On Stratified Flows in a Gravitational Field

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Abstract

This paper, which is restricted to the flow of inviscid fluids, is divided into three parts. In the first part existing formulas by A. R. Richardson, Hans Lewy, and K. N. Tong for constructing free-surface flows are identified with one another so that potential flows of one fluid in contact with a stagnant layer of another fluid can be constructed exclusively from one general formula, with proper modification of the gravitational acceleration. Further, a method for constructing potential flows of two fluids having a common interface (which may or may not be prescribed) is devised. It is hoped that this method will be of some use in the investigation of internal gravity waves. In the second part, it is shown that, for flows of a fluid system of discrete layers, Long's equation of motion for two-dimensional flow of an inhomogeneous fluid reduces to the Laplace equation for each layer (if the flow is irrotational) and the usual boundary conditions at the interfaces, so that flows with discontinuous density variations can be properly considered as limiting cases of flows with continuous ones. For the sake of completeness, equations of motion of an inhomogeneous fluid in axisymmetric motion in cylindrical and spherical coordinates are also given. In the third part the stability of a periodic disturbance present in a parallel flow with continuous density variation is discussed. Sufficient conditions for stability are found, and an upper bound for the amplification factor is given (if instability occurs) for a general class of flows.

I. Introduction

In the study of stratified flows, which have occupied the attention of many hydrodynamicists of the last century and a number of contemporary investigators, a fluid system of many distinct layers has often been considered, partly for the sake of simplicity, partly because essentially distinct layers do occur in nature. On the other hand, flows with continuous density variation have been treated by LOVE (1891) for small disturbances of a vertical stratification, by ROSSBY (1951) and CRAYA (1951), who used the momentum principle and the energy principle (respectively) to obtain the critical regime, and by LONG (1953), who gave the equation for general two-dimensional motion of a heterogeneous inviscid fluid in terms of the stream function. There still appears to be some doubt (BENTON 1953 and LONG 1955) as to whether flows with continuous density variation can be approximated with those of descrete layers.

In this paper, a conformal-mapping method for generating potential flows of two distinct layers of fluid with a common interface (prescribed or unprescribed) will first be presented, after a brief discussion of the essential identity of existing methods for generating free-surface flows of a homogeneous fluid. Then it will be shown that for discontinuous stratification Long's equation reduces to the Laplace equation within each layer, and to boundary conditions on the interfaces according to the Bernoulli equation. Consequently, flows with discrete density variation can be properly considered as limiting cases of those with a continuous one. The reverse is very likely also true, though a rigorous demonstration is still lacking. The two modes of approach in the study of stratified flows are thus tied together, and at

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least a partial justification is provided for using discrete layers to approximate a continuous density variation. For the sake of completeness, equations of axisymmetric motion of an inhomogeneous fluid will be given in cylindrical and spherical coordinates, in terms of Stokes' stream function. Finally, the stability of parallel flows of a heterogeneous fluid will be discussed.

II. Potential Flows with an Interface in a Gravitational Field

Problems of potential flow of a single fluid with one or more free surfaces (on which the pressure is constant) are solved by demanding the velocity potential or the stream function to satisfy the Laplace equation and certain boundary conditions. Because the boundary condition on the free surfaces is a non-linear one, very few physically significant problems of this category have been solved exactly. Reversing the normal order of affairs, A. R. RICHARDSON (1920) devised a method for obtaining potential flows possessing a free surface, or for constructing one with a preassigned surface as the free surface. His method was later rediscovered by HANS LEWY (1951) and K. N. TONG (1954) and was described by these authors in formulas which differ more or less in form from Richardson's, but not in essence. In this section, the identity of the formulas of the three authors mentioned above will first be shown, and then Richardson's reverse method will be extended to apply to potential flows of two layers of immiscible liquids with a common interface, which may or may not be prescribed. Only two-dimensional flows will be considered.

1. *Identification of existing formulas for free-surface flows*

Richardson's first formula for constructing free-surface flows is

$$-\frac{dz}{dw} = [3gG(w)]^{-1/3}\{[1 - G'^2(w)]^{1/2} + iG'(w)\} \tag{1}$$

in which z is the complex variable $x + iy$ with y measured in a direction opposite to that of the gravitational force, w is the complex potential, g is the gravitational acceleration, primes denote differentiation with respect to w, and $G(w)$ is a function of w such that $G^{1/3}$, G', and $(1 - G'^2)^{1/2}$ are real and finite on the free surface. His second formula is for constructing a free-surface flow for which the pressure on the free surface is a function of w, and has the form[1]

$$\frac{dz}{dw} = \left[\frac{1}{c^2 + 2gh - 2gF(w) - 2P(w)} - F'^2(w)\right]^{1/2} + iF'(w) \tag{2}$$

in which c and h are constants, F and P are functions of w, and F' and the square root of the bracket are real over the free surface.

