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The incompatibility of ice-core CO_2 data with reconstructions of biotic CO_2 sources

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ABSTRACT

Analyses of the compatibility between postulated releases of biotic carbon and changes in atmospheric CO_2 concentration must properly consider the history of changes in each quantity. A history of atmospheric CO_2 changes obtained from ice-cores is compared to a history of biotic CO_2 releases obtained from ecosystem modelling. It is shown that these two records are incompatible with any linear time-invariant model of oceanic uptake of CO_2.

1. Introduction

For many years, there has been a disagreement concerning the net contribution of the terrestrial biota to the balance of atmospheric carbon dioxide. Direct estimates of the rate of release of biotic carbon seem to be much larger than can be accounted for by conventional models of oceanic uptake of CO_2. Most of the discussion of this disagreement has been restricted to the current balance and has ignored the fact that much of the response of the atmosphere-ocean system to injections of CO_2 takes place on time scales of many decades. As pointed out by Oeschger and Heimann (1983), the carbon balance of the atmosphere must be expressed in terms of the history of past inputs. Hence any test of the degree of compatibility of estimates of biotic releases with observed changes in atmospheric CO_2 cannot be confined to current changes but must compare the recent histories of such changes.

In recent years, two records appropriate for such comparisons have been obtained: the estimates of biotic carbon release of Houghton et al. (1983) and the record of the last two centuries of atmospheric CO_2 changes, obtained from ice cores by Neftel et al. (1985). The aim of this paper is to test the consistency of these records.

The first question to be answered is whether the two records are consistent within the context of conventional models of the oceanic uptake of CO_2, calibrated using ^{14}C distributions. The best-known model of this type is the box-diffusion model of Oeschger et al. (1975). Enting and Pearman (1986) gave a detailed study of possible biotic changes using a model with oceanic uptake mechanisms more general than those of the box-diffusion model. They concluded that even with this more general model, the history of biotic changes given by Houghton et al. (1983) was incompatible with even the recent CO_2 changes observed at Mauna Loa, assuming that the ocean model remained compatible with ^{14}C data.

The analysis by Enting and Pearman (1986) does not rule out the possibility that some more special ocean model could prove to be compatible with the biotic release estimates of Houghton et al. and the CO_2 history of Neftel et al. A number of possible ocean models have been proposed in order to account for discrepancies in the contemporary CO_2 balance and it is conceivable that some such model could be compatible with both the biotic release estimates and the CO_2 history from ice-cores. Among the more general models that have been proposed are the outcrop model of Siegenthaler (1983), the regional model of Broecker et al. (1980) and the 12-box model of Bolin et al. (1983). The analysis in this paper shows that the Houghton et al. biotic release

estimates and the Neftel et al. ice-core data cannot both be compatible with any of these ocean models or indeed with any other ocean model in which the ocean uptake processes are time-invariant and the CO_2 uptake is essentially linear. This conclusion is reached by applying linear programming techniques (Gass, 1969) to the solution of integral equations based on the relations given by Oeschger and Heimann (1983).

The layout of the remainder of this paper is as follows. Section 2 takes the integral relations given by Oeschger and Heimann (1983) and derives from them the mathematical formulation of the problem of determining the compatibility of the biotic release estimates with the ice-core data. Section 3 shows how this compatibility problem can be expressed in terms of linear programming and Section 4 gives the results of this analysis. The concluding Section, 5, lists the various possible ways in which the discrepancy could be resolved. An appendix gives details of the relation between the continuous and discrete representations of the problem.

2. Mathematical description

In models of ocean uptake of CO_2 that can be linearised, the response of atmospheric CO_2 concentration, $c(t)$, to a source of strength $s(t)$, can be expressed as

$$c(t) = c(t_0) + \int_{t_0}^{t} R(t - t')s(t')\,dt' \qquad (1)$$

(see Oeschger and Heimann, 1983; Enting, 1985a). The response function $R(t)$ gives the amount of CO_2 remaining the atmosphere t years after a source pulse of unit strength. For most carbon cycle modelling studies t_0 is taken as about the year 1800, i.e. prior to most of the CO_2 sources from fossil fuel use and before the most rapid agricultural changes. The form of eq (1) implies the assumption of carbon cycle equilibrium prior to t_0.

