On the Diffusion of Heated Jets

By F. H. SCHMIDT, Royal Netherlands Meteorological Institute, De Bilt, The Netherlands

(Manuscript received, October 13, 1956)

Abstract

A general expression is obtained for the propagation of a heated jet in a stagnant environment. *It is shown that W. Schmidt's and Tollmien's solutions are contained in the general one.*

A new cross-sectional velocity distribution is proposed which seems to be preferable above the generally accepted Gaussian distribution.

1. Introduction

TOLLMIEN (1926) and W. SCHMIDT (1941) were the first to consider the turbulent subsonic flow of a fluid jet through a stagnant environment. Tollmien restricted himself to the case of an incompressible fluid of the same density as the environment, i.e. no buoyancy forces were taken into account. W. Schmidt extended Tollmien's theory to the case of a heated jet rising due to buoyancy. The relative density differences were connected with temperature differences.

Recently PRIESTLEY and BALL (1955) arrived at a solution that can be considered as a more general one, containing as it does both Tollmien's and Schmidt's results.

The authors do not seem to have recognized their solution as such a generalization, however. Moreover, it was obtained by the introduction of special assumptions as to the contribution of velocity and temperature in a plane perpendicular to the jet's axis. These assumptions include the same error function for both velocity and temperature profiles, which seems to be inconsistent with all experimental evidence.

It will be shown here that it is possible to arrive at a general solution without introducing any special assumptions about the velocity and temperature distributions in the jet. Afterwards a velocity profile will be proposed that does not show some of the disadvantages of profiles introduced so far.

2. Two-dimensional jet

First consider a two-dimensional jet, emerging from an infinitely long aperture and assume that density differences only play a role in as far as they effect buoyancy forces. The z-axis is taken vertical, the y-axis perpendicular to the aperture and molecular friction and conduction are ignored. Then the following equations hold for stationary two-dimensional flow:

$$\frac{\partial}{\partial z}(\overline{w} + w') + \frac{\partial}{\partial y}(\overline{v} + v') = 0 \qquad (1)$$

$$\frac{\partial}{\partial z}(\overline{w} + w')^2 + \frac{\partial}{\partial y}(\overline{w} + w')(\overline{v} + v') = -\frac{1}{\varrho}\frac{\partial p}{\partial z} - g \qquad (2)$$

$$\frac{\partial}{\partial z}(\overline{w} + w')(\overline{v} + v') + \frac{\partial}{\partial y}(\overline{v} + v')^2 = -\frac{1}{\varrho}\frac{\partial p}{\partial y} \qquad (3)$$

$$\frac{\partial}{\partial z}(\bar{w}+w')(\bar{\vartheta}+\vartheta')+\frac{\partial}{\partial y}(\bar{v}+v')(\bar{\vartheta}+\vartheta')=0 \tag{4}$$

namely the equation of continuity (1), two equations of motion (2) and (3), and an equation expressing the conservation of heat (4). The symbols are listed at the end of the paper.

We assume the motion to be quasistatic, i.e. p equals the pressure in the undisturbed environment. Or $-\frac{1}{\varrho}\frac{\partial p}{\partial z}-g=g\frac{\varrho_0-\varrho}{\varrho}=g\frac{\bar{\vartheta}+\vartheta'}{\Theta}$ where ϱ_0 denotes the density of the undisturbed environment with temperature Θ and where all density differences are assumed to be caused by the excess temperature $\bar{\vartheta}+\vartheta'$ of the jet.

