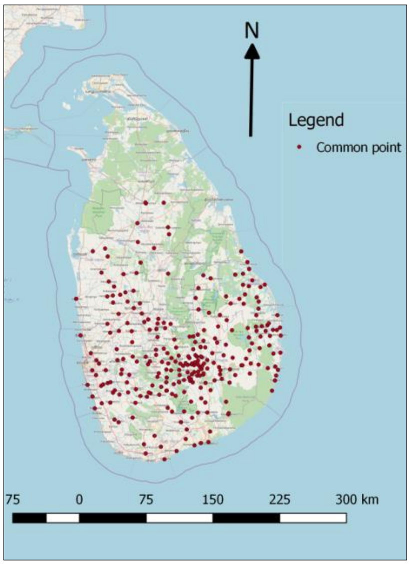

Fig. 2: Training data set

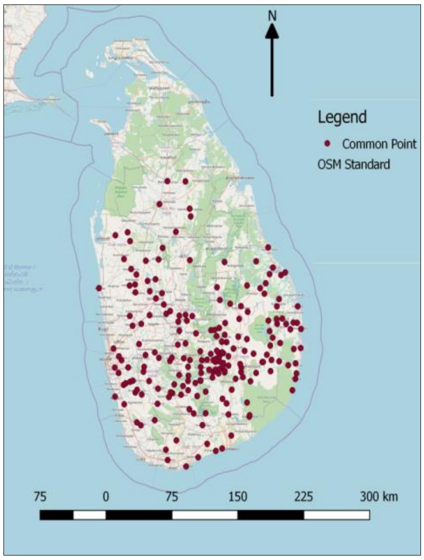

Fig. 3: Test data set

3.7.2 Leave-One-Out Cross-Validation

Leave-one-out cross-validation is a special case of k-fold cross-validation where k equals to the number of instances in the data (Ziggah et al., 2019). In other words, each iteration use nearly all the data except for a single observation for training and the model which is tested on that single observation (Prasanna and Chen, 2012). In this process, all the common points are selected except one as the subset which is used to calculate the coefficients. Then, by using the coefficients calculated, the value of the common point which is not in the computational subset is estimated. This process is repeated to all the common points in the dataset one by one. After applying this process to all common points, estimation errors can be obtained from the differences between estimated and actual values for latitude, longitude and height components.

3.8 Accuracy Assessment and Reverse Techniques Performance Evaluation

In this study, Everest (1830) $(\phi, \lambda, h)_E$ coordinates were calculated using computed transformation parameters and published transformation parameters from the Survey Department of Sri Lanka. A comparison was done against the Everest (1830)$(\phi, \lambda, h)_E$ data which were derived from Abridged Molodensky Transformation Model.

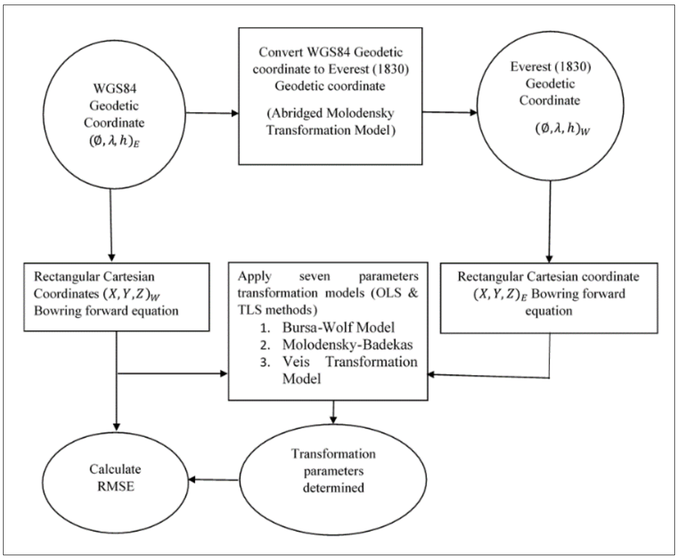

Fig. 4: Computational Procedure of the study

The reverse transformation methods were evaluated based on the residuals generated from WGS84 projected grid coordinates and the calculated WGS84 projected coordinates using three transformation models and two least squares techniques. The Root Mean Square Error (RMSE) statistical indicator was used as a Performance Criteria Index (PCI)

$$\text{RMSE}=\sqrt{\frac{1}{n}\sum_{i=1}^{n} (O_i - P_i)^2} \quad (52)$$

where n is the total number of observations used in the transformation model, O_i and P_i are the existing projected coordinates and calculated projected coordinates of WGS84.

The flow chart in Fig. 4 shows the summary of the computation methodology of this study.

