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A Financial Engineering of Risk-Free Rate of Interest Determination: An Empirical Measurement and Implications for Life Insurance Valuation Under the Uniform Distribution of Death Assumption

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ABSTRACT

To build confidence in the results of computational actuarial models, a key regulatory requirement is to establish a mathematically dependable process. The method employed for the estimation of interest intensities is analytically involving but only open to avid pricing actuaries. Understanding the investment process is essentially critical as actuaries seek alternative ways to develop new methodologies for the insurance industry. This paper deploys intuitive analytical and numerical representations to describe how interest rate intensities could be estimated for a broad range of configurations. The objective is carry-out an independent investigation in the analysis of nominal interest rate by constructing models for the continuous force of interest applicable in life insurance valuation. The main result consists of a generalisation of Bernoulli polynomial in modelling continuous interest rate intensity and the presentation of a series of analytically rigorous arguments. The central contribution includes the derivation of procedure for determining both lower and upper bound for the interest rate intensity. Given a tolerance limit of $\varepsilon = 0.04\%$, empirical evidence shows that the non-decreasing absolute difference of the estimated force of interest $\left|\delta_{exaxct} - \delta_{estimated}\right| < 0.4\%$. This difference is an indication of fair performance of the model when computing the actuarial present value of expected claims and benefits.

Keywords: Bernoulli Polynomials, Financial Market, Present Value, Force of Interest, Tolerance limit.

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INTRODUCTION

The mathematical solution to the present value problem in the financial market is complex with deep economic implications. Therefore, this paper serves as a pedagogical presentation of the interest rate modelling process to insurance professionals and actuaries to deepen an intuitive knowledge of the concept and the analytically innovative techniques involved. The concept of force of interest currently applied in actuarial science defines a measure of interest which can change continuously across time horizons. It is particularly useful for modelling continuously compounded interest rates, and it is defined as the instantaneous rate of interest. It describes the continuous measure of interest accumulation over time, and it is central to the computation of present and future values in compounding interest frameworks.

Following arguments in Aalaei (2022) and Sari, Deautama, and Febrisutisyanto (2023), the researchers deduce the following implications of the force of interest (i) Again, mortality rates and present value calculations heavily depend on the force of interest hence, it is applied in computing life insurance premiums and reserves. (ii) Investors regularly deploy the force of interest to analyse the returns on investments, particularly in cases where insurance assets are expected to generate continuous returns. (iii) The force of interest can adapt to fluctuating conditions in financial markets allowing for modelling of sophisticated risk and returns.

LITERATURE REVIEW

Recent literature explores short-term interest rates where the force of interest is treated as a random process. Short-term models such as the Vasicek model or the Hull-White model analyse the dynamics of interest rates over time. According to Binsbergen, Diamond and Grotteria (2022) the force of interest is used in many areas of finance for instance bond pricing, insurance and pension planning. However, actuarial literature often examines its implications for risk management and portfolio optimisation. There is ongoing research into how the force of interest relates to yield curves and its implications for predicting economic conditions.

According to Sari, Deautama, and Febrisutisyanto (2023), changes in the force of interest usually signal shifts in monetary policy or economic outlook. With the advent of computational finance, many papers focus on numerical methods to approximate solutions involving the force of interest, particularly in scenarios with complex financial instruments. Some studies in behavioural finance investigate how psychological factors influence perceptions of interest rates and investment decisions, indirectly touching on the concept of the force of interest.

Current actuarial literature assumes constant interest rate intensity when conducting life insurance valuation. Although the application of Taylor's series expansion for the force of interest is very common, closed-form expressions that are universally or reasonably acceptable have not been provided for the estimation of the force of interest. For example, Okam, Garba, Mamu, Oluwaseun and Kanti (2018) deployed the Euler's sequence to study interest rates. Although, the authors proved the Bernoulli inequality, there was no closed form expression for the force of interest or the interest rate intensity.

Actuaries and insurance professionals adopt conservative interest rates a priori to soothe life insurance valuation. This constitutes a problem in the quest for the development of functional forms for the interest rate intensity function $\delta = \ln(1+i)$. This study overcomes this problem by deploying the Euler-Maclaurin series formula. This innovative analytical approach is developed to model the force of interest in line with the valuation of life insurance products.

Alternatively, the force of interest is usually defined through the accumulation function $A(s,t)$ for instance insurance transaction taking place between two different Investment times s and t in future. However, a major problem is that the accumulation function does not have any known explicit, analytically developed models. Therefore, the Euler-Maclaurin series formula will be deployed in creating a market model and testing its proximity to specified functional values. Since the force of interest function is continuous, the estimated values can be employed to generate a functional form in the neighbourhood of a small number to obtain the deviation between the exact value and the estimated functional value.

The derivations obtained here are driven from the theoretical perspectives and are meant to set aside the comparatively intractable process of deterministic interest rate intensity computations. In life contingencies, estimating analytical models have rarely been constructed for the force of interest function $\delta(\xi)$ principally applied in the valuation of life insurance schemes whereas all other key actuarial functions such as $\mu_{x+\xi}$; ${}_{\xi}p_x$; $e^{-\delta}$ v and $\frac{1}{1+i}$ seem to have been numerically modelled.

The problem of obtaining the interest rate at which discounting is performed is of crucial implication in actuarial valuation. The choice of this rate is technically based on the analysis of the financial market state and the accuracy of evaluating the present value of the future economic amounts depends on the right choice of the rate. Following this, the study examines the present value dynamics by applying the power series polynomial to model the interest rate intensity.

In life insurance, definite assumptions have to be made by the life office about the interest rates, mortality rates and other expenses that could be incurred in future year for the purpose of setting the premium rates or assessing the liabilities under the insurance policies. Therefore, it becomes necessary to ensure a correct estimate of these factors is obtained, otherwise the actuarial results deduced from those initial assumptions will not be proximal to the actual experience of the insurer. The insurer depends on the mortality experience of the policy holders observed in the recent past as a basis for approximating the probabilities of survival and death. Consequently, interest rates modelling and determination assume a functional role in setting the premium rates for any life insurance policy.

In long term financial modelling, compound interest is employed to accumulate the principal amount. Following Kellison (1991), Kellison (2009), in the computation of compound interest, the principal amount is not actually considered as the base but the value obtained after the interest computation and after adding the interest to the amount of the balance at the preceding period. The aggregate of the interest and the original base is described as the interest capitalization. The capitalization evolves in line with the following identities.

$$C + Ci = C(1+i) \tag{1}$$

$$C(1+i) + C(1+i)i = C(1+i)(1+i) \tag{2}$$

$$C(1+i) + C(1+i)i = C(1+i)^2 \tag{3}$$

$$C(1+i)^2 + C(1+i)^2 i = C(1+i)^2(1+i) \tag{4}$$

$$C(1+i)^2 + C(1+i)^2 i = C(1+i)^3 \tag{5}$$

$$C(1+i)^{n-1} + C(1+i)^{n-1} i = C(1+i)^n \tag{6}$$

The accumulation with compound interest is defined by

$$S = C(1+i)^n \tag{7}$$

Following McCutcheon and Scott (1986), Howard (1995), Anggraeni, Rahmadani Utama and Handayani (2023), when the terms of financial transaction are announced, the annual interest rate is usually defined, and the number of interest payments per annum is often specified, for instance, as quarterly or monthly. The nominal interest rate is defined by j. Suppose the number of interest payments per annum is m. Then the interest computation will be performed at the rate $\frac{j}{m}$ and the number of payment intervals n years is mn. The accumulated amount is given by

$$S = C\left(1+\frac{i}{m}\right)^{mn} \tag{8}$$

Hence, the nominal rate defines the annual interest rate computation m times a year. The bigger the number m, the faster the process of the principal sum accumulation. To compare the different conditions of interest rate computations at various nominal rates and different number of computations, the effective rate of interest is applied. Consequently, the effective rate of interest is the annual rate of interest calculated once a year which provides the same financial result as m single calculations per year with the use of nominal rate j, hence

$$(1+i) = \left(1+\frac{j}{m}\right)^m \tag{9}$$

where i is the effective interest rate, hence

$$i = \left(1+\frac{j}{m}\right)^m - 1 \tag{10}$$

The exchange of the nominal rate j at interest calculation m times a year for the effective rate consistent with formula (10) will not invalidate the financial obligations of either party. In financial mathematics, the continuous nominal interest rate as it applies to life insurance underwriting can be modelled through sophisticated numerical tools such as Bernoulli polynomials.