From the foregoing the equations for the mean motion can be found by rearranging and omitting all first powers of the turbulent quantities. Neglecting $-\frac{1}{\varrho}\frac{\partial p}{\partial y}$ with respect to the turbulent frictional forces the following equations are obtained:

$$\frac{\partial \bar{w}}{\partial z}+\frac{\partial \bar{v}}{\partial y}=0 \tag{1 a}$$

$$\frac{\partial}{\partial z}(\bar{w})^2+\frac{\partial}{\partial y}(\bar{w}\,\bar{v})= =g\frac{\bar{\vartheta}}{\Theta}-\frac{\partial}{\partial z}\overline{(w')^2}-\frac{\partial}{\partial z}\overline{(w'v')} \tag{2 a}$$

$$\frac{\partial}{\partial z}(\bar{w}\,\bar{v})+\frac{\partial}{\partial y}(\bar{v})^2=-\frac{\partial}{\partial z}\overline{(w'\,v')}-\frac{\partial}{\partial y}\overline{(v')^2} \tag{3 a}$$

$$\frac{\partial}{\partial z}(\bar{w}\,\bar{\vartheta})+\frac{\partial}{\partial y}(\bar{v}\,\bar{\vartheta})=-\frac{\partial}{\partial z}\overline{(w'\,\vartheta')}-\frac{\partial}{\partial y}\overline{(v'\,\vartheta')} \tag{4 a}$$

Now the following assumptions are introduced:

1. At $y=0$, $\bar{w}=w_m$, $\bar{v}=0$, $\bar{\vartheta}=\vartheta_m$, w_m and ϑ_m being functions of z only.
2. At $y=\infty$, $\bar{w}=\bar{v}=\bar{\vartheta}=0$.
3. $\bar{w}$ as well as $\bar{\vartheta}$ show similar distributions in every level z:
   $\bar{w}=w_m f(\eta)$, $\bar{\vartheta}=\vartheta_m g(\eta)$ with $\eta=\frac{y}{R}$, R being a measure for the "width" of the jet and a function of z only.
4. $\overline{(w')^2}$, $\overline{(w'v')}$ and $\overline{(v')^2}$ are functions of η and proportional to w_m^2, $\overline{w'\vartheta'}$ and $\overline{v'\vartheta'}$ are functions of η and proportional to $w_m\vartheta_m$.
5. For $y\to\infty$ the turbulent values become $=0$ together with the mean values.

Next $\bar{v}$ is eliminated from the equations by integrating the equation of continuity:

$$\bar{v}=-\int\frac{\partial}{\partial z}\bar{w}dy=-\int R\frac{dw_m}{dz}f(\eta)\,d\eta+ +\int w_m\frac{df(\eta)}{d\eta}\eta\frac{dR}{dz}dy=R\frac{dw_m}{dz}\varphi(\eta)+ +\frac{dR}{dz}w_m\psi(\eta)$$

The additional integration constant must be $=0$ as $\bar{v}=0$ for $y=0$ as well as for $y=\infty$, independent of $R(z)$ and $w_m(z)$ (assumptions 1 and 2).

Introducing the expression for $\bar{v}$ into (2a)—(4a) and integrating over y from $-\infty$ to $+\infty$ one obtains:

$$A\frac{d}{dz}[Rw_m^2]=B\frac{g}{\Theta}\vartheta_m R \tag{2 b}$$

$$C\frac{d}{dz}\left[R^2\frac{dw_m}{dz}w_m\right]+D\frac{d}{dz}\left[R\frac{dR}{dz}w_m^2\right]= =E\frac{d}{dz}[Rw_m^2] \tag{3 b}$$

$$F\frac{d}{dz}[Rw_m\vartheta_m]=0 \tag{4 b}$$

where the constants A—F are found from the integration of $f(\eta)$ etc. over η from $-\infty$ to $+\infty$.

Assuming R, w_m and ϑ_m to be powers of z, one immediately finds $R\sim z^1$; $w_m\sim z^0$ and $\vartheta_m\sim z^{-1}$, in agreement with W. Schmidt.

Tollmien's solution is obtained by putting $\vartheta_m=0$, resulting in: $R\sim z^1$, $w_m\sim z^{-1/2}$.