4 Results and Discussion

The transformation parameters and their corresponding standard deviations obtained from the OLS and TLS techniques with the holdout method are shown in Table 1.

Table 1: Transformation parameters derived from Bursa-Wolf, Molodensky-Badekas and Veis Transformation Models from OLS and TLS techniques with holdout validation

Parameter	OLS	TLS	Transformation Model
$T_x(m)$	0.284469 ±0.048126	0.284579 ±0.048127	Bursa-Wolf Model
$T_y(m)$	-766.868269 ±0.036273	-766.868947 ±0.036273	
$T_z(m)$	-87.706415 ±0.052437	-87.70649 ±0.052437	
$R_x(sec)$	0.000342 ±0.00169	0.000342 ±0.00169	
$R_y(sec)$	0.002197 ±0.001178	0.002198 ±0.001178	
$R_z(sec)$	0.000645 ±0.001559	0.000643 ±0.001559	
η(ppm)	0.012706 ±5.585558E-09	0.012603 ±5.585554E-09	
$T_x(m)$	0.283109 ±0.000438	0.283109 ±0.000438	Molodensky-Badekas Model
$T_y(m)$	-766.949543 ±0.000438	-766.949543 ±0.000438	
$T_z(m)$	-87.71597 ±0.000438	-87.71597 ±0.000438	
$R_x(sec)$	0.000342 ±0.001691	0.000343 ±0.001691	
$R_y(sec)$	0.002197 ±0.001178	0.0021971 ±0.0011778	
$R_z(sec)$	0.00065 ±0.001560	0.00065 ±0.001559	
η(ppm)	0.012706 ±5.585558E-09	0.012706 ±5.585558E-09	
$T_x(m)$	0.283108 ±0.000438	0.283108 ±0.000438	Vies Model
$T_y(m)$	-766.949542 ±0.000438	-766.949543 ±0.000438	

$T_z(m)$	-87.71597 ±0.000438	-87.71597 ±0.000438	
$R_x(sec)$	0.002287 ±0.001152	0.002287 ±0.0011528	
$R_y(sec)$	0.00000277 ±0.001703	0.00000277214 ±0.0017038	
$R_z(sec)$	-0.000272 ±0.001565	-0.000272 ±0.001568	
η(ppm)	0.0127062 ±5.585749E-09	0.012706 ±5.5857498E-09	

Table 2 shows the transformation parameters and their corresponding standard deviations obtained from OLS and TLS techniques with leave-one-out cross-validation method

Table 2: Transformation parameters derived from Bursa-Wolf, Molodensky-Badekas, Vies Transformation Models with OLS, TLS techniques and leave-one-out cross-validation

Parameter	OLS	TLS	Transformation Model
$T_x(m)$	0.276701 ±0.042141	0.276788 ±0.042142	Bursa-Wolf Model
$T_y(m)$	-766.887125 ±0.031879	-766.8878226 ±0.031879	
$T_z(m)$	-87.70539503 ±0.046832	-87.705465 ±0.046832	
$R_x(sec)$	0.000692 ±0.001511	0.000693 ±0.001511	
$R_y(sec)$	0.003682 ±0.00103	0.003682 ±0.00103	
$R_z(sec)$	0.001004 ±0.001365	0.000998 ±0.001365	
η(ppm)	0.00965 ±4.905372E-09	0.00954 ±4.905368E-09	
$T_x(m)$	0.283566 ±0.000385	0.283566 ±0.000385	Molodensky -Badekas Model
$T_y(m)$	-766.949681 ±0.000389	-766.94968 ±0.000388	
$T_z(m)$	-87.715953 ±0.000385	-87.715953 ±0.000384	
$R_x(sec)$	0.000692 ±0.001511	0.000692 ±0.001508	
$R_y(sec)$	0.003682 ±0.001034	0.003682 ±0.001033	
$R_z(sec)$	0.001004 ±0.001365	0.001004 ±0.001363	
η(ppm)	0.009646 ±4.905371E-09	0.009646 ±4.896296E-09	
$T_x(m)$	0.283566 ±0.000386	0.283566 ±0.000385	Vies Model
$T_y(m)$	-766.949681 ±0.000388	-766.949681 ±0.000388	
$T_z(m)$	-87.715953 ±0.000384	-87.715953 ±0.000384	
$R_x(sec)$	0.003842 ±0.00101	0.003842 ±0.00101	
$R_y(sec)$	-0.000118 ±0.001519	-0.000118 ±0.001519	
$R_z(sec)$	-0.000452 ±0.001367	-0.000452 ±0.001367	
η(ppm)	0.009648 ±4.896482E-09	0.009648 ±4.896481E-09	

The precision of the model transformed coordinates can be assessed through the standard deviations. It indicates how well the transformed WGS84 coordinates fit with the known Everest (1830) system. The comparison of standard deviation of all parameters derived from OLS and TLS are presented in Fig. 5. It is notable that both OLS and TLS have produced closely related standard deviations.