The Continuously Compounded Interest Rate

Suppose the initial capital sum is invested for s years at an annual rate i where the interest rate is compounded n times per year, then the interest rate of each conversion period is $\frac{i}{n}$, consequently, there will be $n \times s$ conversion periods in all. Therefore, the compounded amount at the end of s years become.

$$a = C\left(1+\frac{i}{n}\right)^{ns} \tag{11}$$

It is observed that when the interest rate is compounded more frequently, the compounded amount becomes very large for a fixed sum capital, duration and annual interest rate. Although the compounded amount increases indefinitely as the frequency of compounding increases, there exists an

upper bound on the compounded interest.
In this case, the amount a is defined by,

$$a=\lim_{n\to\infty}\left(C\left(1+\frac{i}{n}\right)^{ns}\right) \tag{12}$$

$$\Rightarrow a=C\lim_{n\to\infty}\left(\left(1+\frac{i}{n}\right)^{ns}\right) \tag{13}$$

$$\Rightarrow a=C\lim_{n\to\infty}\left(\left(1+\frac{i}{n}\right)^{\frac{n}{i}\times is}\right) \tag{14}$$

$$\Rightarrow a=C\lim_{n\to\infty}\left(\left(1+\frac{i}{n}\right)^{\frac{n}{i}}\right)^{is} \tag{15}$$

$$\Rightarrow a=C\left(\lim_{n\to\infty}\left(1+\frac{i}{n}\right)^{\frac{n}{i}}\right)^{is} \tag{16}$$

$$\Rightarrow a=C\left(\lim_{n\to\infty}\left(1+\frac{1}{\left(\frac{n}{i}\right)}\right)^{\frac{n}{i}}\right)^{is} \tag{17}$$

Using the substitution $m=\frac{n}{i}$, observe that $m\to\infty$ as $n\to\infty$. Therefore

$$a=C\left(\lim_{n\to\infty}\left(1+\frac{1}{(m)}\right)^{m}\right)^{is} \tag{18}$$

$$\lim_{m\to\infty}\left(1+\frac{i}{m}\right)^{m}=e \tag{19}$$

$$\Rightarrow a=Ce^{is} \tag{20}$$

Bernoulli Power Series

In actuarial mathematics domain, the Bernoulli polynomials represent a generalisation of the Bernoulli numbers in the modelling of interest rates. According to Tuenter (2001), Si (2019) and Apostol (2008), the Bernoulli polynomials have the following representations. Given that $n \geq 1$ and

$$(m)_n = m(m-1)(m-2)(m-3)...(m-n+1) \tag{20a}$$

$$(m)_0 = 1 \tag{20b}$$

$$\sum_{m=0}^{\infty} \frac{B_m(y)x^m}{(m)_n} = \frac{xe^{xy}}{\sum_{u=1}^{\infty} e^{-ux}} \tag{21}$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{B_m(y)x^m}{(m)_n} = \frac{xe^{xy}}{e^x - 1} \tag{22}$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{B_m(y)x^m}{(m)_n} = \frac{x}{e^x - 1} \times e^{xy} \tag{23}$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{B_m(y)x^m}{(m)_n} = \left(\sum_{m=0}^{\infty} \frac{B_m x^m}{(m)_n}\right)\left(\sum_{m=0}^{\infty} \frac{(xy)^m}{(m)_n}\right) \tag{24}$$

$$\sum_{m=0}^{\infty} \frac{B_m(y)x^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(xy)^{m-n}}{(m-n)!}\left(\frac{B_n x^n}{n!}\right) \tag{25}$$

$$\sum_{m=0}^{\infty} \frac{B_m(y)x^m}{m!} = \sum_{m}^{\infty} \sum_{n=0}^{m} \frac{y^{m-n} B_m}{(m-n)!n!} x^m \tag{26}$$

$$\sum_{m=0}^{\infty}\frac{B_m(y)x^m}{m!}=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n}y^{m-n}B_n\frac{x^m}{m!} \tag{27}$$

Comparing the terms in x at the left and right-hand summations, we obtain.

$$B_m(y)=\sum_{n=0}^{m}\binom{m}{n}B_n y^{m-n} \tag{28}$$

Applying these recurrence relations, we obtain

$$B_0(y)=1 \tag{29}$$

$$B_1(y)=y-\frac{1}{2} \tag{30}$$

$$B_2(y)=y^2-y+\frac{1}{6} \tag{31}$$

$$B_3(y)=y^3-\frac{3}{2}y^2+\frac{1}{2}y \tag{32}$$

$$B_4(y)=y^4-2y^3+y^2-\frac{1}{30} \tag{33}$$

$$B_5(y)=y^5-\frac{5}{2}y^4+\frac{5}{3}y^3-\frac{1}{6}y \tag{34}$$

$$B_6(y)=y^6-3y^5+\frac{5}{2}y^4-\frac{1}{2}y^2+\frac{1}{42} \tag{35}$$

Observe that the polynomials are of the form

$$B_n(y)=\sum_{k=0}^{n}C_n^k B_k y^{n-k} \tag{36}$$

$$\frac{d}{dy}B_n(y)=nB_{n-1}(y) \tag{37}$$

$$\frac{d}{dy}B_n(y)=\sum_{k=0}^{n}C_n^k(n-k)B_k y^{n-k-1} \tag{38}$$

$$\frac{d}{dy}B_n(y)=n\sum_{k=0}^{n-1}C_{n-1}^k B_k y^{n-k-1} \tag{39}$$

$$\frac{d}{dy}B_n(y)=nB_{n-1}(y) \tag{40}$$

Evaluating the values at $y=0$ yields the rational Bernoulli numbers

$$B_0(0)=1 \tag{41}$$

$$B_1(0)=-\frac{1}{2} \tag{42}$$

$$B_2(0)=\frac{1}{6} \tag{43}$$

$$B_3(0)=0 \tag{44}$$

$$B_4(0)=-\frac{1}{30} \tag{45}$$

$$B_5(0)=0 \tag{46}$$

$$B_6(0)=\frac{1}{42} \tag{47}$$

For $0<y<1$,

$$B_1(y)=y-\frac{1}{2} \tag{48}$$

The Fourier series of $B_1(y)$ can be computed as

$$B_1(y)=y-\frac{1}{2}=\frac{-1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2\pi kx}{k} \tag{49}$$

Since the $B_n(y)$ are analytic, their Fourier series converge and hence can be integrated to obtain

$$B_{2n}(y)=(-1)^{n+1}\frac{2(2n)!}{(2n)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2\pi ky}{k^{2n}} \tag{50}$$

$$B_{2n+1}(y)=(-1)^{n+1}\frac{2(2n+1)!}{(2n)^{2n+1}}\sum_{k=1}^{\infty}\frac{\cos 2\pi ky}{k^{2n+1}} \tag{51}$$

By combining these two equations for $0<y<1$, we have

$$B_n(y)=\frac{-(n!)}{(2\pi i)^n}\sum_{k\neq 0}\frac{e^{2\pi iky}}{k^n} \tag{52}$$

$B_n(y)$ is bounded as $n \to \infty$ and $|B_n(y)| \to 0$ uniformly on the unit interval. The objective of this paper is to carry out an independent investigation into the analysis of nominal interest rate by constructing models for the force of interest applicable in life insurance valuation.

The main result consists of a generalisation of the Bernoulli polynomial in modelling continuous interest rate intensity and the presentation of a series of mathematically rigorous arguments. The central contribution includes the derivation of a procedure for determining both lower and upper bounds for the interest rate intensity.

Currently extant literatures, such as Kellison (1991) and Kellison (2009) adopted Taylor's expansion as an infinite series to model the force of interest $\log_e(1+i)$ without obtaining any closed-form expression, and this accounts for why we deviate from this line of reasoning.

Bernoulli polynomials exhibit interest rates periodic behaviour which makes them particularly useful for estimating periodic interest rate intensities functions. In contrast, Taylor series are typically centred on a point and may not capture the periodicity behaviour well. For sums of powers of interest rates intensities, Bernoulli polynomials provide closed-form expressions, allowing more efficient computations compared to using Taylor series, which often require too many terms to attain accuracy. While Bernoulli polynomials can provide a better global approximation for the interest rate intensity functions, Taylor series offer local approximations around a point.

Bernoulli polynomials facilitate the numerical integration of life insurance benefit functions, especially in the context of the Euler-Maclaurin formula, making it easier to compute intractable benefits integrals and evaluate sums. For certain life annuity problems, especially in life contingencies, Bernoulli polynomials can reduce the number of computations required compared to using a higher-order Taylor series. Bernoulli polynomials may converge faster than the Taylor series for specific life insurance functions. Bernoulli polynomials are closely linked to immediate annuity-theoretic identities, offering additional insights that are not apparent from Taylor series alone. These advantages make Bernoulli polynomials a powerful tool in financial modelling, particularly in contexts where summation is required.