A generalized solution can now be arrived at in the following way:
In equation (3 b) the variations of kinetic and frictional energies are interconnected. No other form of energy contributing to the horizontal motion, integration with respect to z gives:

$$CR^2\frac{dw_m}{dz}w_m+DR\frac{dR}{dz}w_m^2=ERw_m^2 \tag{3 c}$$

From (4b) and (2b) one derives:

$$A\frac{d}{dz}(Rw_m^2)=\frac{B}{F}\frac{g}{\Theta}\frac{Q}{w_m} \qquad (2c)$$

where

$$Q=FRw_m\vartheta_m \qquad (4c)$$

Integration of equation (2c) with respect to z results in:

$$w_m^3=\frac{1}{R^{3/2}}\left\{\int\frac{3BgQ}{2AF\Theta}\sqrt{R}\,dz+K\right\} \qquad (5)$$

where K is a constant.
Introducing this result into (3c) the following differential equation is obtained for R:

$$qR'+r(R')^2+sRR''=t$$

q, r, s and t being functions of the original constants A, B, C, D, E, F and Q, and a prime denoting a differentiation to z. Finally the following solution is obtained for R':

$$q(R')+r(R')^2-t=C_0R^{-2s}e^{h(R')} \qquad (6)$$

with $h(R')=\int\frac{q\,dR'}{r(R')^2+qR'-t}$ and C_0 an integration constant. A special solution is arrived at by putting $C_0=0$:

$$R'=\text{const} \quad \text{or} \quad R=\frac{R_1}{z_1}z,$$

any additional integration constant resulting in another position on the z-axis of the origin only.

R_1 represents the value of R in a reference level, the origin lying at a distance z_1 upstream.

Introduction of $R=\frac{R_1}{z_1}z$ into (2c) and (4c) then leads to the general solution:

$$w_m=\left[\frac{Bgz_1}{AF\Theta R_1}Q+\frac{K}{R_1{}^{3/2}}\left(\frac{z}{z_1}\right)^{-3/2}\right]^{1/3} \qquad (7)$$

$$\vartheta_m=\frac{Qz_1}{FR_1z}\left[\frac{Bgz_1}{AF\Theta R_1}Q+\frac{K}{R_1{}^{3/2}}\left(\frac{z}{z_1}\right)^{-3/2}\right]^{-1/3} \qquad (8)$$

Obviously K is connected with the extra velocity with which the jet may be forced to emerge from the nozzle. It is immediately evident that for $K=0$, i.e. for the case of a jet rising exclusively due to buoyancy, W. Schmidt's solution: $R\sim z^1$, $w_m\sim z^0$ and $\vartheta_m\sim z^{-1}$ results, whereas for the case of a cold jet, i.e. a jet where no buoyancy forces play a role, $Q=0$ and $R\sim z^1$, $w_m\sim z^{-1/2}$, which is Tollmien's solution.

The general expressions (7) and (8) pertain to a general two-dimensional heated jet that is forced to leave an infinitely long aperture. It may be stressed that it follows from (7) that the resulting velocity cannot be obtained by simple addition of a Tollmien-velocity and a Schmidt-velocity, but that the third powers of the velocities are additive.

3. Axially symmetric jet

The same reasoning holds for the axially symmetric jet. It leads to the following general expressions for w_m and ϑ_m:

$$w_m=\left[\frac{3}{4}\frac{Bgz_1^2}{AF\Theta R_1^2}Qz^{-1}+\frac{K}{R_3^1}\left(\frac{z}{z_1}\right)^{-3}\right]^{1/3} \qquad (9)$$

$$\vartheta_m=\frac{Qz_1^2}{FR_1^2z^2}\left[\frac{3}{4}\frac{Bgz_1^2}{AF\Theta R_1^2}Qz^{-1}+\frac{K}{R_3^1}\left(\frac{z}{z_1}\right)^{-3}\right]^{-1/3} \qquad (10)$$

where the constants have the same meaning as before. We have again a generalization containing Tollmien's and W. Schmidt's solutions:

$R\sim z^1$, $w_m\sim z^{-1}$ and $R\sim z^1$, $w_m\sim z^{-1/3}$ and $\vartheta_m\sim z^{-5/3}$, respectively.

