Steady Convective Motion in a Horizontal Layer of Fluid Heated Uniformly from Above and Cooled Non-uniformly from Below

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Abstract

A gravitationally stable fluid layer is heated non-uniformly (sinusoidally) at the bottom (or top) and a thermal convection regime is established. Examples are worked out for cases (i) without rotation (ii) with uniform rotation, and (iii) with non-uniform rotation (beta-plane).

1. Introduction

The study described in this paper is an outgrowth of a very interesting Russian paper by LINEYKIN (1955) in which he attempted to determine whether geostrophic currents, produced in a stable sea by wind, extend only to a certain limiting depth. There is, of course, ample observational evidence to indicate that currents decrease with depth because of the baroclinity of the ocean. The essential features which Lineykin incorporated into his model to limit the depth of currents were stratification of the ocean (in equilibrium, a linear increase of density with depth) and the Coriolis parameter (uniform rotation). The scale length of the model was taken to be 100 km so that his results could not be applied directly to large-scale oceanic motions. Evidently Lineykin had the Caspian Sea in mind, because attempts to extend his results to systems of horizontal oceanic scale lead to essentially barotropic (vertically uniform) velocities for depths of 5 km. It appears therefore that some additional physical factor—such as the variation of the Coriolis parameter—must be added in order to limit the depth of the currents much more drastically than the mechanisms envisaged by Lineykin. We have incorporated the variation of the Coriolis parameter into a simple model (§ 5) and have shown that it does indeed limit the currents to a depth which is much more realistic.

However, the basic objection to Lineykin's model and to the one discussed in this short paper is that the mean density structure of the ocean cannot be approximated by a constant vertical potential temperature gradient. Indeed, the problem appears to be essentially non-linear with advective processes playing a fundamental role in the formation of the main thermocline. For this reason we hesitate to offer the following results as a real explanation of the observed phenomena in the ocean.

Recently we attended a series of lectures on tropical meteorology by Drs. Herbert Riehl, Eric Palmén, and Joanne S. Malkus, and we realized that in some respects the atmosphere in the vicinity of the trade-wind regions is a more suitable phenomenon to which to apply Lineykin's analysis. In particular, it is much more nearly uniformly stable, and therefore, Lineykin's linearization procedure is more justifiable. It occurred to us that Lineykin's model might be formulated in such a way as to give a crude theory of the Trades. We are not qualified to judge whether the model is truly applicable to the Trades, but we have worked out the analytical details involved and present them here for the convenience of other workers. Also, of course, the results are in themselves of some interest.

2. Basis of the Model

Consider a fluid in a horizontal layer heated uniformly at the top and cooled below. Because of the thermal expansion of the fluid

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the density decreases upward, and the fluid is gravitationally stable. A uniform gradient of temperature $\partial\vartheta/\partial z$ (where z is directed positively upward and ϑ indicates temperature) is established; the heat flux $\varkappa \frac{\partial\vartheta}{\partial z}$ is determined by thermometric conductivity ($\varkappa$) alone and there is no fluid motion.

If the fluid is cooled non-uniformly below by a temperature perturbation $\vartheta_0 \cos ly$ impressed at the boundary $z = 0$, then we may anticipate horizontal differences of density in the fluid. Buoyancy forces produce convective motions which must be maintained against viscous and conductive dissipation; hence the amplitude and vertical extent of the convection must be determined. The present communication endeavors to describe these convective motions in the case where the motions are very slow and the disturbances of the temperature field are very slight (as compared to the basic uniform temperature field); in other words, this is a linearized perturbation theory.