RESEARCH METHODS

The death density is defined as

$$f_{T_x}(\xi) = \frac{d}{d\xi}\left(-{}_{\xi}p_x\right) \tag{52a}$$

$${}_{\xi}p_x\mu_{x+\xi} = \frac{d}{d\xi}\mathbf{P}\left(T_x = \xi\right) \tag{52b}$$

$${}_{\xi}q_x = \xi \times q_x \tag{52c}$$

$${}_{\xi}p_x\mu_{x+\xi} = \frac{d}{d\xi}\left({}_{\xi}q_x\right) = \frac{d}{d\xi}\left(\xi \times q_x\right) = q_x \tag{52d}$$

Replacing x by $x+\kappa$ in (52d) to get

$${}_{\xi}p_{x+\kappa}\mu_{x+\kappa+\xi} = q_{x+\kappa} \tag{52e}$$

The continuous whole life insurance function implies that

$$\overline{A}_x = \int_0^{\Omega-x} e^{-\delta\xi}\left({}_{\xi}p_x\right)\mu_{x+\xi}d\xi = \sum_{\kappa=0}^{\infty}\left(\int_{\kappa}^{\kappa+1} e^{-\delta\xi}\left({}_{\xi}p_x\right)\mu_{x+\xi}d\xi\right) \tag{52f}$$

The uniform distribution of death (UDD) assumption implies that the whole life insurance function for a life aged x becomes

$$\overline{A}_x = \sum_{\kappa=0}^{\infty}\left({}_{\kappa}p_x\right)v^{\kappa+1}e^{\delta}\int_0^1 e^{-\delta\xi}\left({}_{\xi}p_{x+\kappa}\right)\mu_{x+\kappa+\xi}d\xi \tag{52g}$$

Apply (52e) to the integrand in (52g) to get

$$\overline{A}_x = \sum_{\kappa=0}^{\infty}v^{\kappa+1}\left({}_{\kappa}p_x\right)q_{x+\kappa}e^{\delta}\int_0^1 e^{-\delta\xi}d\xi = \sum_{\kappa=0}^{\infty}v^{\kappa+1}{}_{\kappa|1}q_x e^{\delta}\int_0^1 e^{-\delta\xi}d\xi \tag{52h}$$

$$\overline{A}_x = A_x \times e^{\delta}\int_0^1 e^{-\delta\xi}d\xi = e^{\delta}\left[\frac{1}{\delta} - \frac{e^{\delta}}{\delta}\right] = \frac{e^{\delta}-1}{\delta} \tag{52i}$$

$$\frac{\overline{A}_x}{A_x} = \frac{e^{\delta}-1}{\delta} \Rightarrow \frac{A_x}{\overline{A}_x} = \frac{\delta}{e^{\delta}-1} \tag{52j}$$

Observe the relation,

$$\delta \times \bar{a}_{\overline{m|}} = i \times a_{\overline{m|}} \tag{52k}$$

Therefore

$$\frac{m}{a_{\overline{m|}}} = \frac{m \times i}{\left(a_{\overline{m|}}\right) \times i} = \frac{m\left(e^{\delta}-1\right)}{\left(\bar{a}_{\overline{m|}}\right) \times \delta} = \frac{m}{\left(\bar{a}_{\overline{m|}}\right)} \times \frac{\left(e^{\delta}-1\right)}{\delta} \tag{52l}$$

$$\frac{m}{a_{\overline{m|}}} = \frac{m}{\bar{a}_{\overline{m|}}} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} = \frac{m}{\bar{a}_{\overline{m|}}} \times \frac{\bar{A}_x}{A_x} \tag{52m}$$

$$\frac{m}{a_{\overline{m|}}} = \frac{m}{\bar{a}_{\overline{m|}}} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} = \frac{m}{\left\{\frac{1-e^{-m\delta}}{\delta}\right\}} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} = \frac{m\delta}{\left(1-e^{-m\delta}\right)} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} \tag{52n}$$

$$= \frac{\bar{A}_x}{A_x} \times \frac{-m\delta}{\left(e^{-m\delta}-1\right)}$$

The fourth term in (52n) can be expressed as

$$\frac{m\delta}{\left(1-e^{-m\delta}\right)} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} = \frac{-m\delta}{\left(e^{-m\delta}-1\right)} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} = \frac{1}{\left\{\frac{\delta}{e^{\delta}-1}\right\}} \times \left\{\frac{-m\delta}{e^{-m\delta}-1}\right\} \tag{52o}$$

Equation (52o) is now in the form of Bernoulli polynomial and the expansions are given in equation (53)

$$\frac{1}{\left\{\frac{\delta}{e^{\delta}-1}\right\}} \times \left\{\frac{-m\delta}{e^{-m\delta}-1}\right\} =$$

$$\frac{\left\{\begin{aligned}&1+\frac{1}{2}m\delta+\frac{1}{12}m^2\delta^2-\frac{1}{720}m^4\delta^4+\frac{1}{30240}m^6\delta^6-\frac{1}{1209600}m^8\delta^8\\&+\frac{1}{47900160}m^{10}\delta^{10}+\ldots\end{aligned}\right\}}{\left\{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2-\frac{1}{720}\delta^4+\frac{1}{30240}\delta^6+\frac{1}{1209600}\delta^8-\frac{1}{47900160}\delta^{10}+\ldots\right\}} \tag{53}$$

The expression in the left hand-side of (54) can be expanded as follows:

$$\frac{\delta}{e^{\delta}-1}=1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}+\frac{1}{30240}\delta^{6}+\frac{1}{1209600}\delta^{8}+\frac{1}{47900160}\delta^{10}+\ldots \tag{54}$$

Using the quartic estimation, the fifth term and beyond in the numerator and denominator can be ignored to obtain the following relations.

$$\frac{m}{a_{\overline{m|}}}=\frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^{2}\delta^{2}-\frac{1}{720}m^{4}\delta^{4}}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}} \tag{55}$$

Subtract 1 from both sides to get

$$\frac{m}{a_{\overline{m|}}}-1=\frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^{2}\delta^{2}-\frac{1}{720}m^{4}\delta^{4}}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}}-1 \tag{56}$$

$$\Rightarrow\frac{m}{a_{\overline{m|}}}-1=\frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^{2}\delta^{2}-\frac{1}{720}m^{4}\delta^{4}-\left(1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}\right)}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}} \tag{57}$$

$$\Rightarrow\frac{m}{a_{\overline{m|}}}-1=\frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^{2}\delta^{2}-\frac{1}{720}m^{4}\delta^{4}-1+\frac{1}{2}\delta-\frac{1}{12}\delta^{2}+\frac{1}{720}\delta^{4}}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}} \tag{58}$$

Let $\alpha=\frac{m}{a_{\overline{m|}}}-1$;

then

$$\alpha=\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}} \tag{58a}$$

$$\Rightarrow \alpha = \frac{\frac{1}{2}m\delta + \frac{1}{12}m^2\delta^2 - \frac{1}{720}m^4\delta^4 + \frac{1}{2}\delta - \frac{1}{12}\delta^2 + \frac{1}{720}\delta^4}{1 - \frac{1}{2}\delta + \frac{1}{12}\delta^2 - \frac{1}{720}\delta^4} \tag{59}$$

$$\begin{aligned} &\Rightarrow \alpha\left(1 - \frac{1}{2}\delta + \frac{1}{12}\delta^2 - \frac{1}{720}\delta^4\right) \\ &= \frac{1}{2}m\delta + \frac{1}{12}m^2\delta^2 - \frac{1}{720}m^4\delta^4 + \frac{1}{2}\delta - \frac{1}{12}\delta^2 + \frac{1}{720}\delta^4 \end{aligned} \tag{60}$$

$$\begin{aligned} &\Rightarrow \alpha - \frac{1}{2}\alpha\delta + \frac{1}{12}\alpha\delta^2 - \frac{1}{720}\alpha\delta^4 \\ &= \frac{1}{2}m\delta + \frac{1}{12}m^2\delta^2 - \frac{1}{720}m^4\delta^4 + \frac{1}{2}\delta - \frac{1}{12}\delta^2 + \frac{1}{720}\delta^4 \end{aligned} \tag{61}$$

$$\begin{aligned} &\Rightarrow \frac{1}{720}m^4\delta^4 - \frac{1}{720}\alpha\delta^4 - \frac{1}{720}\delta^4 - \frac{1}{12}m^2\delta^2 \\ &+ \frac{1}{12}\alpha\delta^2 + \frac{1}{12}\delta^2 - \frac{1}{2}m\delta - \frac{1}{2}\alpha\delta - \frac{1}{2}\delta + \alpha = 0 \end{aligned} \tag{62}$$

$$\begin{aligned} &\Rightarrow m^4\delta^4 - \alpha\delta^4 - \delta^4 - 60m^2\delta^2 + 60\alpha\delta^2 \\ &+ 60\delta^2 - 360m\delta - 360\alpha\delta - 360\delta + 720\alpha = 0 \end{aligned} \tag{63}$$

$$\begin{aligned} &\Rightarrow \left(m^4 - \alpha - 1\right)\delta^4 - \left(60m^2 - 60\alpha - 60\right)\delta^2 \\ &- \left(360m + 360\alpha + 360\right)\delta + 720\alpha = 0 \end{aligned} \tag{64}$$

Substitute (58a) into (64) to get

$$\begin{aligned} &\Rightarrow \left(m^4 - \left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) - 1\right)\delta^4 - \left(60m^2 - 60\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) - 60\right)\delta^2 \\ &- \left(360m + 360\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) + 360\right)\delta + 720\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) = 0 \end{aligned} \tag{65}$$

This equation (64) is of the form;

$$A\delta^4 + B\delta^3 + C\delta^2 + D\delta + E = 0, \tag{66}$$

where

$A = \left(m^4 - \alpha - 1\right)$; $B = 0$; $C = -\left(60m^2 - 60\alpha - 60\right)$;