Equations (9) and (10) were given for the first time by PRIESTLEY and BALL (1955). A complete derivation along the lines of the preceding section has been given elsewhere (SCHMIDT 1957).

4. The effect of Q

In both cases the quantity Q plays an important role. It is a measure for the source strength. For the two-dimensional flow Q represents the source strength per unit length of the orifice; in the axially symmetric case the total source strength equals $2\pi Q$.

It follows from the equations (7) to (10) that in the case of heated jets rising according to W. Schmidt's laws the vertical velocity increases according to $Q^{1/3}$, the temperature excess according to $Q^{2/3}$, or an increase of the total output by a factor 8 doubles the vertical velocity, whereas ϑ_m increases by a factor 4. This result is of importance for the theory of the diffusion of stack gases in the atmosphere where the rise of the plume due to buoyancy forces is quite essential. The method used here to derive the general expressions for w_m and ϑ_m shows, that it is not necessary to introduce any assumptions on the nature of $f(\eta)$ and $g(\eta)$, as was done by Priestley and Ball. The constants A, B, and F depend on the functions of η that have been introduced and they remain arbitrary, therefore, in the present developments.

More general solutions may be obtained by starting from the complete equation (6), i.e. without putting $C_0 = 0$. All experiments seem to indicate, however, that $R' =$ cst is in close agreement with reality except when the environment is in stable equilibrium in which case the jet spreads rapidly above a certain level (MORTON c.s. 1956).

5. The velocity profile

It may be worth while to make some remarks with respect to the functions $f(\eta)$ and $g(\eta)$ pertaining to the distribution of velocity and temperature over a cross section perpendicular to the axis of the jet.

Tollmien and W. Schmidt tried to determine both functions from the equations of motion by expanding the stream function into an infinite series. To obtain $f(\eta)$ they introduced as boundary conditions $\bar{v} = 0$ $\bar{u} =$ finite for $\eta = 0$ and $\bar{u} = 0$ for $\eta = \eta_r$, the "boundary" of the jet for the two-dimensional case and $\bar{v} = 0$, $\bar{u} =$ finite for $\eta = 0$ for the axially symmetric jet.

The method, though seemingly exact, has disadvantages. In the first place the solution can be applied to values of $\eta > \eta_r$, but it seems to have no physical significance there. Secondly it does not seem quite correct that under stationary conditions, only attained over an infinitely long time after the motion started, the jet should have a definite "boundary" and should not show any effect up to an infinite distance from the axis, at least theoretically. It is for these reasons that some authors adopted an error function for $f(\eta)$ and $g(\eta)$, e.g. ROUSE, YIH and HUMPHREYS (1952), and PRIESTLEY and BALL (1955). Though the error function may represent a