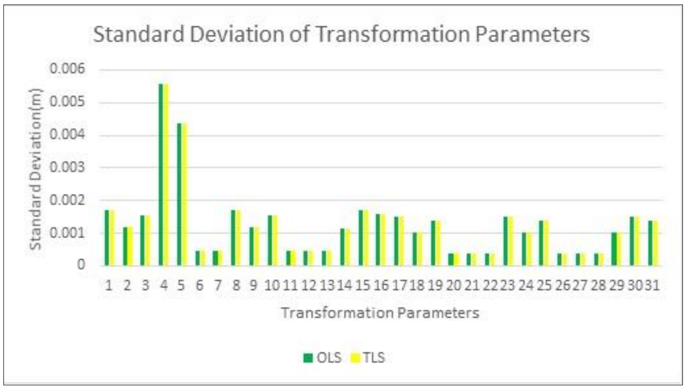

Fig. 5: Standard deviation (m) of all transformation parameters produced by TLS and OLS

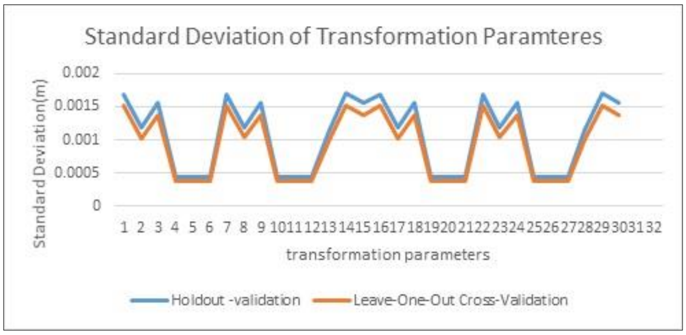

Fig. 6: Standard deviation (m) of all transformation parameters produce from leave-one-out cross-validation and hold-out validation

Standard deviation of transformation parameters produced from holdout and leave-one-out cross-validation methods were plotted in Fig. 6. It can be seen that the standard deviation of transformation parameters produced from holdout method is slightly higher than the leave-one-out cross-validation method.

The standard deviation of transformation parameters were obtained using Bursa-Wolf, Molodensky Badekas and Veis Transformation models with TLS and OLS technique, and holdout and leave-one-out cross validation methods are separately plotted and shown in Fig. 7, Fig. 8 and Fig. 9.

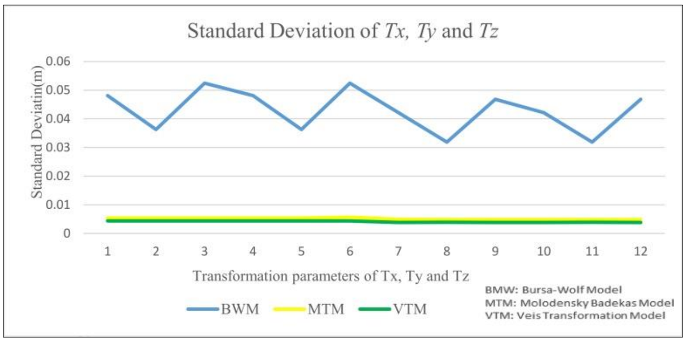

Fig. 7: Standard deviation (m) of translated vector in three axes (T_X, T_Y, T_Z)

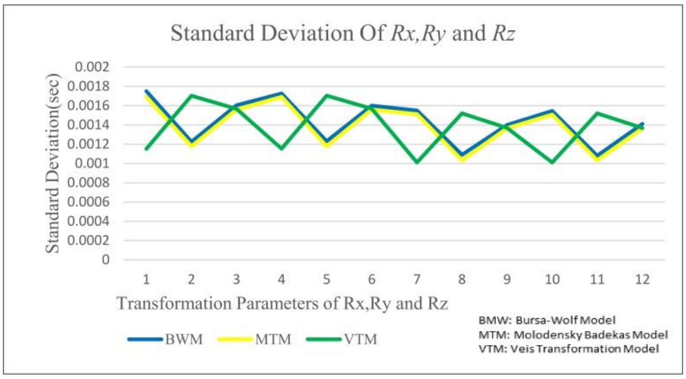

Fig. 8: Standard deviation (seconds) of rotation in three axes (R_X, R_Y, R_Z)