The perturbation equations which we shall use are:

$$-f\varrho_0 v = -\frac{\partial p}{\partial x} + A\varrho_0 \frac{\partial^2 u}{\partial z^2} \qquad (1)$$

$$f\varrho_0 u = -\frac{\partial p}{\partial y} + A\varrho_0 \frac{\partial^2 v}{\partial z^2} \qquad (2)$$

$$g\varrho = -\frac{\partial p}{\partial z} \qquad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

$$w\,\overline{\frac{\partial\vartheta}{\partial z}} = \varkappa \frac{\partial^2\vartheta}{\partial z^2} \qquad (5)$$

$$\varrho = \varrho_0 (1 - \alpha\vartheta) \qquad (6)$$

Equations (1) and (2) are the linearized steady state equations of motion for a rotating system (f is the Coriolis parameter). Since the vertical scale of the perturbation fields in the examples discussed is much less than the horizontal scale, there appears to be no reason to carry the full Laplacian in the frictional term and we have kept only the vertical viscous shearing stress term. Equation (3) is the hydrostatic equation. Equation (4) is the Boussinesq approximation to the conservation of mass equation. Equation (5) is the steady state perturbation heat flow equation linearized by taking $w \overline{\frac{\partial\vartheta}{\partial z}}$ (where the bar indicates the mean unperturbed thermal gradient) as the only important heat advection term. We suppose this to be balanced by the divergence of the vertical conductive perturbed heat transport. Equation (6) is a simple linear equation of state.

We now proceed to discuss solutions of these equations for those cases (§ 3) no rotation, (§ 4) uniform rotation, and (§ 5) non-uniform rotation.

3. No rotation

Consider solutions independent of the x-direction, and in which $f = 0$. Eliminate all dependent variables except ϑ, thus obtaining the equation

$$\frac{\partial^6\vartheta}{\partial z^6} = -\frac{g\varrho_0\alpha b}{A\varkappa}\frac{\partial^2\vartheta}{\partial y^2} \qquad (7)$$

where we have introduced b to represent the basic temperature gradient $\overline{\partial\vartheta/\partial z}$. Consider a simple bottom temperature perturbation of the form $\vartheta = \vartheta_0 \cos ly$, and a solution of the form $\vartheta = \Theta(z) \cos ly$, in which case equation (7) reduces to an ordinary differential equation

$$\frac{d^6\vartheta}{dz^6} = F^6\vartheta \qquad (8)$$

where

$$F = \sqrt{\frac{g\varrho_0 abl^2}{A\varkappa}}$$

The quantity F^{-1} is the characteristic vertical scale of the convection field. We can call it the "frictional depth".

The solution is of the form

$$\vartheta = \vartheta_0 \cos ly \sum c_i e^{p_i Fz} \qquad (9)$$

where the p_i are the sixth roots of unity. If we consider only the three terms which become negligibly small for large z, we need two boundary conditions at $z = 0$ in addition to the condition $\Theta(0) = \vartheta_0$. Physically it is clear that at the bottom $w = v = 0$ are these conditions; in terms of Θ these two additional conditions are

$$\frac{d^2\Theta}{dz^2} = \frac{d^3\Theta}{dz^3} = 0,$$

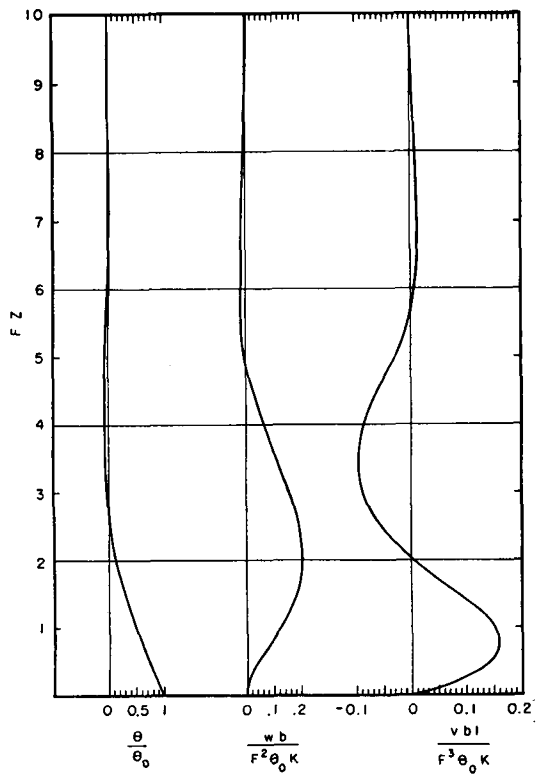


Fig. 1. Amplitude of perturbation temperature, and of the vertical and horizontal components of velocity as a function of altitude for the non-rotating case. Both coordinates are dimensionless.