$D = -\left(360m + 360\alpha + 360\right)$; $E = 720\alpha$

The transformation is used to arrive at the following depressed expression for positive integral power of δ ;

$$\delta = \gamma - \frac{B}{4A} \tag{67}$$

$$\begin{aligned}\delta^2 &= \left(\gamma - \frac{B}{4A}\right)\left(\gamma - \frac{B}{4A}\right) = \gamma\left(\gamma - \frac{B}{4A}\right) - \frac{B}{4A}\left(\gamma - \frac{B}{4A}\right) \\ &= \left(\gamma^2 - \frac{B}{4A}\gamma\right) - \left(\frac{B}{4A}\gamma - \frac{B^2}{16A^2}\right)\end{aligned} \tag{68}$$

$$\Rightarrow \delta^2 = \left(\gamma - \frac{B}{4A}\right)\left(\gamma - \frac{B}{4A}\right) = \gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2} \tag{69}$$

$$\Rightarrow \delta^2 = \left(\gamma - \frac{B}{4A}\right)\left(\gamma - \frac{B}{4A}\right)^2 = \left(\gamma - \frac{B}{4A}\right)\left(\gamma^2 - \frac{\gamma B}{2A} + \frac{B^2}{16A^2}\right) \tag{70}$$

$$\Rightarrow \delta^3 = \gamma\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right) - \frac{B}{4A}\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right) \tag{71}$$

$$\Rightarrow \delta^3 = \left(\gamma^3 - \frac{2B}{4A}\gamma^2 + \frac{B^2}{16A^2}\gamma\right) + \left(-\frac{B}{4A}\gamma^2 + \frac{2B^2}{16A^2}\gamma - \frac{B^3}{64A^3}\right) \tag{72}$$

$$\Rightarrow \delta^3 = \gamma^3 - \frac{3B}{4A}\gamma^2 + \frac{3B^2}{16A^2}\gamma - \frac{B^3}{64A^3} \tag{73}$$

$$\Rightarrow \delta^4 = \left(\gamma - \frac{B}{4A}\right)^2 \left(\gamma - \frac{B}{4A}\right)^2$$
$$= \left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right)\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right) \tag{74}$$

$$\Rightarrow \delta^4 = \gamma^2\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right) - \frac{2B}{4A}\gamma\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right)$$
$$+\frac{B^2}{16A^2}\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right) + \tag{75}$$

$$\Rightarrow \delta^4 = \gamma^4 - \frac{2B}{4A}\gamma^3 + \frac{B^2}{16A^2}\gamma^2 - \frac{2B}{4A}\gamma^3 + \frac{4B^2}{16A^2}\gamma^2 - \frac{2B^3}{64A^3}\gamma$$
$$+\frac{B^2}{16A^2}\gamma^2 - \frac{2B^3}{64A^3}\gamma + \frac{B^4}{256A^4} \tag{76}$$

$$\Rightarrow \delta^4 = \gamma^4 - \frac{4B}{4A}\gamma^3 + \frac{6B^2}{16A^2}\gamma^2 - \frac{4B^3}{64A^3}\gamma + \frac{B^4}{256A^4} \tag{77}$$

Substitute expressions (77), (73), (70), (67) in (66) to obtain

$$A\left(\gamma^4 - \frac{4B}{4A}\gamma^3 + \frac{6B^2}{16A^2}\gamma^2 - \frac{4B^3}{64A^3}\gamma + \frac{B^4}{256A^4}\right)$$
$$+B\left(\gamma^3 - \frac{3B}{4A}\gamma^2 + \frac{3B^2}{16A^2}\gamma - \frac{B^3}{64A^3}\right) \tag{78}$$
$$+C\left(\gamma^2 - \frac{2B}{4A}\gamma + \frac{B^2}{16A^2}\right) + D\left(\gamma - \frac{B}{4A}\right) + E = 0$$

$$\Rightarrow A\gamma^4 - \frac{4B}{4}\gamma^3 + \frac{6B^2}{16A}\gamma^2 - \frac{4B^3}{64A^2}\gamma + \frac{B^4}{256A^3} + B\gamma^3 - \frac{3B^2}{4A}\gamma^2$$
$$+\frac{3B^3}{16A^2}\gamma - \frac{B^4}{64A^3} + C\gamma^2 - \frac{2BC}{4A}\gamma + \frac{B^2C}{16A^2} + D\gamma - D\frac{B}{4A} + E = 0 \tag{79}$$

$$\Rightarrow A\gamma^4+\left(C-\frac{3B^2}{8A}\right)\gamma^2+\left(D-\frac{BC}{2A}+\frac{B^3}{8A^2}\right)\gamma-\frac{3B^4}{256A^3}$$
$$+\frac{B^2C}{16A^2}-\frac{BD}{4A}+E=0 \tag{80}$$

Divide through by A to obtain the depressed form of quartic equation:

$$\gamma^4+\left(\frac{C}{A}-\frac{3B^2}{8A^2}\right)\gamma^2+\left(\frac{D}{A}-\frac{BC}{2A^2}+\frac{B^3}{8A^3}\right)\gamma$$
$$+\left(\frac{E}{A}-\frac{BD}{4A^2}+\frac{B^2C}{16A^3}-\frac{3B^4}{256A^4}\right)=0 \tag{81}$$

$$\Rightarrow \gamma^4+P\gamma^2+Q\gamma+R=0 \tag{82}$$

where

$$P=\left(\frac{C}{A}-\frac{3B^2}{8A^2}\right) \tag{83}$$

$$Q=\left(\frac{D}{A}-\frac{BC}{2A^2}+\frac{B^3}{8A^3}\right) \tag{84}$$

$$R=\left(\frac{E}{A}-\frac{BD}{4A^2}+\frac{B^2C}{16A^3}-\frac{3B^4}{256A^4}\right) \tag{85}$$

Equation (82) can be solved using standard methods in Okoli, Oraekie, and Okeke (2019) and Prodanov (2021) assuming the existence of unknown parameters U,V,W such that the zeroes of the auxiliary function

$$h(\gamma,U,V,W)=(V\gamma+W)^2-(\gamma^2+U)^2 \tag{86}$$

solves the quartic equation of the form

$$A\delta^4+B\delta^3+C\delta^2+D\delta+E=0 \tag{87}$$

with $\delta=\gamma-\frac{B}{4A}$ and $B=0$

It follows that

$$h(\gamma,U,V,W)=0\Rightarrow(V\gamma+W)^2-(\gamma^2+U)^2=0 \tag{88}$$

$$(V\gamma+W)^2-(\gamma^2+U)^2=0\Rightarrow(V\gamma+W)^2=(\gamma^2+U)^2 \tag{89}$$

Consequently

Use the $+$ on the right-hand sides of (90) to get

$$\gamma^2+U=\pm(V\gamma+W) \tag{90}$$

$$\gamma^2+U=+V\gamma+W\Rightarrow\gamma^2-V\gamma+U-W=0 \tag{91}$$

$$\gamma^2-V\gamma+U-W=0\Rightarrow\gamma=\frac{V\pm\sqrt{V^2-4(U-W)}}{2} \tag{92}$$

$$\gamma_1=\frac{V-\sqrt{V^2-4(U-W)}}{2};\gamma_2=\frac{V+\sqrt{V^2-4(U-W)}}{2} \tag{93}$$

Next use the $-$ sign on the right-hand sides of (90) to get

$$\gamma^2+U=-(V\gamma+W)=-V\gamma-W \tag{94}$$

$$\Rightarrow\gamma^2+V\gamma+U+W=0 \tag{95}$$

$$\Rightarrow\gamma=\frac{-V\pm\sqrt{V^2-4(U+W)}}{2} \tag{96}$$

$$\Rightarrow\gamma_3=\frac{-V-\sqrt{V^2-4(U+W)}}{2};\gamma_4=\frac{-V+\sqrt{V^2-4(U+W)}}{2} \tag{97}$$

$\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ is the solution set to the depressed form

$$\gamma^4+P\gamma^2+Q\gamma+R=0 \tag{98}$$

Recall equation (89)

$$(V\gamma+W)^2-(\gamma^2+U)^2=0 \tag{99}$$

Equation (99) can be expanded to get

$$V^2\gamma^2 + 2V\gamma W + W^2 - \gamma^4 - 2\gamma^2 U - U^2 = 0 \tag{100}$$

Multiply through by -1 to obtain

$$-V^2\gamma^2 - 2V\gamma W - W^2 + \gamma^4 + 2\gamma^2 U + U^2 = 0 \tag{101}$$

$$\Rightarrow \gamma^4 + \left(2U - V^2\right)\gamma^2 - 2VW\gamma + U^2 - W^2 = 0 \tag{102}$$

Comparing equations (82) and (102) yields

$$P = \left(2U - V^2\right) \Rightarrow U = \frac{P + V^2}{2} \tag{103}$$

$$Q = -2VW \Rightarrow W = -\frac{Q}{2V} \tag{103a}$$

$$R = U^2 - W^2 \Rightarrow R = \left(\frac{P + V^2}{2}\right)^2 - \left(-\frac{Q}{2V}\right)^2 \tag{104}$$

$$\Rightarrow R = \frac{P^2 + 2PV^2 + V^4}{4} - \frac{Q^2}{4V^2} \tag{105}$$

$$\Rightarrow 4V^2\left(\frac{P^2 + 2PV^2 + V^4}{4}\right) - 4V^2 \times \frac{Q^2}{4V^2} = 4V^2 \times R \tag{106}$$

$$\Rightarrow V^6 + 2PV^4 + P^2V^2 - 4RV^2 - Q^2 = 0 \tag{107}$$

$$\Rightarrow \left(V^2\right)^3 + 2P\left(V^2\right)^2 + \left(P^2 - 4R\right)V^2 - Q^2 = 0 \tag{108}$$

