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Accurate numerical solutions for Daisyworld

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ABSTRACT

The numerical solutions of the Daisyworld model of Watson and Lovelock contain significant quantitative errors. We give accurate numerical solutions for the same cases. We also show how the errors may have been caused by failure to enforce computational constraints such as strict tests of steadiness. The errors which we find do not qualitatively alter the main conclusions of Watson and Lovelock, but they illustrate a peril. The Daisyworld model is an example of a mathematical system which is too idealized to be compared with observations but too complex to be solved analytically. Such systems can be probed only by numerical simulations, so it is crucial that the computations be trustworthy.

1. The Daisyworld model

Watson and Lovelock (1983) created an intriguing conceptual model of interactions between the climate and the biota of an idealized planet which they called Daisyworld. Their paper (henceforth referred to as WL) analyzes the climate feedback mechanism of Daisyworld, on which the climate is described by a single variable, temperature, which can change because of a variable planetary albedo. On Daisyworld, the albedo variation is caused entirely by changes in the biota, which are daisies of two different colors. The feedback loop of WL is completed by specifying the growth rates of the daisies as a function of temperature. The equations describing Daisyworld are simple in appearance, but they are intractable analytically, and so the main results of WL are in the form of numerical solutions.

The purpose of this note is to point out that the results of WL are affected by significant numerical errors. These errors do not change the main conclusions of WL, but they serve as a warning. When a theoretical model is so idealized that it cannot be tested by comparison with observation, but is still complicated enough to require recourse to numerical solution, then the validity of the model results depends completely on the adequacy of the numerical procedures.

Using the notation of WL, Daisyworld is governed by:

$$\begin{aligned} \mathrm{d}a_b/\mathrm{d}t &= a_b(x\beta_b - \gamma), \\ \mathrm{d}a_w/\mathrm{d}t &= a_w(x\beta_w - \gamma), \end{aligned} \tag{1}$$

where a_b, a_w are the fractional areas of the planet covered by "black" and "white" daisies. The area of fertile ground not covered by daisies, x, is given by

$$x = p - a_b - a_w, \tag{2}$$

where p is the proportion of the planet's area which is fertile ground. The death rate of the daisies, γ, is a constant while the growth rates of the daisies, β_b and β_w, are assumed to be parabolic functions of the local temperatures, T_b(°C) and T_w(°C):

$$\beta_{b,w} = 1 - 0.003265(22.5 - T_{b,w})^2. \tag{3}$$

These local temperatures are given by

$$T_{b,w} = q'(A - A_{b,w}) + T_e, \tag{4}$$

with q' a parameter expressing the degree to which absorbed solar energy is redistributed among the different types of planetary cover. A_b, A_w are the albedos of the "black" and "white" daisies with $A_b < A_w$, from which it follows that $T_b > T_w$. A is the albedo of the planet and is given by

$$A = a_g A_g + a_b A_b + a_w A_w, \tag{5}$$

where $a_g (= 1 - a_b - a_w)$ and A_g are the area and albedo of bare ground respectively. T_e(°C) is the effective temperature of the planet, and is found by equating absorbed and emitted radiation:

$$\sigma(T_e + 273)^4 = SL(1 - A). \tag{6}$$

Here σ is Stefan's constant, S is a solar constant and L is a dimensionless measure of the luminosity of Daisyworld's sun. WL give further details and assumptions regarding this model.

2. Results and conclusions

The numerical results of WL are given in the form of five figures for five specific cases. Using the same parameter values for each of these cases, we have recomputed these solutions. To perform these calculations, we used two different numerical procedures. One is a standard method for ordinary differential equations, a 5th-order Runge-Kutta scheme with monitoring of the local truncation error and adaptive control of the size of the time step so as to achieve a predetermined accuracy. Our Runge-Kutta routines are based on the ODEINT and RKQC programs given by Press et al. (1986). For each of the five cases of WL, we have repeatedly integrated the model to steady state using a wide range of accuracy criteria for the time-dependent solutions and a number of severe tests of steadiness.

We have also computed solutions for the same five cases using Euler's method, "the simplest of all difference equations, and probably the least accurate" (Strang, 1986). This method is not generally recommended for practical applications. It is weakly unstable, it is formally only first-order accurate, and there are far more efficient techniques, such as the Runge-Kutta methods, which offer a better balance of accuracy and stability without much penalty in the form of complexity. Nevertheless, the Euler method is adequate for systems of equations as simple as those of Daisyworld, for which computational efficiency is not a stringent requirement. In fact, these equations can readily be solved on a personal computer. Furthermore, for our purposes the Euler method has the virtue of providing not only an independent check on the Runge-Kutta results, but also some insight into how computational errors may have affected the solutions of WL.