from equations (5) and (4) respectively. The solutions of the problem are

$$\vartheta = \vartheta_0 \cos ly \left\{ \frac{1}{2} e^{-Fz} + \frac{1}{2} e^{-\frac{Fz}{2}} \left(\cos \frac{\sqrt{3}}{2} Fz + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} Fz \right) \right\} \tag{10}$$

$$w = \frac{F^2 \vartheta_0 \varkappa}{2b} \cos ly \left\{ e^{-\frac{Fz}{2}} + e^{-\frac{Fz}{2}} \left(-\cos \frac{\sqrt{3}}{2} Fz + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} Fz \right) \right\} \tag{11}$$

$$v = \frac{F^3 \vartheta_0 \varkappa}{2bl} \sin ly \left\{ -e^{-Fz} + e^{-\frac{Fz}{2}} \left(\cos \frac{\sqrt{3}}{2} Fz + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} Fz \right) \right\} \tag{12}$$

These relations are plotted in Fig. 1. The convective motion is in the form of long strip cells, or rolls, with axes in the x-direction. Fluid rises over warm portions of the bottom, and sinks over colder portions. The amplitude of the motion is proportional to the temperature perturbation at the bottom. The vertical extent of the motion is directly proportional to viscosity, conductivity, and horizontal scale. It varies inversely as the gravitational stability of the mean density field.

4. Uniform Rotation

In this case f is a constant and there is an x-component of velocity, u, but all variables are independent of the x-coordinate. Elimination of variables leads to an equation analogous to equation (8).

$$-\frac{1}{E^4} \frac{d^7 \vartheta}{dz^7} + \frac{d^3 \vartheta}{dz^3} - L^2 \frac{d\vartheta}{dz} = 0 \tag{13}$$

where

$$E^4 = -\frac{f^2}{A^2}, \quad L^2 = \frac{g \varrho_0 \cancel{q} b l^2}{f^2} \frac{A}{\varkappa}$$

α

The quantity $2\pi/E$ is the well-known "Ekman depth" and the quantity L^{-1} can be called the "Lineykin depth".

In the case where the only terms admitted in the solution are those which vanish at infinity and $L \gg E$ the solutions are

$$\frac{\vartheta}{\vartheta_0 \cos ly} = 2e^{-\frac{E}{\sqrt{2}} z} \left[D_1 \cos z \frac{E}{\sqrt{2}} + D_2 \sin z \frac{E}{\sqrt{2}} \right] + C e^{-Lz} \tag{14}$$

$$\frac{u}{\vartheta_0 \sin ly} = \frac{\sqrt{2} g \alpha l}{Ef} \left[(D_1 - D_2) \sin \frac{E}{\sqrt{2}} z - (D_1 + D_2) \cos \frac{E}{\sqrt{2}} z \right] e^{-\frac{E}{\sqrt{2}} z} - \frac{g\alpha l}{Lf} C e^{-Lz} - \frac{\sqrt{2} A \varkappa E^5}{flb} \left[(D_1 - D_2) (\cos \frac{E}{\sqrt{2}} z + (D_1 + D_2) \sin \frac{E}{\sqrt{2}} z \right] e^{-\frac{E}{\sqrt{2}} z} + \frac{A\varkappa}{flb} C L^5 e^{-Lz} + \text{const.} \tag{15}$$

$$\frac{v}{\vartheta_0 \sin ly} = \frac{\sqrt{2}E^3\varkappa}{lb}\left\{(D_2 - D_1)\sin\frac{E}{\sqrt{2}}z - \right.$$
$$\left. -(D_1 + D_2)\cos\frac{E}{\sqrt{2}}z\right\}e^{-\frac{E}{\sqrt{2}}z} +$$
$$+\frac{\varkappa L^3}{lb}Ce^{-Lz} \tag{16}$$

$$\frac{w}{\vartheta_0 \cos ly} = \frac{2E^2\varkappa}{b}\left\{D_1 \sin\frac{E}{\sqrt{2}}z - \right.$$
$$\left. - D_2 \cos\frac{E}{\sqrt{2}}z\right\}e^{-\frac{E}{\sqrt{2}}z} + \frac{\varkappa L^2}{b}Ce^{-Lz} \tag{17}$$