Letting

$$V^2 = T \tag{109}$$

$$\Rightarrow T^3 + 2PT^2 + \left(P^2 - 4R\right)T - Q^2 = 0 \tag{110}$$

T can be solved for to obtain V

Equations (103), (103a) and (104) yields

$$U+W=\frac{P+V^2}{2}-\frac{Q}{2V}=\frac{VP+V^3-Q}{2V} \tag{111}$$

$$U-W=\frac{P+V^2}{2}+\frac{Q}{2V}=\frac{VP+V^3+Q}{2V} \tag{112}$$

Then using equations (93) and (97) we have partially solved equation (82) as follows

$$\gamma_1=\frac{V-\sqrt{V^2-4(U-W)}}{2}\Rightarrow\gamma_1=\frac{V-\sqrt{V^2-4\left(\frac{VP+V^3+Q}{2V}\right)}}{2} \tag{113}$$
$$=\frac{V}{2}-\sqrt{\frac{V^2}{4}-\left(\frac{VP+V^3+Q}{2V}\right)}$$

$$\gamma_2=\frac{V+\sqrt{V^2-4(U-W)}}{2}\Rightarrow\gamma_2=\frac{V+\sqrt{V^2-4\left(\frac{VP+V^3+Q}{2V}\right)}}{2} \tag{114}$$
$$=\frac{V}{2}+\sqrt{\frac{V^2}{4}-\left(\frac{VP+V^3+Q}{2V}\right)}$$

$$\gamma_3=\frac{-V-\sqrt{V^2-4(U+W)}}{2}\Rightarrow\gamma_3=\frac{V-\sqrt{V^2-4\left(\frac{VP+V^3-Q}{2V}\right)}}{2} \tag{115}$$
$$=-\frac{V}{2}-\sqrt{\frac{V^2}{4}-\left(\frac{VP+V^3-Q}{2V}\right)}$$

$$\gamma_4 = \frac{-V+\sqrt{V^2-4\left(U+W\right)}}{2} \Rightarrow \gamma_4 = \frac{-V+\sqrt{V^2-4\left(\dfrac{VP+V^3-Q}{2V}\right)}}{2} \tag{116}$$

$$= -\frac{V}{2}+\sqrt{\frac{V^2}{4}-\left(\frac{VP+V^3-Q}{2V}\right)}$$

$$P = \left(\frac{C}{A}\right) \Rightarrow P = \frac{-\left(60m^2-60\alpha-60\right)}{\left(m^4-\alpha-1\right)} \tag{117}$$

$$Q = \left(\frac{D}{A}\right) \Rightarrow Q = \frac{-\left(360m+360\alpha+360\right)}{\left(m^4-\alpha-1\right)} \tag{118}$$

$$R = \frac{720\alpha}{\left(m^4-\alpha-1\right)} \tag{119}$$

Since $B=0$ in (66), then the equations (66) and (82) have the same partial solutions and $\delta \equiv \gamma$

We need to solve the cubic equation (110) in T completely in order to solve equation (82) completely

$$T^3+2PT^2+\left(P^2-4R\right)T-Q^2=0 \tag{120}$$

Let

$$T = y-\frac{b}{3a} \Rightarrow T = y-\frac{2P}{3} \tag{121}$$

$$T = \left(y-\frac{2P}{3}\right) \tag{122}$$

$$\Rightarrow T^2 = \left(y-\frac{2P}{3}\right)\left(y-\frac{2P}{3}\right) = y\left(y-\frac{2P}{3}\right)-\frac{2P}{3}\left(y-\frac{2P}{3}\right) \tag{123}$$

$$\Rightarrow T^2 = \left(y^2-\frac{2P}{3}y\right)-\left(\frac{2P}{3}y-\frac{4P^2}{9}\right) \tag{124}$$

$$\Rightarrow T^2 = y^2 - \frac{4P}{3}y + \frac{4P^2}{9} \tag{125}$$

$$\Rightarrow T^3 = \left(y - \frac{2P}{3}\right)\left(y^2 - \frac{4P}{3}y + \frac{4P^2}{9}\right) \tag{126}$$

$$\Rightarrow T^3 = \left(y^3 - \frac{4P}{3}y^2 + \frac{4P^2}{9}y\right) - \left(\frac{2P}{3}y^2 - \frac{8P^2}{9}y + \frac{8P^3}{27}\right) \tag{127}$$

$$\Rightarrow T^3 = y^3 - \frac{6P}{3}y^2 + \frac{12P^2}{9}y - \frac{8P^3}{27} \tag{128}$$

Substituting (122), (125), (128) into equation (120) yields

$$\begin{aligned} &y^3 - \frac{6P}{3}y^2 + \frac{12P^2}{9}y - \frac{8P^3}{27} + 2P\left(y^2 - \frac{4P}{3}y + \frac{4P^2}{9}\right) \\ &+\left(P^2 - 4R\right)\left(y - \frac{2P}{3}\right) - Q^2 = 0 \end{aligned} \tag{129}$$

$$\begin{aligned} &\Rightarrow y^3 + \frac{12P^2}{9}y - \frac{24P^2}{9}y + P^2 y - 4Ry - \frac{8P^3}{27} \\ &+\frac{8P^3}{9} - \frac{2P^3}{3} + \frac{8RP}{3} - Q^2 = 0 \end{aligned} \tag{130}$$

Leading to the depressed cubic equation

$$y^3 - \left(\frac{3P^2}{9} + 4R\right)y + \left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) = 0 \tag{131}$$

Following Simakov (2011), Okoli, Laisin, Nsiegbe and Eze, (2020), we assume the existence of unknown parameters U, V, W such that the zeros of the auxiliary function

$$H\left(\gamma, U, V, W\right) = \left(Vy + W\right)^3 - \left(y + U\right)^3 \tag{131a}$$

solve the cubic equation (120) where $V^2 = T$ and

$$T = \left(y - \frac{2P}{3}\right) \tag{132}$$

Note that

$$H\left(\gamma, U, V, W\right)=0 \tag{133}$$

$$\Rightarrow\left(Vy+W\right)^{3}-\left(y+U\right)^{3}=0 \tag{134}$$

$$\Rightarrow\left(Vy+W\right)^{3}=\left(y+U\right)^{3}\Rightarrow\left(Vy+W\right)=\left(y+U\right) \tag{135}$$

$$\Rightarrow\left(V-1\right)y=U-W \tag{136}$$

Therefore, the solution to the depressed equation (131) are obtained using

$$y=\frac{\left(U-W\right)}{\left(V-1\right)} \tag{137}$$

Expanding equation (134),

$$\left(Vy+W\right)^{3}=\left(Vy+W\right)\left(Vy+W\right)^{2}=\left(Vy+W\right)\left(V^{2}y^{2}+W^{2}+2VyW\right) \tag{138}$$

$$\Rightarrow\left(Vy+W\right)^{3}=V^{3}y^{3}+3W^{2}Vy+3V^{2}y^{2}W+W^{3} \tag{139}$$

Similarly

$$\left(y+U\right)^{3}=y^{3}+3U^{2}y+3y^{2}U+U^{3} \tag{140}$$

Equation (134) transforms to

$$V^{3}y^{3}+3W^{2}Vy+3V^{2}y^{2}W+W^{3}-\left(y^{3}+3U^{2}y+3y^{2}U+U^{3}\right)=0 \tag{141}$$

$$\Rightarrow V^{3}y^{3}+3W^{2}Vy+3V^{2}y^{2}W+W^{3}-y^{3}-3U^{2}y-3y^{2}U-U^{3}=0 \tag{142}$$

$$\left(V^{3}-1\right)y^{3}+\left(3V^{2}W-3U\right)y^{2}+\left(3W^{2}V-3U^{2}\right)y+W^{3}-U^{3}=0 \tag{143}$$

Compare (131) and (143) to get

$$V^{3}-1=1\Rightarrow V^{3}=2 \tag{144}$$

$$3V^{2}W-3U=0\Rightarrow U=V^{2}W \tag{145}$$

$$W^2V - U^2 = -\left(\frac{P^2}{9} + \frac{4R}{3}\right) \tag{146}$$

$$\Rightarrow W^3 - U^3 = \left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) \tag{147}$$

From equation (145),

$$U = V^2W \tag{148}$$

Plug (148) into (146) to obtain

$$W^2V - V^4W^2 = -\left(\frac{P^2}{9} + \frac{4R}{3}\right) \tag{149}$$

$$\Rightarrow \left(V - V^4\right)W^2 = -\left(\frac{P^2}{9} + \frac{4R}{3}\right) \Rightarrow V\left(V^3 - 1\right)W^2 = \left(\frac{P^2}{9} + \frac{4R}{3}\right) \tag{150}$$