There is some degree of uncertainty concerning each of the functions c, R and s involved in eq. (1). The most common ways of obtaining estimates of each function are:

(i) $R(t)$ is usually estimated from an ocean model of CO_2 uptake. Such models have been calibrated by the rate at which they take up natural radiocarbon (Oeschger et al., 1975), tritium (Broecker et al., 1980), weapons produced radiocarbon (Enting and Pearman, 1986) or other tracers (Peng and Broecker, 1985).

(ii) The net source $s(t)$ is assumed to be due to anthropogenic influences and is the sum of fossil carbon release $f(t)$ and net biotic release $b(t)$. The fossil carbon component is believed to be relatively well known (Keeling, 1973; Rotty, 1981). The biotic component has been estimated by Houghton et al. (1983) using ecosystem modelling of the biotic response to changing levels of agriculture. Their population-based reference estimate of $b(t)$ is used throughout this paper. An alternative approach has been to determine $s(t)$ by a deconvolution of $^{13}C:^{12}C$ ratios in tree-rings (Peng et al., 1983). This approach requires the assumption of a model response function $r(t)$ for isotopic variations. More seriously, there appear to be many physiological mechanisms that cause the isotopic ratios in tree-rings to respond to other environmental effects as well as the $^{13}C:^{12}C$ ratio in the atmosphere (Francey and Farquhar, 1982; Francey et al., 1985).

(iii) The variation of CO_2 concentration over the last 200 years has recently been determined by Neftel et al. (1985) by measuring the CO_2 concentrations in air trapped in polar ice.

Since there is some uncertainty in each of R, s and c and since the estimates of the functions $s(t)$ and $c(t)$ are relatively recent, many carbon cycle studies have used two of the functions in equation (1) to determine the third.

The problem of using eq. (1) to deduce $s(t)$ from $R(t)$ and $c(t)$ is a Volterra integral equation of the first kind. This can be converted to a Volterra equation of the second kind by differentiating with respect to t to give:

$$\dot{c}(t) = R(0)s(t) + \int_{t_0}^{t} \dot{R}(t - t')s(t')\,dt'. \qquad (2)$$

The solution of such a problem is relatively stable (Enting, 1985b) and so the stability of the solution of (1) for $s(t)$ is determined by the stability of the numerical differentiation of $c(t)$. Siegenthaler and Oeschger (1987) have described the deconvolution of the record of CO_2 from ice-cores obtained by Neftel et al. (1985). The results for the biotic contribution were strikingly different from the estimates of Houghton et al. (1983) in that they failed to show a large increase over the last 50 years.

The use of $R(t)$ and $s(t)$ to produce $c(t)$ corresponds to direct integration of an ocean model and is the most stable of the three problems arising from eq. (1). The difficulties that have been encountered with this approach are that most models over-estimate the current rate of CO_2 increase, i.e., $\dot{c}(t)$, even if $b(t)$ is set to zero. Adding a biotic release of the form estimated by Woodwell et al. (1978) or Houghton et al. (1983) merely exacerbates the discrepancy between model estimates of $c(t)$ and the rate of increase determined by direct observations at Mauna Loa (Keeling et al., 1982). This discrepancy (the so-called missing carbon problem) has prompted many attempts at constructing modified ocean models, generally by increasing the resolution, thus allowing more detailed representations of CO_2 uptake (Broecker, et al., 1980; Siegenthaler, 1983; Bolin et al., 1983).