where

$$D_1 = \frac{L^2(2L^2 - E^2)}{4L^4 - 2L^2E^2 + 2\sqrt{2}LE^3 + 2E^4}$$

$$D^2 = \frac{L^2(E^2 + \sqrt{2}LE)}{4L^4 - 2L^2E^2 + 2\sqrt{2}LE^3 + 2E^4}$$

$$C = \frac{E^3}{E^3 - EL^2 + \sqrt{2}L^3}$$

These relations are plotted in Fig. 2. The Ekman layer is noticeable only in the lowest portion of the figure, the remainder of the profile being controlled essentially by the Lineykin depth. Qualitatively gravitational stability and horizontal scale have the same kind of influence on the vertical extent of the convective motion as in the non-rotating case, but their influence is stronger. On the other hand, the viscosity and conductivity enter as a ratio: if, in large-scale problems one were to suppose that they are to be interpreted as eddy-coefficients of comparable magnitude, the Lineykin depth must be largely *independent* of their amplitude. The ordinates are plotted non-dimensionally for Lz since L^{-1} is the controlling depth, but if E is changed the shape of the lower portions of the curves is modified. Thus the curves in Fig. 2 are not universal curves in the same sense as those in Fig. 1. The origin for the u-curve in Fig. 2 is arbitrary as indicated by the constant in equation (15). The constant can be determined by specifying the pressure at the level $z=0$. If one imposes $u=0$ at $z=0$, u approaches a constant value with height. If u is supposed to vanish at infinity it is finite at $z=0$. In either case, however, $\partial u/\partial z$ vanishes at $z=0$.

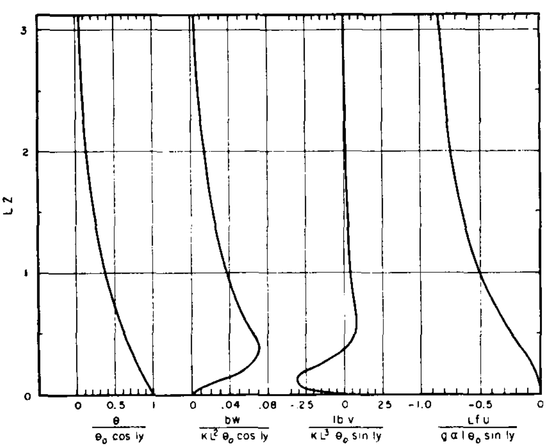


Fig. 2. Amplitude of the perturbation temperature and components of velocity as functions of altitude for the case of uniform rotation. The ordinate is the dimensionless quantity Lz expressed in terms of the Lineykin-depth. However, since there is an additional characteristic depth E^{-1}, it is not possible to present universal curves, particularly for the lowest values of Lz. The abscissae are dimensionless.

5. Non-uniform Rotation

Let us now consider the case where the Coriolis parameter varies with latitude, i.e., $\beta = \partial f/\partial y \neq 0$. For reasons that will become evident, the geometry of the problem is also changed; the perturbed boundary condition is taken as a function of x instead of y; at $z=0$, $\vartheta = \vartheta_0 \cos lx$; and also, in order to correspond to the ocean, the top instead of the bottom is taken at $z=0$. We are thus concerned with the solution in the region $z \leq 0$. It can be shown that for a certain choice of parameters (in particular large horizontal scale of oceanic dimensions) the main features of the temperature field come from the following approximate equation for ϑ:

$$\frac{\partial^4\vartheta}{\partial z^4} = \frac{\beta g\alpha b}{f^2\varkappa}\frac{\partial\vartheta}{\partial x} \tag{18}$$