Put (144) into (150) to get

$$VW^2 = \left(\frac{P^2}{9} + \frac{4R}{3}\right) \Rightarrow V = \frac{1}{W^2}\left(\frac{P^2}{9} + \frac{4R}{3}\right) \tag{151}$$

Plug (148) into (147) to yield

$$W^3 - V^6W^3 = \left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) \tag{152}$$

$$\Rightarrow \left(1 - V^6\right)W^3 = \left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) \tag{153}$$

$$\Rightarrow \left(1 - \left(V^3\right)^2\right)W^3 = \left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) \tag{154}$$

$$\Rightarrow \left(1 - V^3\right)\left(1 + V^3\right)W^3 = \left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) \tag{155}$$

$$\Rightarrow \left(V^3 - 1\right)\left(1 + V^3\right)W^3 = \left(-\frac{8RP}{3} + \frac{2P^3}{27} + Q^2\right) \tag{156}$$

Substitute (144) into (156) to obtain

$$\left(1+V^3\right)W^3=\left(-\frac{8RP}{3}+\frac{2P^3}{27}+Q^2\right) \tag{157}$$

Plug (151) into (157) to secure

$$\left\{1+\frac{1}{W^6}\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3\right\}W^3=\left(-\frac{8RP}{3}+\frac{2P^3}{27}+Q^2\right) \tag{158}$$

$$\Rightarrow\left\{W^3+\frac{1}{W^3}\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3\right\}=\left(-\frac{8RP}{3}+\frac{2P^3}{27}+Q^2\right) \tag{159}$$

$$\Rightarrow W^6+\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)W^3+\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3=0 \tag{160}$$

Let

$$S=W^3 \tag{161}$$

Substitute (161) in (160) to obtain

$$S^2+\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)S+\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3=0 \tag{162}$$

$$S_{\mathbf{i}}=\frac{-\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)+(-1)^{\mathbf{i}}\sqrt{\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)^2-4\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3}}{2} \tag{163}$$

$;\mathbf{i}\in\{1,2\}$

$$S_{\mathbf{i}} = \frac{-\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)+\left(-1\right)^{\mathbf{i}-1}\sqrt{\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)^2-4\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3}}{2} \tag{163a}$$

$;\mathbf{i}\in\{1,2\}$

From (161), we have

$$W_{\mathbf{i}} = \sqrt[3]{\frac{-\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)+\left(-1\right)^{\mathbf{i}}\sqrt{\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)^2-4\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3}}{2}} \tag{164}$$

$;\mathbf{i}\in\{1,2\}$

$$W_{\mathbf{i}} = \sqrt[3]{\frac{-\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)+\left(-1\right)^{\mathbf{i}-1}\sqrt{\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)^2-4\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3}}{2}} \tag{165}$$

$;\mathbf{i}\in\{1,2\}$

From equation (151)

$$V-1=\frac{1}{W^2}\left(\frac{P^2}{9}+\frac{4R}{3}\right)-1=\frac{\left(\frac{P^2}{9}+\frac{4R}{3}\right)-W^2}{W^2} \tag{166}$$

$$\Rightarrow W^2\left(V-1\right)=\left(\frac{P^2}{9}+\frac{4R}{3}\right)-W^2$$

Put (145) and (151) into (137) to secure

$$y=\frac{\left(V^2W-W\right)}{\left(\frac{1}{W^2}\left(\frac{P^2}{9}+\frac{4R}{3}\right)-1\right)}=\frac{\left(V^2-1\right)W^3}{\left(\frac{P^2}{9}+\frac{4R}{3}\right)-W^2}=\frac{\left(V^2-1\right)W^3}{\left(\frac{P^2}{9}+\frac{4R}{3}\right)-W^2} \tag{167}$$

$$\Rightarrow y=\frac{\left(V-1\right)\left(V+1\right)W^3}{W^2\left(V-1\right)}=\left(V+1\right)W \tag{168}$$

$$\Rightarrow y=\left(\frac{1}{W^2}\left(\frac{P^2}{9}+\frac{4R}{3}\right)+1\right)W \tag{169}$$

$$\Rightarrow y=W+\frac{1}{W}\left(\frac{P^2}{9}+\frac{4R}{3}\right) \tag{170}$$

plugging the values of W_i in (164) into (170) yield the following solution to the depressed equation (131) as follows

$$y_{\mathbf{i}}=\sqrt[3]{\frac{-\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)+(-1)^{\mathbf{i}}\sqrt{\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)^2-4\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3}}{2}}+\left(\frac{P^2}{9}+\frac{4R}{3}\right)\left(\sqrt[3]{\frac{-\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)\pm\sqrt{\left(\frac{8RP}{3}-\frac{2P^3}{27}-Q^2\right)^2-4\left(\frac{P^2}{9}+\frac{4R}{3}\right)^3}}{2}}\right)^{-1};\mathbf{i}\in\{1,2\} \quad (171)$$

Consequently, by using (132), two solutions of (120) become

$$T_i = \sqrt[3]{\frac{-\left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) + (-1)^i \sqrt{\left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right)^2 - 4\left(\frac{P^2}{9} + \frac{4R}{3}\right)^3}}{2}}$$
$$+\left(\frac{P^2}{9} + \frac{4R}{3}\right)\left(\sqrt[3]{\frac{-\left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right) + (-1)^i \sqrt{\left(\frac{8RP}{3} - \frac{2P^3}{27} - Q^2\right)^2 - 4\left(\frac{P^2}{9} + \frac{4R}{3}\right)^3}}{2}}\right)^{-1} \quad (172)$$
$$-\frac{2P}{3}$$

$i \in \{1, 2\}$

Let the solutions $T_i = r_1$

The other r_2 and r_3 of equation (120) can be found using the expressions for the sum of roots and product of roots. Using equation (131) as our reference point yields

$$r_1 + r_2 + r_3 = \frac{-b}{a} \quad (173)$$

$$r_1 r_2 + r_2 r_3 + r_1 r_3 = \frac{c}{a} \quad (174)$$

$$r_1 + r_2 + r_3 = 0 \quad (175)$$

$$r_1r_2 + r_2r_3 + r_1r_3 = -\left(\frac{3P^2}{9} + 4R\right) \tag{176}$$

Since we know r_1 already, we can make r_2 the subject of the formula in (172) (175) to get

$$r_2 = -(r_1 + r_3) \tag{177}$$

Plug (177) into (176) to secure

$$-r_1(r_1 + r_3) - r_3(r_1 + r_3) + r_1r_3 = -\left(\frac{3P^2}{9} + 4R\right) \tag{178}$$

$$\Rightarrow -r_3^2 - r_1r_3 - r_1^2 = -\left(\frac{3P^2}{9} + 4R\right) \tag{179}$$

$$\Rightarrow r_3^2 + r_1r_3 + \left(r_1^2 - \frac{3P^2}{9} - 4R\right) = 0 \tag{180}$$

$$\Rightarrow r_3 = \frac{-r_1 \pm \sqrt{r_1^2 - 4\left(r_1^2 - \frac{3P^2}{9} - 4R\right)}}{2} \tag{181}$$

$$r_2 = -(r_1 + r_3) \tag{182}$$

$$\Rightarrow r_2$$

$$=\left\{\begin{array}{l} -\sqrt[3]{\dfrac{-\left(\dfrac{8RP}{3}-\dfrac{2P^3}{27}-Q^2\right)\pm\sqrt{\left(\dfrac{8RP}{3}-\dfrac{2P^3}{27}-Q^2\right)^2-4\left(\dfrac{P^2}{9}+\dfrac{4R}{3}\right)^3}}{2}} \\ -\left(\dfrac{P^2}{9}+\dfrac{4R}{3}\right)\left(\sqrt[3]{\dfrac{-\left(\dfrac{8RP}{3}-\dfrac{2P^3}{27}-Q^2\right)\pm\sqrt{\left(\dfrac{8RP}{3}-\dfrac{2P^3}{27}-Q^2\right)^2-4\left(\dfrac{P^2}{9}+\dfrac{4R}{3}\right)^3}}{2}}\right)^{-1} \\ +\dfrac{2P}{3}-\dfrac{-r_1\pm\sqrt{{r_1}^2-4\left({r_1}^2-\dfrac{3P^2}{9}-4R\right)}}{2} \end{array}\right\} \tag{183}$$

Therefore using equation (109): $V^2 = T = r$, the solutions defined by (113), (114), (115), (116) have been completely obtained to model the force of interest δ

FINDINGS AND DISCUSSION

The coefficients in equation (64) have been as presented in Table 1.