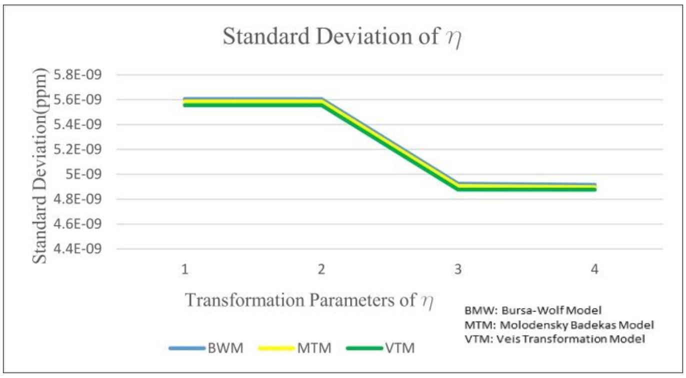

Fig. 9: Standard deviation (ppm) of scale factor (η)

When analyzing these three figures, it is evident that the standard deviation of translated vector of three axes (T_X, T_Y, T_Z) in Bursa-Wolf model is higher than the other two models, and the standard deviations of the rotation of three axes (R_X, R_Y, R_Z) and scale-factor (η) are almost the same for the three models.

Table 3 shows that the average difference in latitude, longitude and height of Everest (1830) $(\phi, \lambda, h)_E$ coordinates that were derived from the Abridged Molodensky Transformation Model and Everest (1830) $(\phi, \lambda, h)_E$ data calculated from transformation parameters published by the Survey Department of Sri Lanka, and the transformation parameters calculated from leave-one-out cross-validation method as shown in Table 2.

Table 3: Coordinate comparison with Abridged Molodensky Transformation Model

		$\Delta\phi^0{}_E$	$\Delta\lambda^0{}_E$	Δh_E(m)
Publish data of SD		0.000386584	0.00090656	5.336104189
Bursa-Wolf Model	OLS	6.09704E-07	3.30821E-11	-0.12932914
	TLS	6.09692E-07	4.582E-11	-0.1278576
Molodensky-Badekas Model	OLS	-9.34247E-13	1.01352E-11	5.54875E-07
	TLS	-9.42565E-13	1.02928E-11	6.63891E-07
Veis Transformation Model	OLS	2.67202E-13	2.75554E-11	1.17697E-09
	TLS	2.59663E-13	2.77122E-11	1.09329E-07

NGA provides transformation parameters from WGS84 to Kandawala (Everest 1830) from the standard Abridged Molodensky Transformation Model, In Table 3, it can be observed that the average residual of Everest 1830 $(\Delta\phi, \Delta\lambda, \Delta h)_E$ calculated from transformation parameters published by the Survey Department are higher than the other three models.

Average residuals of Everest (1830) $(\Delta\phi, \Delta\lambda, \Delta h)_E$ calculated from Bursa-Wolf, Molodensky-Badekas and Veis Transformation Models (Table 3) were plotted separately and depicted in Fig. (10), (11), and (12).

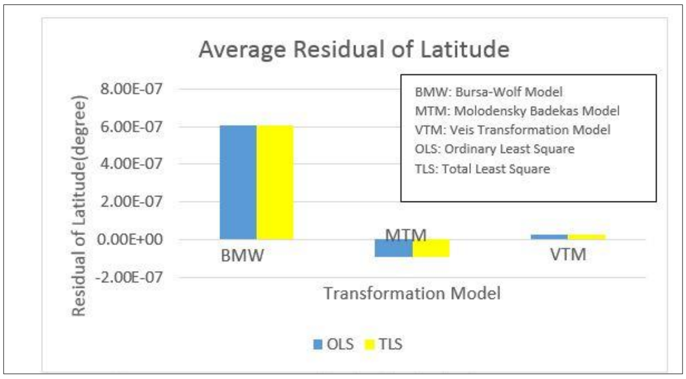

Fig. 10: Average residual of latitude of Everest (1830)

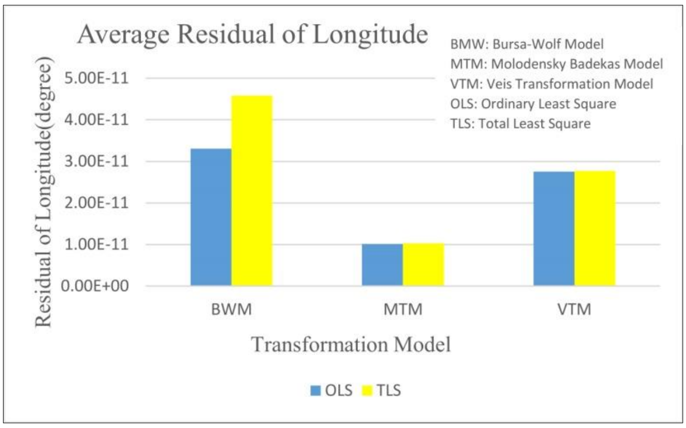

Fig. 11: Average longitude difference of Everest (1830)

