Tellus (1991), *43B*, 76–79

SHORT CONTRIBUTION

An improved approach to one-dimensional modelling of meridional transport in the atmosphere

By I. G. ENTING[1] and G. N. NEWSAM[2]*, [1] *CSIRO, Division of Atmospheric Research, Private Bag 1, Mordialloc, Victoria 3195, Australia;* [2] *Centre for Mathematical Analysis, Australian National University*

(Manuscript received 17 May 1990; in final form 30 July 1990)

ABSTRACT

Many of the errors involved in applications of one-dimensional modelling of meridional transport of atmospheric constituents arise from the practice of approximating vertically-averaged concentrations with surface data. We propose a simple correction that greatly reduces these errors.

The purpose of this note is to give an expanded description of a correction term that we recently proposed (Newsam and Enting, 1988) for use with one-dimensional (latitude-only) models of atmospheric transport. Despite their limitations, one-dimensional models including only the north-south transport of atmospheric constituents continue to be used in the interpretation of atmospheric constituent data (see, e.g., Bolin and Keeling, 1963; Keeling and Heimann, 1986; Levin et al., 1987). Their appeal lies in their simplicity and ease of use, and in the possibility of obtaining semi-analytic solutions which may give greater insight than might be got from a "black-box" numerical model with two or three spatial dimensions.

The drawback of one-dimensional models is, of course, their limited accuracy. Moreover, as noted by Bolin and Keeling (1963), their use in the inverse problem of deducing surface sources from surface concentration data suffers from catastrophic amplification of small-scale noise. (For more recent discussion of this instability see Enting (1985), Enting and Mansbridge (1989), Newsam and Enting (1988) and Enting and Newsam (1990)). Thus, the catastrophic amplification of inherent model errors would appear to preclude the use of one-dimensional models for the inverse problem.

Our recent analyses, however, show that the major source of error in such applications is not due to the model itself but rather to errors in the way in which the data are substituted into the model. We present here two derivations of a simple correction that reduces such errors. These may be able to be generalized to provide similar correction procedures for use with other models whose dimensions have been reduced by averaging.

The need for the correction arises from the discrepancy between the available data and the variables appearing in the model. One-dimensional models work with vertically averaged concentrations, $\bar{c}(\theta, t)$. However, such data are usually unavailable; most measurements are of surface concentrations, $c_s(\theta, t)$. The usual procedure is simply to substitute c_s for $\bar{c}$; as we shall see, this approximation can introduce serious errors so we propose, as an alternative, the following higher-order approximation:

$$\bar{c}(\theta, t) = c_s(\theta, t) - s(\theta, t)(p_0 - p_1)/(3K\kappa_p), \qquad (1)$$

where t is time, θ is co-latitude, $s(\theta, t)$ is the zonally averaged surface source, κ_p is a vertical diffusion

* Present address: Electronics Research Laboratory, D.S.T.O., P.O. Box 1600, Salisbury, S.A. 5108, Australia.

coefficient, $p_0 - p_1$ is the pressure difference across the troposphere and $K = M/(Ap_0)$, where A is the area of the earth and M is the size of the atmosphere in moles (1.773×10^{20} moles; Trenberth, 1981). In eq. (1) and throughout the paper, concentrations are expressed as molar mixing ratios and sources are in moles per unit area, using area units consistent with those of A. The parameter κ_p does not usually appear in one-dimensional models; we have proposed (Enting and Newsam, 1990) $\frac{4}{9} \times 10^3\ \mathrm{Pa^2 s^{-1}}$ as a tentative value, but it may be more appropriate to include the estimation of κ_p in the model calibration process.

The first method of deriving (1) is to simply postulate a reasonable form for the pressure-dependence of the zonally-averaged concentration $c(p, \theta, t)$ that satisfies the boundary conditions. These are:

$$c(p_0, \theta, t) = c_s(\theta, t), \tag{2a}$$

$$K\kappa_p \frac{\partial}{\partial p} c(p, \theta, t) = s(\theta, t) \quad \text{at} \quad p = p_0, \tag{2b}$$

$$\frac{\partial}{\partial p} c(p, \theta, t) = 0 \quad \text{at} \quad p = p_1. \tag{2c}$$

Relation (2c) implies that all the variation in space and time is confined to the troposphere. For constituents where this approximation is unreasonable, one-dimensional modelling is inappropriate with or without our correction.