The deconvolution of eq. (1) using the ice-core $c(t)$ and the ecosystem modelling for $b(t)$ (plus the fossil fuel release, $f(t)$) in order to determine the required ocean model response $R(t)$ is formally equivalent to the problem of deducing $s(t)$ from c and R. Eq. (1) transforms under change of variable to

$$c(t) - c(t_0) = \int_0^{t-t_0} R(t')\, s(t-t')\, \mathrm{d}t'. \tag{3}$$

Of course since $c(t)$ is at best available only for $t_0 \leqslant t \leqslant t_1$, where t_1 represents the present, eq. (3) contains no information concerning $R(t)$ for $t > t_1 - t_0$. Even for $t < t_1 - t_0$, the stability of the deconvolution is mainly determined by the extent to which $c(t)$ can be differentiated. With $c(t)$ only determined for a limited number of times by Neftel et al. (1985), this direct approach is not yet practical. What we have investigated is the extent to which the available values of $c(t)$ are consistent with estimates of $s(t)$, without assuming any particular model for $R(t)$. However, because we assume that the ocean response is dependent only on the CO_2 source, we can derive three restrictions on the response function, $R(t)$:

(i) $R(t)$ is always positive. This follows because the ocean must respond to a source in atmospheric CO_2 by absorbing some of the CO_2 but not more than was injected into the atmosphere. In some experiments we consider the stronger condition that $R(t) \geqslant 9/99.7 = 0.162$. (The units for $R(t)$ are dimensionless because we have converted the sources to ppmv equivalent.) This constraint represents the limit of an equilibrium distribution of CO_2 between the atmosphere and the whole ocean. We have assumed a buffer factor of 9 and a ratio of atmospheric to ocean carbon of 1:55.7.

(ii) $\mathrm{d}R(t)/\mathrm{d}t$ is never positive. The response function must monotically decrease with time, otherwise the ocean would, at some time, give more CO_2 to the atmosphere than it takes up. This can be seen by considering a pulse source of CO_2-corresponding to replacing $s(t)$ in eq. (1) by a Dirac delta function.

(iii) $\mathrm{d}^2R(t)/\mathrm{d}t^2$ is never negative. It follows from this restriction that the rate of oceanic uptake from a pulse source of CO_2 cannot increase with time. Although this condition is quite reasonable physically, it is not very restrictive in practice; we perform some calculations without it.

3. Linear programming formulation

In order to explore the consistency of the records of $c(t)$ and $s(t)$, we have formulated the problem in terms of linear programming (Gass, 1969). We put

$$c_i = c(t_0 + 4i) = c_0 + \sum_{j=1}^{i} R_{i-j} S_j, \tag{4}$$

where we have discretised eq. (1) in 4-year blocks starting from 1800. The errors in this discretisation are discussed in the appendix. The source function is obtained by adding estimates of $f(t)$ from Rotty (1981 and personal communication) to the estimate of $b(t)$ from the population-based reference case of Houghton et al. (1983). We have taken $f(t)$ as increasing linearly from 1800 to 1860 (the first value given by Keeling, 1973) and $b(t)$ as increasing linearly from 1800 to 1826 (the earliest estimate available to us).

We seek a discretised set of responses R_i that give c_i which are as close as possible to the values from Neftel et al. (1985) which we denote $\hat{c}(t)$. We require

$$c_0 \leqslant 279 + \delta \tag{5a}$$

$$c_0 \geqslant 279 - \delta, \tag{5b}$$

and

$$c_i \leqslant \hat{c}(t_0 + 4i) + \delta \tag{6a}$$

$$c_i \geqslant \hat{c}(t_0 + 4i) - \delta, \tag{6b}$$

for $i = (2, 6, 16, 24, 29, 33, 37, 39, 41, 44, 45)$ with $\hat{c} = (280, 284, 288, 297, 300, 306, 311, 312, 318, 328, 334)$ respectively. The restrictions on the response function, which are discussed in Section 2, are discretised to give:

$$R_i \geqslant 0 \qquad \text{for } i = 0 \text{ to } 44 \tag{7a}$$

$$R_i - R_{i+1} \geqslant 0 \qquad \text{for } i = 0 \text{ to } 43 \tag{7b}$$

$$R_i - 2R_{i+1} + R_{i+2} \geqslant 0 \qquad \text{for } i = 0 \text{ to } 42. \tag{7c}$$

The validity of the discretisation is discussed in the appendix.