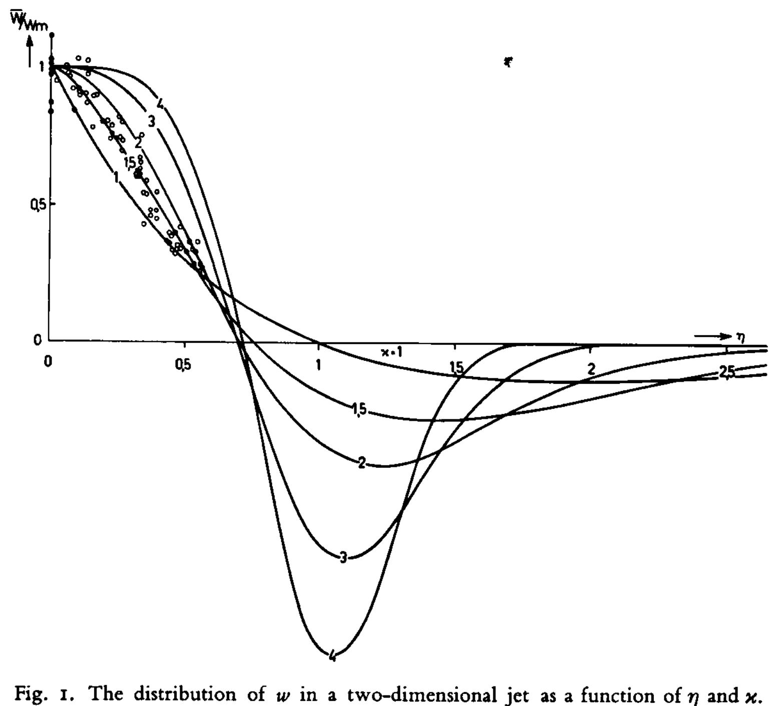

Fig. 1. The distribution of w in a two-dimensional jet as a function of η and $\varkappa$. The small circles pertain to the experimental results of Rouse c.s.

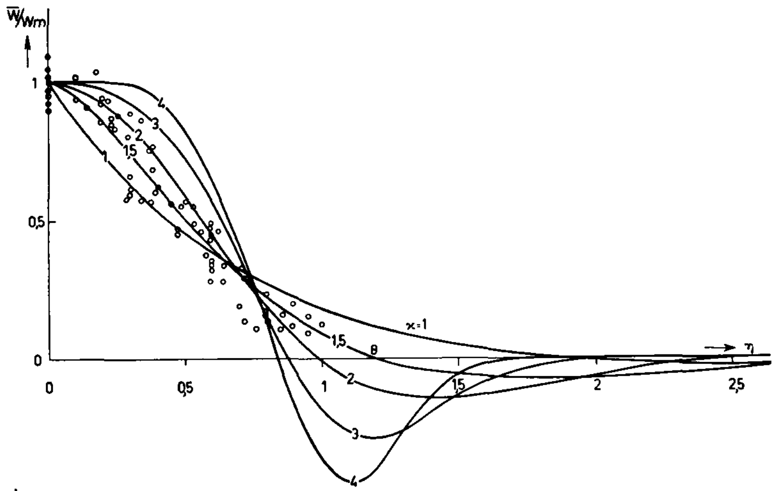

Fig. 2. The distribution of $\overline{w}$ in an axially symmetric jet as a function of η and $\varkappa$. The small circles pertain to the experimental results of Rouse c.s.

reasonable approximation of the temperature profile, it cannot be used to describe the velocity profile.

For, introducing $f(\eta) = \exp\{-y^2/R^2\}$ in the case of the two-dimensional jet, one finds from equation (1 a):

$$\overline{v}_\infty - \overline{v}_0 = -\frac{\partial}{\partial z}\int_0^\infty \overline{w}dy = -\frac{\partial}{\partial z}\left[\frac{\sqrt{\pi}}{2} Rw_m\right] \quad (11)$$

This result shows that $\overline{v}$ cannot be $=0$ for $y=0$ as well as for $y=\infty$ except in case Rw_m is independent of z, which is certainly not true for free jets. However, $\overline{v}\,(y=0)=0$ is obviously necessary from a physical point of view as otherwise there would exist a discontinuity for $y=0$, the plane of symmetry. On the other hand a finite horizontal velocity at $y=\infty$ would involve an infinitely large total kinetic energy per slice of unit thickness and per unit length of the aperture:

$$\int_0^\infty \frac{1}{2}\varrho\,(\overline{w}^2+\overline{v}^2)\,dy.$$

Moreover, all experimental investigations on the propagation of jets have been performed in a finite room with fixed walls at a distance $y>R$ from the symmetry plane. At these walls $\overline{v}$ must be zero of necessity. The same reasoning holds for convective currents in the atmosphere.

It follows then from (11), that $\int_0^\infty \overline{w}dy$ must be constant. From the properties of Γ-function it can be shown easily that

$$\overline{w} = w_m(z)\{1-\varkappa\eta^\varkappa\}e^{-\eta^\varkappa}, \quad (12)$$

where $\varkappa$ is a constant is an example of a function satisfying this condition.