It can be readily verified that the Bernoulli equation

$$\left|\frac{dw}{dz}\right|^2 + 2gy = \text{constant}$$

is satisfied on the free-surface by the flow given by Eq. (1), so that the pressure is constant on the free surface. With equal ease one can verify that on the free surface of the flow given by Eq. (2) one has

$$y = F(w), \qquad p = \varrho P(w)$$

except possibly for an additive constant. If the pressure on the free surface is constant, one can take

$$c^2 + 2gh - 2P = 0$$

and Eq. (2) reduces to

$$\frac{dz}{dw} = \left[-\frac{1}{2gF(w)} - F'^2(w)\right]^{1/2} + iF'(w) \tag{3}$$

Lewy's formula is, in the same notation and with g equal to 1,

$$z = \int \left[\frac{1}{2\lambda(w)} - \lambda'^2(w)\right]^{1/2} dw - i\lambda(w) \tag{4}$$

in which both $\lambda(w)$ and the integrand of the integral are real along the free surface, whereas Tong's formula is

$$\frac{dw}{dz} = [2c - 2gf(\gamma)]^{1/2} \left[\frac{1 - if'(\gamma)}{1 + if'(\gamma)}\right]^{1/2} \tag{5}$$

in which γ is a complex variable connected with z through the equation

$$z = \gamma + if(\gamma) \tag{6}$$

[1] The multiplier 2 of the function P is added by the writer to make Richardson's statement about the pressure on the free surface correct.

in which both γ and $f(\gamma)$ are real on the free surface. It is a matter of straightforward calculation to show that the Bernoulli equation is satisfied on the free surface.

It will now be shown that Eqs. (1), (3), (4), and (5) are essentially identical. First, it is obvious that if Eq. (4) is differentiated with respect to w and $\lambda(w)$ is set equal to $-F(w)$ the resulting equation is identical with Eq. (3) if g is set equal to 1 (as Lewy did) in the latter. The condition of reality of the several quantities involved is not affected by the substitution of $-F$ for λ. Next, if one sets

$$F(w) = -(2g)^{-1/3} H(w)$$

in Eq. (3), and remembers that one is free to choose the sign of the square root therein, one has

$$-\frac{dz}{dw} = (2g)^{-1/3}\{[H^{-1}(w) - H'^2(w)]^{1/2} + iH'(w)\} \tag{7}$$

which is the same as Eq. (1) if one writes

$$H(w) = [3G(w)/2]^{2/3}$$

Thus Eqs. (1) and (3) are essentially the same. Finally, setting

$$f(\gamma) = -(2g)^{-1/3}H(w) + \frac{c}{g}$$

one can show, with the aid of Eq. (6) that Eqs. (5) and (7) are the same. Thus Eqs. (1), (3), (4), and (5) are essentially identical, and are equivalent to Eq. (7). Since among the five equations Eq. (7) is the simplest one that contains g explicitly, it will be used for further discussion.

Existing formulas for constructing free-surface potential flows are applicable to potential flows of one liquid in contact with a stagnant layer of another liquid. If γ_1 denotes the specific weight of the upper and stagnant layer, and γ_2 that of the moving layer, the Bernoulli equation for the moving layer at the interface is

$$\varrho_2 \left|\frac{dw}{dz}\right|^2 + 2(\gamma_2 - \gamma_1)\, y = \text{constant} \tag{8}$$

if ϱ_2 is the density of the moving fluid. Equation (14) can be written as

$$\left|\frac{dw}{dz}\right|^2 + 2g'y = \text{constant}$$

if

$$g' = \frac{\gamma_2 - \gamma_1}{\varrho_2}$$

Thus all the existing formulas for constructing free-surface potential flows can be used to construct potential flows having an interface with a stagnant layer, if g is changed to g'. Since these formulas have been shown to be identical and replaceable by Eq. (7), one may use the formula

$$-\frac{dz}{dw} = (2g')^{-1/3}\{[H^{-1}(w) - H'^2(w)]^{1/2} + iH'(w)\} \tag{9}$$

exclusively for constructing potential flows having an interface with a stagnant layer.

2. *Steady potential flows of two layers with a common interface*

If w_1 denotes the complex potential of the upper layer and w_2 that of the lower layer, and if primes denote ordinary differentiation, the appropriate formulas for potential flows of two layers of fluid with a common interface are, with k as the density ratio ϱ_1/ϱ_2,

$$-\frac{dz}{dw_1} = \frac{1}{E}[(kF_1^{-1} - G_1'^2)^{1/2} + iG_1'] \tag{10}$$

and

$$-\frac{dz}{dw_2} = \frac{1}{E}\left\{[(F_1 + G_1)^{-1} - K_2'^2]^{1/2} + iK_2'\right\} \tag{11}$$

in which E is a constant to be determined, the argument of F_1 and G_1 is w_1, and that of K_2 is w_2. Integration of Eqs. (10) and (11) along a streamline on which the square roots and the multipliers of i in these equations are real yields the following relationships:

$$y = -G_1/E, \qquad y = -K_2/E \tag{12}$$

$$kq_1^2 = E^2F_1, \qquad q_2^2 = E^2(F_1 + G_1) \tag{13}$$

where q_1 is the magnitude of the velocity vector for the upper layer, and q_2 is that for the lower layer. In Eq. (12) the constants of integration have been absorbed in G_1 and K_2. If now E^3 is taken to be $2g'$, one has

$$\varrho_1 q_1^2 - \varrho_2 q_2^2 = 2g(\varrho_2 - \varrho_1)\, y \tag{14}$$

which is the condition of the continuity of pressure across the interface – the only condition aside from the one that the interface must be a streamline if steady flows are considered.