The first-order approximation $\bar{c} \approx c_s$ is equivalent to approximating $c(\theta, t)$ by a function that is independent of pressure and which satisfies (2a) only. Approximating $c(\theta, t)$ by a quadratic that satisfies (2a–c) gives the alternative approximation

$$c(p, \theta, t) \approx c_s(\theta, t) + s(\theta, t) \times \frac{(p - p_1)^2 - (p_0 - p_1)^2}{2K(p_0 - p_1)\kappa_p}. \tag{3}$$

Vertical averaging then leads to approximation (1). This approximation is now used in the one-dimensional transport equation:

$$\frac{\partial \bar{c}}{\partial t} = \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta}\left[\kappa_\theta \sin\theta \frac{\partial \bar{c}}{\partial \theta}\right] + \frac{s(\theta, t) + \bar{\sigma}(\theta, t)}{K(p_0 - p_1)}, \tag{4}$$

where R is the radius of the earth and $\bar{\sigma}(\theta, t)$ is the vertically-integrated contribution of sources in the free atmosphere, expressed in moles per unit area per unit time. Approximation (1) can be used to substitute for $\bar{c}$ in terms of c_s and s in (4) which can then be solved for c_s in terms of s and $\bar{\sigma}$, or for s in terms of c_s and $\bar{\sigma}$.

To understand better how the correction significantly reduces errors, we examine the derivation of the one-dimensional model from a purely diffusive two-dimensional model (which in turn can be regarded as a zonal average of a three-dimensional model). For quantitative comparisons, it is necessary to modify our earlier derivation (Newsam and Enting, 1988) and (following Enting and Newsam, 1990) use pressure coordinates rather than height coordinates. The zonally-averaged transport equation is

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial p}\left[\kappa_p \frac{\partial c}{\partial p}\right] + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \times \left[\kappa_\theta \sin\theta \frac{\partial c}{\partial \theta}\right] + K^{-1}\sigma(p, \theta, t), \tag{5}$$

subject to boundary conditions (2a–c). Here σ is a source in the free troposphere in units of moles per unit time per unit surface area per unit pressure interval. Multiplying by $(p_0 - p_1)^{-1}$ and integrating over the pressure co-ordinate, gives (4) immediately, where $\bar{c}$ is the mass-weighted mean:

$$\bar{c}(\theta, t) = \frac{1}{p_0 - p_1} \int_{p_1}^{p_0} c(p, \theta, t)\, \mathrm{d}p, \tag{6}$$

and the source terms are related by:

$$\bar{\sigma}(\theta, t) = \int_{p_1}^{p_0} \sigma(p, \theta, t)\, \mathrm{d}p. \tag{7}$$

Restricting ourselves to the case where $\sigma(p, \theta, t) = 0$, we examine the relations between the surface sources $s(\theta, t)$, the vertically-averaged concentrations, $\bar{c}(\theta, t)$, and the surface concentrations $c_s(\theta, t)$. We consider only periodic solutions and expand these quantities in terms of Legendre polynomials, $P_n(\cos\theta)$, as:

$$s(\theta, t) = \sum_{kn} s_{kn} P_n(\cos\theta) \mathrm{e}^{2\pi i k t/T}, \tag{8a}$$

$$\bar{c}(\theta, t) = \sum_{kn} a_{kn} P_n(\cos\theta) e^{2\pi ikt/T}, \quad (8b)$$

$$c_s(\theta, t) = \sum_{kn} b_{kn} P_n(\cos\theta) e^{2\pi ikt/T}. \quad (8c)$$

The relations between sources and concentrations involve the constant

$$\alpha = \frac{K\kappa_p}{p_0 - p_1}, \quad (9)$$

and suitably scaled eigenvalues of the differential operator in (4):

$$\gamma_{kn}^2 = [\kappa_0 n(n+1)/R^2 + 2\pi ik/T](p_0 - p_1)^2/\kappa_p$$
$$= 1.06n(n+1) + 2.85ik \quad (10)$$

using the transport coefficients in Newsam and Enting (1988) and Enting and Newsam (1990).

As shown in our earlier studies, the solution of (5) gives the relation between surface sources and surface concentrations as:

$$s_{kn} = \alpha b_{kn} \gamma_{kn} \tanh(\gamma_{kn}). \quad (11)$$

This relation shows that, asymptotically, the factor relating source components and concentration components increases linearly with the meridional wave-number, n. Similar linear growth is found in more realistic models that include both advective and diffusive transport (Enting and Mansbridge, 1989). The implications of this growth rate for the inverse problem of deducing surface sources from surface concentrations are discussed by Newsam and Enting (1988), Enting and Mansbridge (1989) and Enting and Newsam (1990).