and if we let $\vartheta = \Theta e^{ilz}$ we obtain

$$\frac{d^4\Theta}{dz^4} = iG^4\Theta$$

$$\text{where } G = \sqrt[4]{\frac{\beta g\alpha bl}{f^2\varkappa}}$$

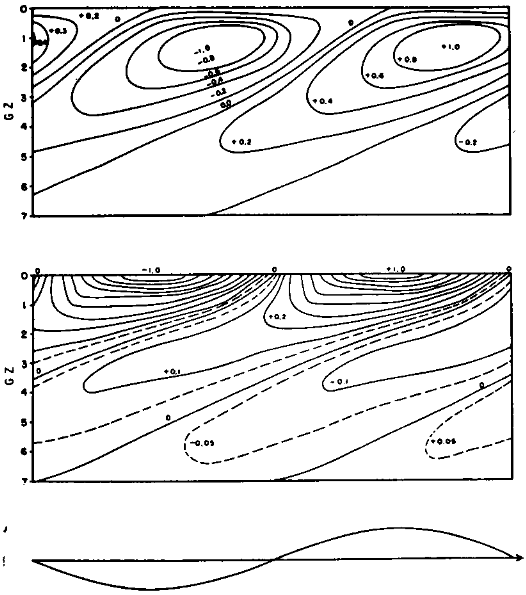


Fig. 3. (a) Contours of equal perturbation temperature (Ocean example with the non-dimensional depth Gz measured positive downward) in the case of non-uniform rotation. The horizontal coordinate is directed eastward toward the right of the figure. The perturbation temperature at the surface ($z=0$) is a sine function. The origin of the x-coordinate is in the center of the figure. The numbers on the contours represent the fraction of the amplitude of the surface perturbation.

$$2\sqrt{2}\,\frac{w f^2 r^2}{\beta g \alpha l T_0}$$

(b) Contours of the vertical velocity component in dimensionless form

The vertical scale in both (a) and (b) is exaggerated.

These equations can be obtained directly from equations (1)—(6) if the viscous terms in (1) and (2) are omitted. (It can be shown that for large-scale disturbances the effect of A is negligible below the Ekman layer.) The quantity defines a new characteristic depth which depends upon a different combination of parameters. There is a phase shift toward the west with depth, so that regions (or cells) whose perturbation temperature has the same sign tilt toward the west. Figures similar to Figs. 1 and 2 are not so easily interpretable. Therefore the full non-dimensional field of ϑ and w in the x—z plane is shown in Fig. 3, according to the solutions of equation (18):

$$\vartheta = \frac{\vartheta_0}{2}\left\{\cos lx\left[e^{-Gz\cos\frac{\pi}{8}}\cos\left(Gz\sin\frac{\pi}{8}\right) + e^{-Gz\cos\frac{3\pi}{8}}\cos\left(Gz\sin\frac{3\pi}{8}\right)\right] + \sin lx\left[e^{-Gz\cos\frac{\pi}{8}}\sin\left(Gz\sin\frac{\pi}{8}\right) - e^{-Gz\cos\frac{3\pi}{8}}\sin\left(Gz\sin\frac{3\pi}{8}\right)\right]\right\}. \tag{19}$$

$$w = \frac{\sqrt{2}}{4}\frac{G^2\varkappa\vartheta_0}{b}\left\{\cos lx\left[e^{-Gz\cos\frac{\pi}{8}}\left(\cos\left(Gz\sin\frac{\pi}{8}\right) + \sin\left(Gz\sin\frac{\pi}{8}\right)\right) - e^{-Gz\cos\frac{3\pi}{8}}\left(\cos\left(Gz\sin\frac{3\pi}{8}\right) - \sin\left(Gz\sin\frac{3\pi}{8}\right)\right)\right] + \sin lx\left[-e^{-Gz\cos\frac{\pi}{8}}\left(\cos\left(Gz\sin\frac{\pi}{8}\right) - \sin\left(Gz\sin\frac{\pi}{8}\right)\right) + e^{-Gz\cos\frac{3\pi}{8}}\left(\cos\left(Gz\sin\frac{3\pi}{8}\right) + \sin\left(Gz\sin\frac{3\pi}{8}\right)\right)\right]\right\} \tag{20}$$

Table 1.