We have expressed the problem of fitting the sources to the observations as a linear programming problem connecting the variables c_0, δ (both of which we are taking as positive) and the 45 R_i which are also positive (relation 7a). Relations (7b, c) plus the relations obtained by substituting (4) into (6a, b) give a set of 111 inequalities in the linear programming problem. Computer routines to perform the calculation of finding the smallest δ consistent with all these inequalities are readily available.

4. Results

The results of this calculation show that the constraints (6a, b), (7a–c) required $\delta \geqslant 3.92$ ppmv. The values of c_i from (4) lie on the continuous line in the figure and are to be compared to the $\hat{c}$ values from Neftel et al. (1985). It must be emphasized that our study represents a test of the consistency of the point values and no special assumptions are made about the values between the points.

In achieving the minimum δ solution all but one of the 43 constraints (7c) are 'active', i.e., the equalities are satisfied. As a result, the response functions drop linearly to achieve their limiting values at $i = 24$. Even though the constraints on the second derivative are active, they do not influence the minimum value of δ greatly; removing them allows the minimum value of δ to become 3.75 ppmv. The constraints (7c) are reasonable in that they imply that the rate of oecan uptake of CO_2 due to a pulse source will not increase in time. Indeed the uptake rate would be expected to decrease as the atmosphere-ocean gradient decreases with time.

The active constraints on the concentrations are for $i = 0$, 24 and 44. It is also worth emphasizing many of the constraints that are inactive. The last i value was not actually given by Neftel et al. (1985) but, as was done by Siegenthaler and Oeschger (1987), the gradient from the Mauna Loa record was used to extend the ice core record to 1980. Since the constraints involving this point are inactive, this procedure has not influenced our result; the discrepancy arises solely from the ice core data. Similarly inactive constraints arise from the values $\hat{c} = 311$, 312 ppmv at points 9 years apart which give a much smaller gradient than the rest of the record; our results do not rely on these values which Neftel et al. indicated might be influenced by impermeable ice layers.

The $\delta \geqslant 3.92$ result lies outside the two standard deviation range given by Neftel et al. (1985) for the ice-core data and this discrepancy is required for 3 of the 11 points. In addition, the deviations of the other values are required to show a consistent sign of the type unlikely to have arisen by chance from random measurement errors and 5 of the 11 points lie outside the 2 s.d. range. The measure of the discrepancy rises to $\delta \geqslant 4.24$ if we add the stricter constraint

$$R_{44} \geqslant 9/55.7 = 0.162 \tag{8}$$

discussed in Section 2. (In the absence of constraint (8), the minimum δ is achieved by a response function which drops linearly to zero at $i = 24$).

The error bars shown in Fig. 1 refer only to CO_2 measurement errors in the ice-cores and not to the dating of extracted air; Neftel et al. (1985) estimate that 2 standard deviations of this error cover 11 years. Such dating uncertainty is only important when the CO_2 concentration is changing rapidly, after about 1940, at which time the concentration is well known from direct observations. The resultant uncertainty will include an additional component proportional to dC/dt which can be taken into account by substituting $w_i\delta$ for δ in eqs. (6a) and (6b). The weighting w_i then varies between 1.0 and 4.5 and the revised linear programming problem gives qualitatively the same results as those discussed above. With $R_{44} \geqslant 0.162$ we find $\delta \geqslant 3.22$ and 3 of the points