Our results are shown in Fig. 1, together with those of WL for four of the five cases. To produce these results, we used the procedure of WL, integrating the equations to steady state for a

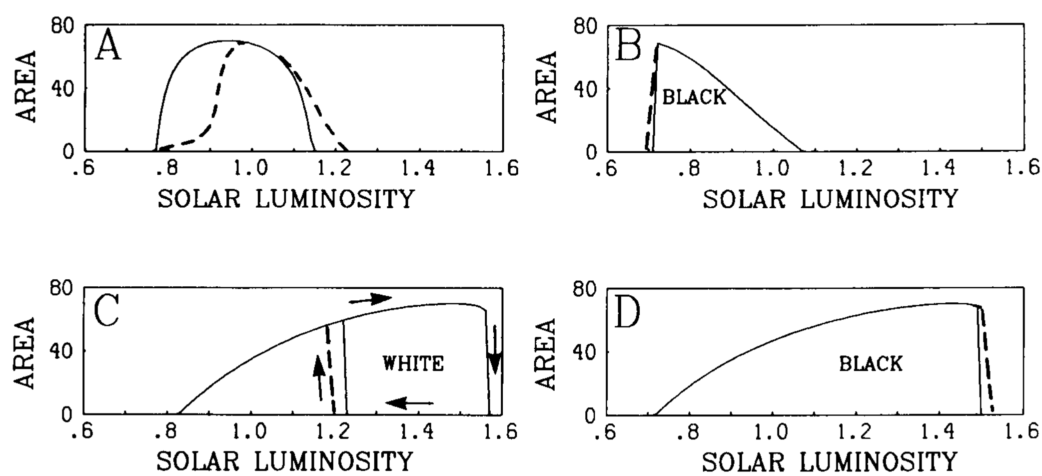

Fig. 1. Our accurate numerical solutions (——), and the results of Watson and Lovelock (1983) (-----) for:

(A) Case 1—a population of only neutral daisies, albedo 0.5;
(B) Case 2—a population of only black daisies, albedo 0.25;
(C) Case 3—a population of only white daisies, albedo 0.75;
(D) Case 4—a population of both black and white daisies, albedos 0.25 and 0.75 with a cloud of albedo 0.8 assumed to obscure the black daisies.

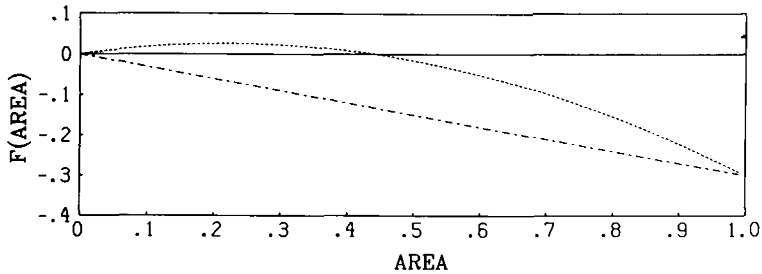

Fig. 2. The zeros of $F(a_n) = a_n(x\beta_n - \gamma)$ correspond to the equilibrium states for the neutral daisies case. For $L = 0.8$ (-----), there are two possible equilibrium states, one at $a_n = 0.00$, another near $a_n = 0.44$. For $L = 1.2$ (—-—), there is only one equilibrium state at $a_n = 0.00$.

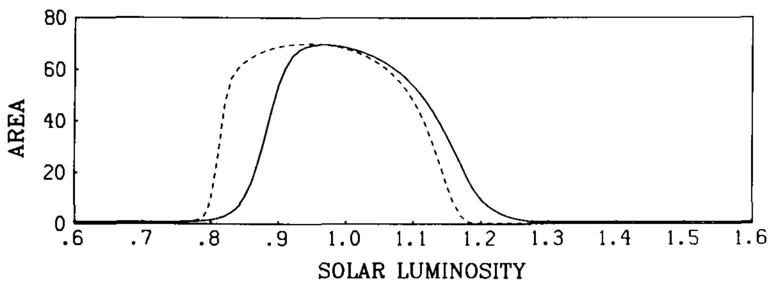

Fig. 3. Response curve of the neutral daisies when solar luminosity was increased after: 10 time steps (——), and 50 time steps (-----), regardless of whether a true steady state had been achieved.

fixed value of luminosity L, then incrementing L by 0.01, integrating to steady state again, and so on. In repeating these steps, we used either 0.01 or the previous steady-state values of area, if these were non-zero, as initial conditions for each successive time integration. We can be confident of the accuracy of our results, not only because we have obtained the same results using two independent techniques, but also because we have varied the time step in the Euler method, the accuracy criteria in the Runge-Kutta method, and the steadiness tests in both methods, to verify that our solutions have indeed converged to true steady states.