Parameters	Experimental Water Model	Atmosphere	Ocean	Units
A	10^{-2}	$5\cdot10^5$	5	cm²/sec
K	10^{-3}	$5\cdot10^5$	5	"
$\frac{1}{\varrho}\frac{\partial\varrho}{\partial z}$	$2.5\cdot10^{-4}$	$2.5\cdot10^{-7}$	$5\cdot10^{-9}$	cm^{-1}
l	10^{-1}	10^{-8}	10^{-8}	cm^{-1}
f	1	10^{-4}	10^{-4}	sec^{-1}
α	$2.5\cdot10^{-4}$	$2.5\cdot10^{-3}$	$2.5\cdot10^{-4}$	$°C^{-1}$
β		$2\cdot10^{-13}$	$2\cdot10^{-13}$	$cm^{-1}\,sec^{-1}$
F^{-1}	0.4	$1.5\cdot10^5$	$9\cdot10^3$	cm
E^{-1}	0.1	$0.7\cdot10^5$	$2.2\cdot10^2$	cm
L^{-1}	7.0	$6.3\cdot10^5$	$45.0\cdot10^5$	cm
G^{-1}		$3.2\cdot10^5$	$0.46\cdot10^5$	cm

$$F^{-1} = \left[\frac{A\varkappa}{g \frac{1}{\varrho} \frac{\partial \varrho}{\partial z} l^2} \right]^{1/6}$$

$$E^{-1} = \left[\frac{A}{f} \right]^{1/2}$$

$$L^{-1} = \left[\frac{f^2}{g \frac{1}{\varrho} \frac{\partial \varrho}{\partial z} l^2} \quad \frac{\varkappa}{A} \right]^{1/2}$$

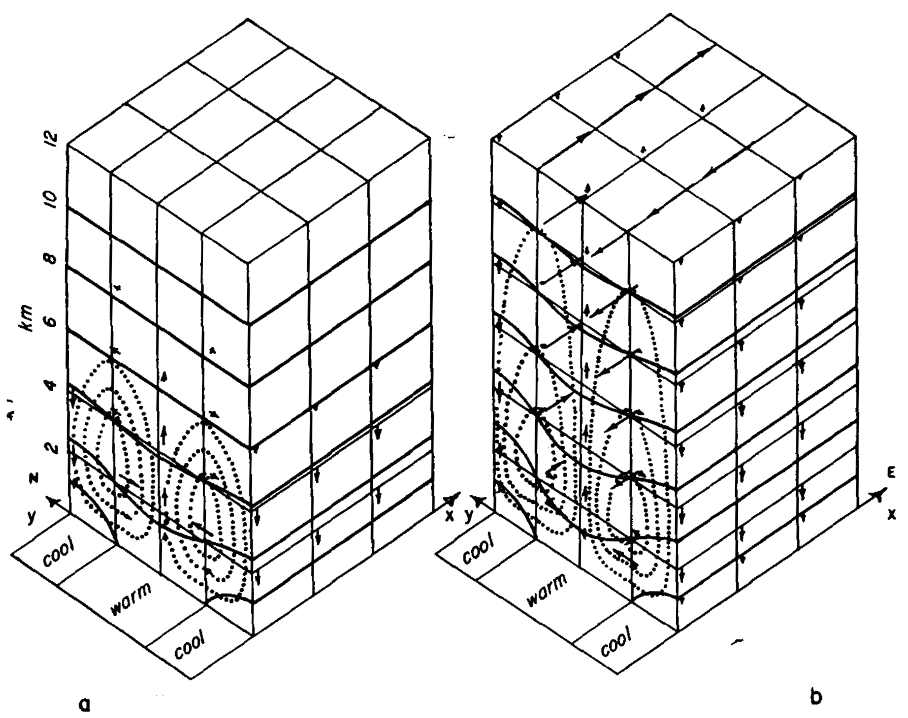
$$G^{-1} = \left[\frac{f^2 \varkappa}{g \frac{1}{\varrho} \frac{\partial \varrho}{\partial z} l\beta} \right]^{1/4}$$

6. Discussion of Results

In order to exhibit the contrast in temperature and velocity structure in the different cases, parameters appropriate to three different physical situations are inserted into the expressions for the various characteristic lengths and the results tabulated in Table I.