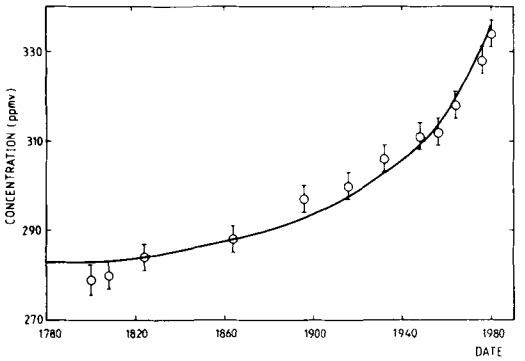

Fig. 1. Discrepancy between CO_2 concentrations obtained from ice-cores (open circles with error bars) and the closest fit (continuous curve) that is possible using biospheric releases from ecosystem modelling and allowing complete freedom in the choice of a linear steady-state model of oceanic CO_2 uptake.

lying outside 2 standard deviations. However, this result probably underestimates δ as the dating error may be less than 11 years (Enting, 1985c).

5. Conclusions

The conclusion that we can draw from this study is that either (i) there is a systematic bias in the ice-core data of Neftel et al. (1985) with the bias changing sign twice over the last 180 years; or (ii) the source strength estimates are incorrect in either the biotic component or, less probably, in the fossil carbon component; or (iii) the assumption of pre-industrial CO_2 equilibrium is incorrect, as might be implied by the ice-core data for the last 1000 years given by Raynaud and Barnola (1985); or (iv) there have been significant non-linear effects in the ocean uptake of CO_2; or (v) the assumption of a steady state ocean response is incorrect.

We believe that of these 5 general possibilities, the most probable explanation is that the source function $s(t)$ is incorrect. The modelling study by Enting and Pearman (1986) shows that a relatively conventional global carbon cycle model can be consistent with a wide range of carbon cycle data (e.g., ^{14}C distributions), but cannot be simultaneously made consistent with a CO_2 source like that considered here. In contrast, the linear programming approach has only considered the discrepancy between the source $s(t)$ and the CO_2 concentrations from polar ice. No attempt has been made to assess the extent to which these data sets are consistent with other data such as ^{14}C changes. (Enting, 1985b describes relations between response functions that might form the basis for such an extended comparison. However a proper comparison using these relations would probably have to be extended to describe regional variations).

There are a number of mechanisms that might account for errors in the net source $s(t)$. One possibility is an increase in biomass due to the fertilising effect of increase CO_2 concentrations. This possibility is not considered by Houghton et al. As well as this there is the inherent uncertainty in the ecosystem modelling process as indicated by the three different estimates produced by Houghton et al. (1983). In particular, their 'FAO' analysis, with a major peak in release just before the commencement of the Mauna Lao record, would largely reduce the discrepancy between the biotic release estimates and the Mauna Load CO_2 record. However, Houghton (personal communication) attributes the peak to a change in FAO definitions rather than an actual conversion of land to agriculture. The fossil fuel estimates from Rotty (1981) are similarly subject to uncertainties although these would seem to be much smaller. Peng and Broecker (1984) analyse a number of proposed anthropogenic influences on the oceanic uptake of CO_2 but conclude that geochemical observations exclude all the cases that they considered (This type of behaviour would correspond to case (v) above in that the amount of anthropogenic influence would be expected to be increasing).

To summarize, we would assert that the ice-core data of Neftel et al. (1985) and the ecosystem modelling estimates of biotic CO_2 sources cannot both be consistent with any conventional linear model of oceanic CO_2 uptake and so attempts to refine conventional models such as those of Siegenthaler (1983), Broecker et al. (1980) and Bolin et al. (1983) will fail to resolve the missing carbon problem. The present study does not distinguish between the five possibilities listed above but, on the basis of the study by Enting and Pearman (1986), we would follow Broecker et al. (1980) and assert that most probably 'the error lies in the biomass estimates'.

6. Appendix

The response function $R(t)$ is discretised by specifying it at the mid-point of each 4-year interval. These mid-point values are denoted $R_i = R(t_i)$, where the t_i are the midpoints of the intervals.