Table 1: Value of Coefficient

m	δ^4	δ^2	δ	α
1	-0.0500000000	3.0000000000	-738.0000000000	36.0000000000
2	14.9243902439	-175.4634146341	-1107.2195121951	54.4390243902
3	79.8983743061	-473.9024583664	-1476.5852498017	73.1704996035
4	254.8719526696	-892.3171601752	-1846.0970389490	92.1940778980
5	623.8451260094	-1430.7075605615	-2215.7546366309	111.5092732618

6	1294.8178951913	-2089.0737114803	-2585.5577311180	131.1154622360
7	2399.7902612709	-2867.4156762526	-2955.5059424844	151.0118849687
8	4094.7622254910	-3765.7335294587	-3325.5988232477	171.1976464954
9	6559.7337892802	-4784.0273568131	-3695.8358591216	191.6717182431
10	9998.7049542503	-5922.2972550207	-4066.2164698756	212.4329397513
11	14639.6757221936	-7180.5433316162	-4436.7400103030	233.4800206061
12	20734.6460950798	-8558.7657047850	-4807.4057712899	254.8115425798
13	28559.6160750528	-10056.9645031692	-5178.2129809850	276.4259619699
14	38414.5856644276	-11675.1398656559	-5549.1608060646	298.3216121292
15	50623.5548656859	-13413.2919411517	-5920.2483530899	320.4967061798
16	65534.5236814724	-15271.4208883415	-6291.4746699512	342.9493399025
17	83519.4921145906	-17249.5268754341	-6662.8387473955	365.6774947909
18	104974.4601679980	-19347.6100798947	-7034.3395206319	388.6790412638
19	130319.4278448030	-21565.6706881648	-7405.9758710112	411.9517420224
20	159998.3951482560	-23903.7088953712	-7777.7466277730	435.4932555460
21	194479.3620817500	-26361.7249050240	-8149.6505698560	459.3011397120
22	234254.3286488120	-28939.7189287056	-8521.6864277666	483.3728555331
23	279839.2948530960	-31637.6911857502	-8893.8528854991	507.7057709981
24	331774.2606983820	-34455.6419029161	-9266.1485825032	532.2971650064
25	390623.2261885680	-37393.5713140512	-9638.5721156931	557.1442313861
26	456974.1913276620	-40451.4796597514	-10011.1220414913	582.2440829827
27	531439.1561197840	-43629.3671870160	-10383.7968779043	607.5937558085
28	614654.1205691480	-46927.2341488965	-10756.5951066212	633.1902132424
29	707279.0846800690	-50345.0808041444	-11129.5151751335	659.0303502669
30	809998.0484569480	-53882.9074168555	-11502.5554988670	685.1109977340
31	923519.0119042690	-57540.7142561126	-11875.7144633238	711.4289266475
32	1048573.9750265900	-61318.5015956289	-12248.9904262266	737.9808524533
33	1185918.9378285600	-65216.2697133896	-12622.3817196625	764.7634393249
34	1336333.9003148500	-69234.0188912969	-12995.8866522187	791.7733044373
35	1500622.8624902500	-73371.7494148152	-13369.5035111087	819.0070222174
36	1679613.8243595400	-77629.4615726198	-13743.2305642814	846.4611285627
37	1874158.7859276000	-82007.1556562483	-14117.0660625100	874.1321250200
38	2085133.7471993300	-86504.8319597571	-14491.0082414574	902.0164829148
39	2313438.7081796600	-91122.4907793814	-14865.0553237117	930.1106474234
40	2559997.6688735500	-95860.1324132015	-15239.2055207909	958.4110415818
41	2825758.6292860100	-100717.7571608150	-15613.4570351107	986.9140702215
42	3111693.5894220500	-105695.3653230140	-15987.8080619141	1015.6161238282

43	3418798.5492866900	-110792.9572014740	-16362.2567911578	1044.5135823156
44	3748093.5088849700	-116010.5330984410	-16736.8014093548	1073.6028187098
45	4100622.4682219400	-121348.0933164390	-17111.4401013685	1102.8802027370
46	4477453.4273026300	-126805.6381579740	-17486.1710521564	1132.3421043127
47	4879678.3861320900	-132383.1679252560	-17860.9924484629	1161.9848969258
48	5308413.3447153300	-138080.6829199240	-18235.9024804571	1191.8049609142
49	5764798.3030573800	-143898.1834427810	-18610.8993433147	1221.7986866294
50	6249997.2611632300	-149835.6697935430	-18985.9812387433	1251.9624774865
51	6765198.2190378400	-155893.1422705920	-19361.1463764488	1282.2927528976
52	7311613.1766861800	-162070.6011707430	-19736.3929755427	1312.7859510855
53	7890478.1341131500	-168368.0467890180	-20111.7192658890	1343.4385317780
54	8503053.0913236400	-174785.4794184350	-20487.1234893898	1374.2469787796
55	9150622.0483225000	-181322.8993497980	-20862.6039012108	1405.2078024217
56	9834493.0051145300	-187980.3068715090	-21238.1587709445	1436.3175418890
57	10555997.9617045000	-194757.7022693810	-21613.7863837119	1467.5727674238
58	11316492.9180971000	-201655.0858264660	-21989.4850412036	1498.9700824073
59	12117357.8742970000	-208672.4578228900	-22365.2530626597	1530.5061253194
60	12959996.8303089000	-215809.8185357020	-22741.0887857884	1562.1775715769
61	13845837.7861373000	-223067.1682387290	-23116.9905676263	1593.9811352526
62	14776332.7417867000	-230444.5072024440	-23492.9567853379	1625.9135706759
63	15752957.6972616000	-237941.8356938400	-23868.9858369582	1657.9716739164
64	16777212.6525663000	-245559.1539763200	-24245.0761420770	1690.1522841540
65	17850621.6077052000	-253296.4623095890	-24621.2261424680	1722.4522849361
66	18974732.5626825000	-261153.7609495560	-24997.4343026622	1754.8686053243
67	20151117.5175025000	-269131.0501482550	-25373.6991104677	1787.3982209354
68	21381372.4721692000	-277228.3301537600	-25750.0190774378	1820.0381548757
69	22667117.4266868000	-285445.6012101190	-26126.3927392876	1852.7854785752
70	24009996.3810593000	-293782.8635572900	-26502.8186562612	1885.6373125223
71	25411677.3352905000	-302240.1174310910	-26879.2954134513	1918.5908269026
72	26873852.2893844000	-310817.3630631550	-27255.8216210728	1951.6432421456
73	28398237.2433447000	-319514.6006808850	-27632.3959146909	1984.7918293818
74	29986572.1971751000	-328331.8305074320	-28009.0169554071	2018.0339108142
75	31640621.1508794000	-337269.0527616660	-28385.6834300034	2051.3668600068
76	33362172.1044610000	-346326.2676581590	-28762.3940510469	2084.7881020938
77	35153037.0579235000	-355503.4754071740	-29139.1475569568	2118.2951139135
78	37015052.0112702000	-364800.6762146610	-29515.9427120346	2151.8854240692
79	38950076.9645047000	-374217.8702822570	-29892.7783064606	2185.5566129212

80	40959995.9176301000	-383755.0578072910	-30269.6531562566	2219.3063125132
81	43046716.8706497000	-393412.2389827970	-30646.5661032182	2253.1322064365
82	45212171.8235666000	-403189.4139975300	-31023.5160148175	2287.0320296351
83	47458316.7763839000	-413086.5830359870	-31400.5017840774	2321.0035681549
84	49787131.7291046000	-423103.7462784300	-31777.5223294205	2355.0446588410
85	52200620.6817317000	-433240.9039009180	-32154.5765944925	2389.1531889850
86	54700811.6342679000	-443498.0560753390	-32531.6635479634	2423.3270959269
87	57289756.5867162000	-453875.2029694490	-32908.7821833065	2457.5643666130
88	59969531.5390791000	-464372.3447469070	-33285.9315185573	2491.8630371145
89	62742236.4913595000	-474989.4815673240	-33663.1105960542	2526.2211921083
90	65609995.4435598000	-485726.6135863060	-34040.3184821617	2560.6369643235
91	68574956.3956826000	-496583.7409555040	-34417.5542669781	2595.1085339561
92	71639291.3477304000	-507560.8638226620	-34794.8170640276	2629.6341280551
93	74805196.2997055000	-518657.9823316770	-35172.1060099406	2664.2120198812
94	78074891.2516104000	-529875.0966226460	-35549.4202641211	2698.8405282423
95	81450620.2034472000	-541212.2068319330	-35926.7590084034	2733.5180168068
96	84934651.1552182000	-552669.3130922170	-36304.1214466990	2768.2428933980
97	88529276.1069255000	-564246.4155325610	-36681.5068046353	2803.0136092706
98	92236811.0585713000	-575943.5142784690	-37058.9143291863	2837.8286583725
99	96059596.0101575000	-587760.6094519510	-37436.3432882970	2872.6865765941
100	99999994.9616862000	-599697.7011715830	-37813.7929705024	2907.5859410048

Using the rows of table 1, there will be 100 equations to be formulated and solved. As we formulate and solve the first two equations to test our model, we discovered that the results were very close to the exact solution. Row 1 of Table 1 is used to formulate equation (184) while row 2 of same Table is used to formulate equation (185) as follows.

$$0.05\delta^4 - 3\delta^2 + 738\delta - 36 = 0; \quad m = 1 \tag{184}$$

$$\begin{aligned} &14.9243902\delta^4 - 175.463415\delta^2 - (1107.21951)\delta \\ &+54.4390244 = 0; \quad m = 2 \end{aligned} \tag{185}$$

The equation (184) was solved in R with the results stated below.

```
> coefficient = c(-36, 738, -3,0, 0.05)

> roots = polyroot(coefficient)
> print(roots)
```

[1]   0.04879016+ 0.00000j
[2] -25.36012003+ 0.00000j

[3] 12.65566493-20.53618j

[4] 12.65566493+20.53618j

```
> coefficient = c(54.4390244, -1170.21951, -175.46341, 0, 14.9243902)

> roots = polyroot(coefficient)

> print(roots)
```

Equation (185) was also solved in R with the results stated below.