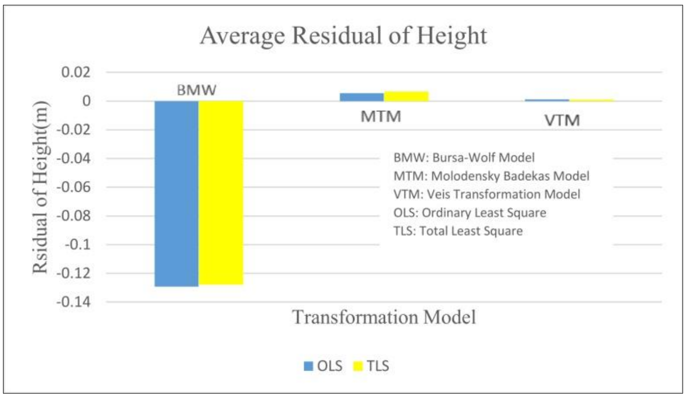

Fig. 12: Average height difference of Everest (1830)

By analyzing the above figures (Fig.10, 11 and 12), it is notable that the average residual of Everest (1830) $(\Delta\phi, \Delta\lambda, \Delta h)_E$ in Bursa-Wolf model is higher than the other two models, while the average residuals of Everest (1830) $(\Delta\phi, \Delta\lambda, \Delta h)_E$ in Molodensky-Badekas and Veis Transformation models are nearly equal.

RMSE showed the degree of fitness of the reverse methods to the existing data. Calculated RMSE values are shown in Table 4.

Table 4: RMSE Statistical Indicator results

Methods	Least-Square Technique	RMSE (m)
Bursa-Wolf Model	OLS	0.0684
	TLS	0.0684
Molodensky-Badekas	OLS	0.0095
	TLS	0.0095
Veis Transformation Model	OLS	0.0093
	TLS	0.0093

It can be concluded from the Table 4 that both OLS and TLS techniques have equal performance, and also Molodensky-Badekas and Veis have better results than the Bursa-Wolf model.

5 Conclusions and Recommendations

Since geodetic datums are diverse as geocentric and non-geocentric, geodetic datum transformation is required and the least squares technique is used as the common adjustment process, among which OLS and TLS are the most commonly used procedures. In the literature, several studies can be found focusing on geodetic networks of their countries to test the efficiency of OLS and TLS methods in coordinate transformation.

In this study, OLS and TLS techniques were used with three near conformal transformation models namely, Bursa-Wolf, Molodensky-Badekas and Veis, and leave–one-out and hold-out cross-validation methods were applied for validating the results. The comparison of the result shows that both the OLS and TLS produce nearly equal transformation results that can be used for the transformation of coordinates in Sri Lankan geodetic reference network. However, the present study results are in contrary to the reported results obtained in other jurisdictions where TLS was the best technique.

Validation is the statistical method of evaluating and comparing learning algorithms by dividing data in two segments; one is used to learn or train a model and the other is used to validate the model. In this study, leave-one-out and holdout cross-validation methods were used to validate the models. It was revealed that the leave-one-out cross-validation method performed better than the holdout validation method. From the comparison of the standard deviations of these three transformation models, it can be said that Molodensky-Badekas and Veis Transformation models have performed better than the Bursa-Wolf model.

NGA provides transformation parameters from WGS84 to Kandawala for the standard Molodensky transformation model. Both Kandawala and SLD99 were used in Everest (1830) ellipsoid and yet, NGA does not provide transformation parameters from WGS84 to SLD99 by using the standard Molodensky transformation model. One of the reasons for it could be that the standard Molodensky transformation model out-performs the SLD99 system which could be conformed from coordinate comparison with Abridged Molodensky Transformation Model in Table 3. It also shows that the calculated result from Molodensky-Badekas and Veis Transformation Model give the least deviation. Therefore, it can be concluded that Molodensky-Badekas or Veis Transformation Model is the best model to drive the transformation parameters in Sri Lankan geodetic network. In view of this, the authors recommend the transformation parameters shown in Table 2 for the coordinate transformation from global to the local SLD99 system.

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Author Contributions

Conceptualization, methodology, analysis, writing - original draft preparation, K.V.C.P. Original concept and methodology, and writing - review and editing, H.M.I.P. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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