Clearly $R(t) \geqslant 0$ for all t (A.1a)

implies $R_i \geqslant 0$ for all i. (A.1b)

The constraint on the derivative is

$$\frac{dR}{dt} \leqslant 0 \quad \text{for all } t \tag{A.2a}$$

which implies

$$\int_{t_i}^{t_{i+1}} \frac{dR}{dt} dt \leqslant 0$$

i.e.,

$$R_{i+1} - R_i \leqslant 0 \quad \text{for all } i. \tag{A.2b}$$

The constraint on the second derivative is

$$\frac{d^2R}{dt^2} \geqslant 0 \quad \text{for all } t. \tag{A.3}$$

This implies

$$\int_a^b \frac{d^2R}{dt^2} dt = \left.\frac{dR}{dt}\right|_b - \left.\frac{dR}{dt}\right|_a \geqslant 0 \quad \text{for all } a \leqslant b. \tag{A.4}$$

The mean value theorem implies

$$(R_{i+1} - R_i)/\Delta t = \left.\frac{dR}{dt}\right|_\beta \quad \text{for some } \beta$$

such that

$$t_i \leqslant \beta \leqslant t_{i+1} \tag{A.5a}$$

and

$$(R_i - R_{i-1})/\Delta t = \left.\frac{dR}{dt}\right|_\alpha$$

for some α such that

$$t_{i-1} \leqslant \alpha \leqslant t_i. \tag{A.5b}$$

Thus

$$R_{i+1} - 2R_i + R_{i-1} = \Delta t\left(\left.\frac{dR}{dt}\right|_\beta - \left.\frac{dR}{dt}\right|_\alpha\right)$$

for some α, β such that

$$\alpha \leqslant t_i \leqslant \beta. \tag{A.6}$$

Therefore from (A.4), the constraint (A.3) on the second derivative implies

$$R_{i+1} - 2R_i + R_{i-1} \geqslant 0 \quad \text{for all } i. \tag{A.7}$$

Each of the descretized constraints (7a, b, c) (i.e. A.1b, A.2b, A.7) is a direct consequence of the corresponding continuous constraints (A.1a, A.2a, A.3). Since the discretized constraints apply only to specific points rather than a continuum of t values, the set of discretized constraints is less restrictive than the corresponding continuous constraints. This would suggest that the size of the discrepancy (as measured by δ) would increase if our discretization interval Δt was reduced. However our experience suggests that the change in δ would be small.

Unlike the inequalities on the response function, the discretization of the integrals over sources involves a 'numerical error' that does not preserve inequalities exactly. The size of the numerical error is most conveniently calculated using the change of variables that connects eq. (1) and (3) above,

$$\int_{t_0}^{t} R(t-t')s(t')\,dt' = \int_0^{t-t_0} R(t')s(t-t')\,dt'. \tag{A.8}$$

The discretization is based on

$$S_j = \int_{t_0+4j-4}^{t_0+4j} s(t')\,dt' \tag{A.9a}$$

$$R_i = R(4i+2). \tag{A.9b}$$

We denote deviations from these discretized values by

$$\sigma(t) = s(t) - \tfrac{1}{4}S_j \quad \text{for } t_0 + 4j - 4 \leqslant t \leqslant t_0 + 4j \tag{A.10a}$$

$$\rho(t) = R(t) - R_i \quad \text{for } 4i \leqslant t \leqslant 4i+4. \tag{A.10b}$$

In this notation (A.8) becomes, for $t = 4i + t_0$,

$$\int_0^{4i} R(t')s(4i + t_0 - t')\,dt'$$

$$= \sum_{j=1}^{i} \int_{4j-4}^{4j} R(t')s(4i + t_0 - t')\,dt'$$

$$= \sum_{j=1}^{i} \int_{4j-4}^{4j} (R_{j-1} + \rho(t'))(\tfrac{1}{4}S_{i-j+1} + \sigma(4i + t_0 - t'))\,dt'$$

$$= \sum_{j=1}^{i} R_{j-1} S_{i-j+1} + \sum_{j=1}^{i} \tfrac{1}{4} S_{i-j+1} \int_{4j-4}^{4j} \rho(t')\,dt'$$

$$+ \sum_{j=1}^{i} \int_{4j-4}^{4j} \rho(t')\,\sigma(4i + t_0 - t')\,dt' \qquad (A.11)$$