[1] 0.04620037-0.000000j

[2] -2.61028482+2.906008j

[3] -2.61028482-2.906008j

[4] 5.17436927+0.000000j

From the above results, the roots of these equations are complex in the form $\delta = \alpha \pm \beta j$. We then pick the correct real part. So for equation (184), the force of interest is $\delta = 0.04879016$ while for equation (185), the force of interest is $\delta = 0.04620037$. When the ruling rate of interest is 5%. The true value of the force of interest is $\delta = \log_e(1+0.05) = 0.0487901642$ using $\delta = \log_e(1+i)$ where i is the rate of interest. From the results obtained, we can now set bounds for both δ and e^{δ}. Consider Taylor's approximation of $A(i)$

$$A(i) = A(\alpha) + \frac{(i-\alpha)^{(1)}}{1!} A^{(1)}(\xi) + \frac{(i-\alpha)^2}{2!} A^{(2)}(\xi) + \frac{(i-\alpha)^3}{3!} A^{(3)}(\xi) + \ldots + \frac{(i-\alpha)^{(m+1)}}{(m+1)!} A^{(m+1)}(\xi) \tag{186}$$

for $\alpha \le \xi \le i$. The exact value of the parameter ξ is undetermined but it is bounded. The bounds on ξ essentially determine the upper and lower bounds on $\log_e(1+i)$

In actuarial practice, the value of interest rate i is conservatively very small and therefore setting $\alpha = 0$ and $m = 0$ we obtain its linear Taylor's approximation of one term and the error term as $A(i) = A(0) + iA'(\xi)$ (187)

for $0 \le \xi \le i$; $A(i)$ is now a function of i

Set

$$A(i) = \log_e(1+i) \tag{188}$$

Then

$$A(i) = \log_e(1+0) + i \times \frac{1}{1+\xi} = \frac{i}{1+\xi} \tag{189}$$

But the interest rate i is always greater than zero, we can find the maximum and minimum value on $0 \le \xi \le i$ as $\max\limits_{0\le\xi\le i}\left(\frac{i}{1+\xi}\right) = i$ and $\min\limits_{0\le\xi\le i}\left(\frac{i}{1+\xi}\right) = \frac{i}{1+i}$ noting that

$$\arg\max_{0\le\xi\le i}\left(\frac{i}{1+\xi}\right) = 0 \text{ and } \arg\min_{0\le\xi\le i}\left(\frac{i}{1+\xi}\right) = 1$$

By definitions $\delta = \log_e(1+i)$; $v = \frac{1}{1+i}$ and $d = iv$, thus $\frac{i}{1+i} \le \log_e(1+i) \le i$

$$\Rightarrow iv \le \delta \le i \tag{190}$$

implying that

$$d \le \delta \le i \tag{191}$$

Exponentiating the last inequality, yields $e^d \le e^\delta \le e^i$

Observe that

$$\frac{d}{di}\left(\log_e\left(1+i\right)\right)=\frac{1}{1+i}=\frac{1}{1-\left(-i\right)}=\sum_{m=0}^{\infty}\left(-i\right)^m \qquad (192)$$

In general, evidence from equations (190) and (191) may support lowering discount rates under a feasible best guess, based on the available financial information that the lower discount rate should be at most i while the upper discount rate should be at least d ; hence, upper and lower bounds can be set for equations (113)-(116) as follows:

(i) $$d\le\left\{\frac{V}{2}\pm\sqrt{\frac{V^2}{4}-\left(\frac{VP+V^3+Q}{2V}\right)}\right\}\le i \qquad (193)$$

and **(ii)**

$$d\le\left\{-\frac{V}{2}\pm\sqrt{\frac{V^2}{4}-\left(\frac{VP+V^3-Q}{2V}\right)}\right\}\le i \qquad (194)$$

where $i^{(m)}-d^{(n)}\le\frac{i^{(2)}}{\min\left(m,n\right)}$ for

$m,n\in\mathbf{Z}^{+}$

The result obtained on the force of interest is applicable in various areas, including bond pricing, insurance and pension planning. Investors can use the results on the force of interest estimated to analyse the returns on investments of life insurance funds, particularly in environments where insurance assets are expected to generate continuous returns. For example, let i_T be the technical rate of interest which is only be issued by the regulatory authorities such as Nigerian National Insurance Commission or Central Bank. Then $\delta\left(i_T\right)=\delta_T$ will be the corresponding technical interest rate intensity calculated from the derived model above. However if an assumed interest rate i_A is used, then $\delta\left(i_A\right)=\delta_A$ will be the corresponding assumed interest rate intensity.Past literatures such asKellison (2009) have adopted Taylor's series expansion which do not offer a closed-form expression to model interest rate intensities and hence Taylor's series is doomed in this context. Finally the result also found out that the model is an alternative expansion for the function $\log_e\left(1+x\right)$ for $0\le x\le 1$.

PRACTICAL IMPLICATION

The estimated force of interest has several important implications for life insurers: It enables precise valuation of future liabilities, ensuring that the present value of expected claims and benefits is accurately calculated. This accuracy is crucial for financial stability and regulatory compliance. Insurers can use the force of interest to set premiums that adequately reflect the time value of money, ensuring that they collect enough funds over time to cover future claims. Understanding the force of interest allows insurers to align their investment portfolios with their liabilities. By anticipating interest rate changes, they can optimise returns and manage risks associated with interest rate fluctuations. By incorporating the force of interest into actuarial models, insurers can better assess the risks associated with different interest rate environments. This informs their strategies for hedging and capital management.

The force of interest influences how insurers determine the reserves needed to meet future obligations. Adequate reserves ensure that the insurer remains solvent and can meet policyholder claims. Regulatory bodies often require insurers to maintain certain solvency levels. The force of interest helps ensure that insurers can demonstrate they have sufficient capital to meet future obligations. The force of interest impacts the design of insurance products, including how benefits accumulate over time.

LIMITATIONS

Based on the rigour of modelling in this study and the experience of the Nigerian financial markets, Bernoulli polynomials have some limitations when modelling interest rates. Interest rates are usually characterised by non-linear behaviour and influenced by factors like inflation and central bank policies. Possibly, Bernoulli polynomials may not effectively capture all these complexities. Financial markets are subject to sudden changes and volatility, which can be challenging for Bernoulli polynomial models that assume smoothness and continuity. Interest rates can switch between two different regimes (low-rate and high-rate environments). Bernoulli polynomials may find it difficult to adapt to these random shifts.

Since the behaviour of short-term rates differs significantly from long-term rates, Bernoulli polynomial models may not capture the nuances of the yield curve effectively. While polynomial fitting can be simple, it may result in overfitting, particularly in scenarios with limited data points. This can reduce the model's predictive power. Bernoulli polynomials assume a level of smoothness and continuity which may not reflect the often-discontinuous nature of the financial market context. The application of higher-order Bernoulli polynomials may complicate the interpretation of results, making it difficult for actuaries to obtain actionable insights. Bernoulli polynomials are primarily used in specific mathematical contexts and may less integrate well with other models used in finance such as those based on stochastic processes.

CONCLUSION

From the results presented, the force of interest represents the instantaneous rate of return on investment. In life insurance, it allows for the calculation of present values of future cash flows using continuous compounding. This is particularly useful when evaluating policies with varying cash flows over time. Life insurance policies promise future benefits, such as death benefits or cash values. The force of interest helps actuaries discount these future cash flows back to their present value, which is essential for determining the policy's fair value.

By incorporating the force of interest into models, insurers can establish premium rates that reflect the time value of money. This ensures that premiums are adequate to cover future liabilities. Insurers need to maintain reserves to meet future claims. The force of interest is used to calculate the present value of expected future claims, helping insurers set appropriate reserves. The force of interest can be varied to assess the impact on valuations under different interest rate scenarios. This is important for risk management and regulatory compliance.

Understanding the force of interest helps life insurers in optimising their investment strategies, as they can align their asset returns with their liabilities based on interest rate projections. In summary, the force of interest is fundamental in determining the present value of future cash flows, establishing premium rates, calculating reserves, and managing the financial aspects of life insurance products effectively. The force of interest is a critical concept in financial engineering, underpinning the theoretical framework for computing interest rates in continuous time. This study explores its implications across various financial and life insurance products and environments, specifically within the advent of complex financial instruments.

FUTURE RESEARCH DIRECTIONS

Future research direction may integrate behavioural insights into the analytical economic models to attain a more holistic understanding of interest rates in emerging economies.

DECLARATION OF CONFLICTING INTERESTS

The authors declared no potential conflict of interest with respect to the research, authorship, and publication of this article.

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