Figure 1 shows the velocity profiles for some values of $\varkappa$. It appears that a downward countercurrent exists for $\eta > \varkappa^{-1/\varkappa}$. This countercurrent has long been recognized in the atmosphere where it occurs around cumuliform clouds. It has not been found in the various experiments on the propagation of jets. It is possible, however, that measurements of $\overline{w}$ have not been performed for sufficiently large values of η.

In figure 1 the points corresponding to the measurements of Rouse c.s. have also been entered. It appears that the $\overline{w}$-curve proposed here is well in accordance with the experimental data for $\varkappa=1.5$.

The same reasoning applies to the axially symmetric jet leading to the velocity distribution:

$$\overline{w} = w_m(z)\left\{1-\frac{\varkappa}{2}\eta^\varkappa\right\}e^{-\eta^\varkappa} \quad (13)$$

Figure 2 shows the various velocity distributions and again the experimental results of Rouse c.s. Theory and experiment are well in accordance for $\varkappa = 1.8$.

6. Final Remarks

It should be kept in mind that the foregoing developments are based on too restrictive assumptions.

No account has been taken of the fact that the similarity of velocity and temperature profiles only comes into being at distances of about ten times the diameter of the aperture from the orifice (Kuethe, 1935).

Moreover, it has been found that the flow in a free jet is a very complicated hydrodynamical phenomenon. For instance, at some distance from the symmetry-axis turbulence appears to be intermittent whereas at large distances from the axis the motion is almost a non-turbulent one (Pai, 1954). Slight improvements in Tollmien's and W. Schmidt's results have been obtained or may be obtained by considering these facts. They have of necessity been ignored here. Nevertheless the present results may be helpful in the interpretation of future experiments as well as in understanding some convective phenomena in the atmosphere.

The author wants to thank Prof. W. Bleeker and Mr. W. J. A. Kuipers for having contributed valuable remarks.

List of symbols

x, y, z	rectangular coordinates, z pointing upwards and y perpendicular to the plane of symmetry in the case of a two-dimensional jet.
$\overline{w}$	mean velocity in the z-direction.
$\overline{v}$	mean velocity in the y-direction.
w', v'	turbulent velocity components in the z- and y-directions.
$\overline{\vartheta}$	mean temperature excess of the jet.
ϑ'	turbulent fluctuation superimposed on $\overline{\vartheta}$.
w_m	mean vertical velocity in the symmetry plane (c.q. axis) of the jet.
ϑ_m	mean temperature excess in the symmetry plane (c.q. axis) of the jet.
Θ	temperature of the environment.
p	pressure.
g	acceleration of gravity.
ϱ	density.
ϱ_0	density of the environment.
R	measure for the width of the jet.
η	y/R.
A—F, q—t,	constants.
Q	measure for the heat output of the jet.

REFERENCES

Kuethe, A. M., 1953: Investigations of the turbulent mixing regions formed by jets; *Journ. Appl. Mech.* **2** p. A 87.

Morton, B. R., Taylor, Sir G., and Turner, J. S., 1956: Turbulent gravitational convection from maintained and instantaneous sources; *Proc. Roy. Soc., A* **234** p. 1.

Pai, Shih-I, 1954: *Fluid dynamics of jets;* Van Nostrand Comp., New York, Chapter V.

Priestley, C. H. B., and Ball, F. K., 1955: Continuous convection from an isolated source of heat; *Qu. J. Roy. Met. Soc.*, **81** p. 144.

Rouse, H., Yih, C. S., and Humphreys, H. W., 1952: Gravitational Convection from a Boundary Source; *Tellus* **4**, p. 201.

Schmidt, F. H., 1957: On the diffusion of stack gases in the atmosphere; *Koninklijk Nederlands Meteorologsch Instituut, Mededelingen en Verhandelingen 68.*

Schmidt, W., 1941: Turbulente Ausbreitung eines Stromes erhitzter Luft; *Z.A.M.M.* **21** p. 265 and p. 351.

Tollmien, W., 1926: Berechnung der turbulenten Ausbreitungsvorgänge; *Z.A.M.M.* **4** p. 468.