The water experimental model is envisaged as being about 10 cm deep, with a basic vertical temperature gradient of 1°/cm. The bottom temperature is perturbed with a wave length of 60 cm, and an amplitude of not much more than 0.1° C (otherwise the perturbation approximation breaks down). The viscosity and conductivity coefficients are molecular. Under these conditions a very thin ($F^{-1} = 0.4$ cm) convective cellular motion arises with rising over the warm areas of the bottom and sinking over the cool areas, and compensating horizontal flows. As can be seen from Fig. 1, the top of the cells is effectively at $Fz = 6$, or about 2.4 cm. If this whole system is uniformly rotated at about one revolution every twelve seconds, the height of the cellular pattern increases about seventeen times, a thin Ekman layer appears close to the bottom, and a large component of geostrophic flow transverse to the horizontal temperature gradients appears. As it is not easy to picture how a non-uniformly-rotating experimental model could be constructed this case is not discussed.

The parameters in the atmospheric case are chosen to coincide as nearly as possible to



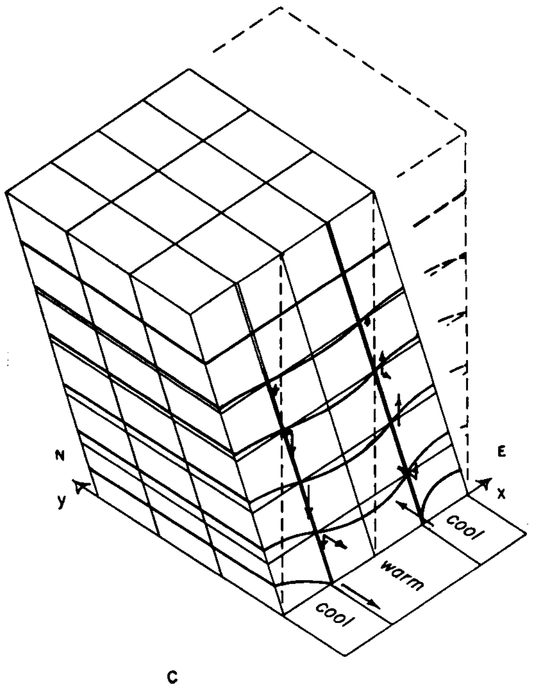


Fig. 4. Three cases using parameters for the atmosphere as given in Table I, with the vertical scale greatly exaggerated. The velocity components at selected points are shown by arrows. The heavy lines delineate iso-thermal surfaces. The dotted curves are the two-dimensional streamfunction in the y—z plane. Case (a) is non-rotating, there is no x-component of velocity as there is in case (b) of uniform rotation. In the case of non-uniform rotation (c) the applied boundary temperature is a function of x (instead of y as in (a) and (b)). The dashed lines represent the vertical, and the regions of perturbation temperature of same sign are enclosed by the tilted lines. Thus in (c) each warm and cold cell is tilted toward the west. On the other hand there is no x-component of velocity: all motion is confined to planes x-constant.

those used by Malkus and Riehl in a discussion of the Trades. The extremely large eddy coefficients are meant to include effects of cumulus convection as turbulent eddies. The mean stability corresponds to a vertical potential temperature lapse rate of $1°/100$ m. The perturbation wave length is 6,000 km. The characteristic lengths are computed in Table I, and a three-dimensional sketch of the three atmospheric cases is shown in Fig. 4. Uniform rotation (b) produces a much higher cell than no rotation (a), and introduces zonal winds. The model for non-uniform rotation, in which the perturbed bottom temperature is a function of longitude (4c) instead of latitude, has a marked westward tilt with height. There is no horizontal component of cross-isobar flow because in the simplified case treated in Section 5 there is no friction. All flow patterns are geostrophic and in vertical meridional planes. The vertical velocity components are associated with the planetary divergence of the meridional winds.

The oceanic model in Table I is very unrealistic on account of the absence of meridional barriers. It is interesting to note that lack of the rotation gives a very shallow depth of penetration of the perturbation temperature (or broadly speaking, depth of the main thermocline) (90 meters); in the uniform rotation case the Lineykin depth of the current is 45 km; non-uniform rotation (c) leads to a thermocline depth of 470 meters.

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