There are several significant differences between our results and those of WL. One discrepancy which is immediately apparent is the strikingly different shape of the area curves in Fig. 1A. It is easy and instructive to show that the results of WL in Fig. 1A are in error. The right-hand sides of eq. (1), which must vanish in the steady state, may be plotted as functions of the area for fixed values of luminosity L. Such plots are shown in Fig. 2. For $L = 0.8$, there are two equilibrium solutions, but the trivial (zero) solution is easily shown to be linearly unstable, and only the non-zero solution corresponding to our result in Fig. 1 is stable. There is no stable solution corresponding to the result of WL. Similarly, for $L = 1.2$, where WL show a non-zero solution in Fig. 1A, only zero is an equilibrium solution. Thus, in this case, even without solving the time-dependent system of equations, it is possible to demonstrate that the results of WL shown in Fig. 1A cannot be correct.

We have used the Euler method to construct deliberately inaccurate solutions in an effort to support a conjecture which may explain how the results of WL shown in Fig. 1A were produced. Fig. 3 shows the results of two integrations in which inaccurate solutions for the same case were produced by incrementing the luminosity L after a fixed number of time steps, regardless of whether or not a true steady state had been reached. There is a strong resemblance between the results of WL in Fig. 1A and our deliberately unsteady result in Fig. 3. We may conjecture, therefore, that the integrations of WL may not

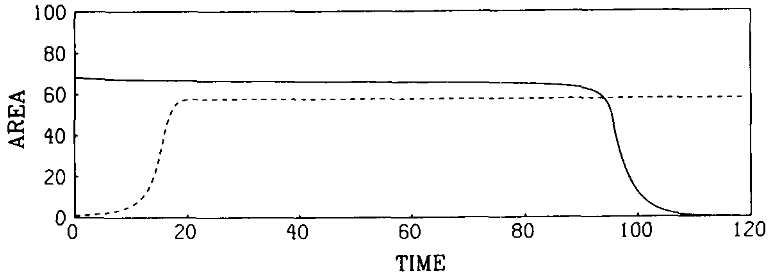

Fig. 4. Time-dependent area growth for: Case 4 at $L = 1.5$ (——), and Case 3 at $L = 1.2$ (-----). The initial behavior of the curves may be mistaken for a steady state if weak steady state conditions were employed.

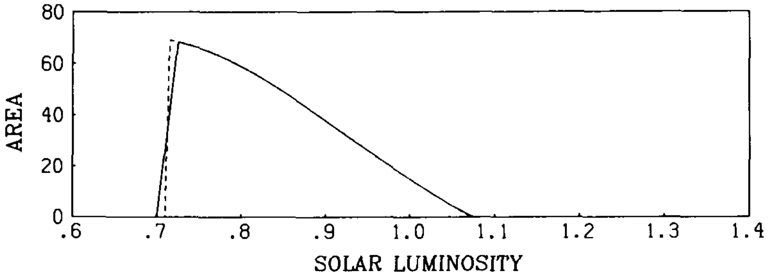

Fig. 5. Steady state responses for Case 2 using: 0.025 luminosity increments (——), and 0.005 luminosity increments (-----). As ΔL approached zero, the slope near $L = 0.71$ became vertical. Similar behavior occurred for Cases 3 and 4 at other specific values of L.

have been continued until all time variability had ceased.

Next, we briefly mention some other inaccuracies in the results of WL and point out possible causes. In comparing the results of WL shown in Fig. 1D with the correct solution, we note that at the luminosity value $L = 1.5$, the area computed by WL is near a maximum value, far from the correct answer of zero. Conversely, in comparing the results of WL shown in Fig. 1C with the correct solution, the correct answer at $L = 1.2$ is around 60% rather than the value of zero obtained by WL. It is instructive to view the time-dependent solutions for these two examples, which are shown in Fig. 4. In both cases, it can be seen that the solutions remained almost constant for a considerable time near the values given by WL, before abruptly shifting to their final steady-state values. Here too, it may be that the integrations of WL were prematurely stopped before a true steady state had been achieved.

Finally, we note an inaccuracy which appears in the results of WL shown in Figs. 1B, 1C and 1D. In each of these cases, the straight-line portion of the area graphs is less vertical than the corresponding part of the correct solutions. This error appears to be due to the use of insufficiently small increments in L. In Fig. 5, we illustrate the effect on the solutions of varying this parameter. It can be seen that as the luminosity increment is decreased, the straight-line segment of the graph becomes more nearly vertical.

We have pointed out a number of respects in which the numerical results of WL are inaccurate. By deliberately relaxing some of the constraints which accurate solutions must satisfy, we have been able to reproduce these inaccuracies and thereby to give probable explanations of their causes. None of the numerical mistakes of WL affects the nature of the conclusions drawn from the WL analysis of the Daisyworld model. The accurate solutions which we have computed display much the same types of behavior as the erroneous ones. The differences between the accurate solutions and the inaccurate ones are not small, but they tend to be quantitative rather than qualitative. That this should be so for this particular set of calculations, however, is purely fortuitous.

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