Both the one-dimensional approximation and the full two-dimensional solution give the relation between surface sources and vertical averages as:

$$s_{kn} = \alpha a_{kn} \gamma_{kn}^2. \quad (12)$$

Direct substitution of c_s for $\bar{c}$ in (4) is thus equivalent to substituting b_{kn} for a_{kn} in (12). When compared with (11), this gives clearly erroneous estimates for s_{kn}: with the numerical values of (10), the estimates are in error by a factor of 2 or more for $n \geqslant 2$. Thus, in this inverse problem, approximating surface concentrations by vertically-averaged concentrations leads to the concentration data being magnified by factors that grow quadratically with n even though, as noted above, the more realistic description given by relation (11) shows that the appropriate factor is linear in n.

If, however, (1) is used to substitute $b_{kn} - s_{kn}/3\alpha$ for a_{kn} in (12), then we derive the approximation

$$s_{kn} \approx \alpha b_{kn} [\gamma_{kn}^2/(1 + \gamma_{kn}^2/3)]. \quad (13)$$

This is in error by factors of more than 2 only for $n \geqslant 5$. Moreover, if (13) is used to estimate s_{kn} from b_{kn}, the inversion is now stabilized in that there is now no longer amplification of high-frequency components (and hence high-frequency errors). Unfortunately, this stabilization is of only limited value in the present case as there is no way of systematically varying its application so as to best match actual error levels in the data.

Finally, the above analysis can be reversed to reproduce approximations such as (1) from known relations in the spectral domain. To see this, we note that (13) can be derived not only by approximating a_{kn} in (12) by $b_{kn} - s_{kn}/\alpha$, but also by approximating $\gamma_{kn}\tanh(\gamma_{kn})$ in (11) by $\gamma_{kn}^2/(1 + \gamma_{kn}^2/3)$. As $\gamma^2/(1+\gamma^2/3)$ is the (2,2) Padé approximant (Baker and Gammel, 1970) to $\gamma\tanh(\gamma)$, it gives more accurate results than the second-order Taylor approximation γ^2. More generally, we could construct any reasonable approximation to $\gamma\tanh(\gamma)$ in the spectral domain and use the duality between (12) and (13) to translate it into an approximation of $\bar{c}$ in terms of c_s and s which can then be substituted in (4) to determine s from c_s. Alternatively, we could reach the same ultimate relationship between s and c_s directly through constructing a modification of (4) by translating the spectral approximation in (13) directly in terms of the associated averaged differential operator. These approaches give more insight into the ways in which one might construct dimensional reductions of two- or three-dimensional models that are more realistic than the purely diffusive model considered here.

In conclusion, we have shown that the accuracy obtainable from one-dimensional transport models can be greatly improved by use of approximations such as (1) in place of direct approximation of unknown vertically-averaged concentrations by the available surface concentration data. We have presented two derivations of this approximation: through averaging a model profile chosen to match the boundary conditions and through approximating the spectral relation between sources and concentrations as determined by a more accurate model. Clearly, the methodology can be extended to other dimensionally-averaged models to derive relations between the unmeasured averaged quantities and the available point data.

Acknowledgement

This work was supported in part by a CSIRO/ANU collaborative research grant.

REFERENCES

Baker, G. A. and Gammel, J. L. 1970. *The Padé approximant in theoretical physics*. Academic Press, New York.

Bolin, B. and Keeling, C. D. 1963. Large scale atmospheric mixing as deduced from seasonal and meridional variations of carbon dioxide. *J. Geophys. Res. 68*, 3899–3920.

Enting, I. G. 1985. A classification of some inverse problems in geochemical modelling. *Tellus 37B*, 216–229.

Enting, I. G. and Mansbridge, J. V. 1989. Seasonal sources and sinks of atmospheric CO_2: Direct inversion of filtered data. *Tellus 41B*, 111–126.

Enting, I. G. and Newsam, G. N. 1990. Inverse problems in atmospheric constituent studies: II. Sources in the free atmosphere. *Inverse Problems 6*, 349–362.

Keeling, C. D. and Heimann, M. 1986. Meridional eddy diffusion model of the transport of atmospheric carbon dioxide. Mean annual cycle. *J. Geophys. Res. 91D*, 7782–7796.

Levin, I., Kromer, B., Wagenbach, D. and Münnich, K. O. 1987. Carbon isotope measurements of atmospheric CO_2 at coastal station in Antarctica. *Tellus 39B*, 89–95.

Newsam, G. N. and Enting, I. G. 1988. Inverse problems in atmospheric constituent studies: I. Determination of surface sources under a diffusive transport approximation. *Inverse Problems 4*, 1037–1054.

Trenberth, K. E. 1981. Seasonal variations in global sea-level pressure and the total mass of the atmosphere. *J. Geophys. Res. 88C*, 5238–5246.

Printed in Belgium