(since by definition $\sigma(t)$ has a zero integral over any 4-year interval). The discretized approximation to this integral expression is

$$\sum_{j=1}^{i} R_{i-j} S_j = \sum_{j=1}^{i} R_{j-1} S_{i-j+1}. \qquad (A.12)$$

Thus the discretion error for C_i is

$$E_i = \sum_{j=1}^{i} \tfrac{1}{4} S_{i-j+1} \int_{4j-4}^{4j} \rho(t')\,dt'$$

$$+ \sum_{j=1}^{i} \int_{4j-4}^{4j} \rho(t')\,\sigma(4i + t_0 - t')\,dt'. \qquad (A.13)$$

Upon expanding $\rho(t')$ in a Taylor's series we find that

$$\int_{4j-4}^{4j} \rho(t')\,dt'$$

$$= 4\rho(4j-2) + \tfrac{8}{3}\rho''(\alpha), \quad \text{with } 4j-4 \leqslant \alpha \leqslant 4j.$$

Thus

$$E_i = \sum_{j=1}^{i} \tfrac{2}{3} S_{i-j+1}\,\rho''(\alpha_j)$$

$$+ \sum_{j=1}^{i} \int_{4j-4}^{4j} \rho(t')\,\sigma(4i + t_0 - t')\,dt'$$

with

$$4j-4 \leqslant \alpha_j \leqslant 4j. \qquad (A.14)$$

The final integral depends on the amount of covariance between ρ and σ. In this case the best that seems possible is to obtain the bound

$$\left| \int_{4j-4}^{4j} \rho(t')\,\sigma(4i + t_0 - t')\,dt' \right| \leqslant 4\,|\rho|_j\,|\sigma|_{i-j}, \quad (A.15)$$

where $|\rho|_j$ is defined as the maximum absolute value of ρ over the interval $[4j-4,\ 4j]$ and $|\sigma|_{i-j}$ is defined as the maximum absolute value of σ over the interval $[4(i-j) + t_0,\ 4(i-j+1) + t_0]$.

Thus,

$$|E_i| \leqslant \sum_{j=1}^{i} \tfrac{2}{3} S_{i-j+1} |\rho''(\alpha_j)| + 4 \sum_{j=1}^{i} |\rho|_j |\sigma|_{i-j}. \quad (A.16)$$

In our extremal (i.e., minimum δ) case, the solution has $\rho'' \equiv 0$ and so the first sum in (A.16) gives no contribution to the error. More realistic responses can be roughly characterized by decay with a time constant of about 50 years and so ρ''/R is of order $(50)^{-2}$. Thus the first sum in (A.13) contributes at most

$$\frac{8}{3 \times 2500} \approx 0.01 \sum_{j=1}^{i} R_{i-j} S_j$$

for any realistic response. In the second sum in (A.16), $|\rho|_j \leqslant R_0/25$ for a wide range of possible response functions, including both the realistic (^{14}C-based model estimates of $R(t)$) and the extremal solutions found by the linear-programming calculations. The variability in $s(t)$ leads to the estimates $|\sigma|_j \leqslant 1/30\ (S_j/4)$. These estimated bounds on σ and ρ give the relation

$$4 \sum_{j=1}^{i} |\rho|_j |\sigma|_{i-j} \leqslant 0.06 \sum_{j=1}^{i} R_{i-j} S_j.$$

Inspection of the release function shows that this last bound will significantly overestimate the actual contribution to the discretization error because the maximum variations in $s(t)$ do not occur in all 44 time-intervals. However we give the estimate

$$E_i \leqslant 0.07 \sum_{j=1}^{i} R_{i-j} S_j \quad \text{for all } i.$$

We believe that a bound of 7% relative error is acceptably small for the purposes of our analysis, particularly since 7% is almost certainly a significant overestimate of the actual discretization error.

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