Geometrical compatibility requires, from Eq. (12), that

$$G_1(w_1) = K_2(w_2) \tag{15}$$

and, from Eqs. (10) to (12), that

$$(kF_1^{-1} - G_1'^2)^{1/2} dw_1 = [(F_1 + G_1)^{-1} - K_2'^2]^{1/2} dw_2 \tag{16}$$

on the interface. Thus if the functional forms G_1 and K_2 are known and w_1 is known on the surface, w_2 will be known on the surface as a function of w_1. Then by analytic continuation w_1 and w_2 are known for both layers. Under these circumstances, the function F_1 is not arbitrary. In fact, since

$$G_1' dw_1 = K_2' dw_2 \tag{15}$$

one has, from Eq. (16)

$$kF_1^{-1} - G_1'^2 = [(F_1 + G_1)^{-1} - K_2'^2] G_1'^2/K_2'^2 = \\ = \frac{G_1'^2}{K_2'^2}(F_1 + G_1)^{-1} - G_1'^2$$

or

$$F_1 = \frac{kK_2'^2 G_1}{G_1'^2 - kK_2'^2} \tag{17}$$

which determines F_1 on the interface since w_2 is a known function of w_1 there. By analytic continuation $F_1(w_1)$ is known throughout the upper layer, and $F_1[w_1(w_2)]$ known throughout the lower layer. The conformal mappings are thus complete.

If one wishes to construct two potential flows with a common interface given by

$$x = C(y) \tag{18}$$

one can set

$$(kF_1^{-1} - G_1'^2)^{1/2}/G_1' = C'(-G_1/E) \tag{19}$$

which, with the prime denoting ordinary differentiation, is the differential form of Eq. (18), as can be shown from Eq. (10). One can thus choose G_1 (to satisfy any other boundary condition if possible in practice), solve Eq. (19) for F_1 and Eq. (17) for K_2'. Thus from Eq. (15) dw_2/dw_1 (and hence w_2) is known as a function of w_1, and $K_2(w_2)$ is simply $G_1[w_1(w_2)]$. The required conformal mappings are then given by Eqs. (10) and (11).

It is hoped that the method just achieved for constructing potential flows of two fluids with a common interface will be of some use in the investigation of many geophysical phenomena, the main features of which can be simulated by potential flows of two layers.

III. Equations of Motion of an Inhomogeneous Fluid

For the motion of a heterogeneous fluid considered to be inviscid, Euler's equations of motion do not yield the persistence of irrotationality as they do for that of a homogeneous inviscid fluid. For this reason, the customary procedure of substituting the Laplace equation (which is kinematic in nature) for the Euler equations (which are dynamic) cannot be used when the fluid is not homogeneous. What, then, is the equation governing the motion of an inviscid fluid with variable density? In the case of two-dimensional steady flows, this question has been answered by LONG (1953). In this section, the connection of Long's equation with the differential equation and boundary conditions at the interfaces governing the flow of distinct layers will be established, in order to provide a link between the two modes of study of stratified flows mentioned in the Introduction. Then the result of Long will be extended to axisymmetric steady flows, and the equations of motion of such flows will be given in cylindrical and spherical coordinates, in terms of Stokes' stream function.

1. *Discussion of Long's equation*

For two-dimensional steady flows of an inviscid and inhomogeneous fluid which is also incompressible and non-diffusive, Long obtained the equation of motion in terms of the stream function ψ:

$$\nabla^2 \psi + \frac{d\ln\varrho}{d\psi}\left(\frac{\psi_x^2 + \psi_y^2}{2} + gy\right) = F(\psi) \tag{20}$$

in which ∇^2 is the Laplacian in two dimensions, ϱ is the density, which depends on ψ alone, x and y are Cartesian coodinates with y measured in the vertical direction, g is (as before) the gravitational acceleration, and subscripts denote partial differentiation.

For the flow of distinct layers, within each layer the density is constant, so that Eq. (20) reduces to

$$\nabla^2 \psi = F(\psi)$$

If the motion starts from rest or any irrotational state, the persistence of irrotationality demands that $F(\psi)$ vanish, and the equation of motion reduces to the usual Laplace equation:

$$\nabla^2 \psi = 0$$

At the interfaces not only the density but also the magnitude q of the velocity undergoes a sudden change, for there usually is slip at the interfaces because viscosity has been neglected. With this in mind, one may proceed from the equation of motion for a continuously stratified fluid and show that, at the interfaces of a discontinuous stratification, it reduces to the usual boundary condition. Since

$$q^2 = \psi_x^2 + \psi_y^2$$

Eq. (20) can be written in the form

$$\nabla^2 \psi + \frac{1}{\varrho} \frac{d\varrho}{d\psi} \left(\frac{q^2}{2} + gy \right) = F(\psi)$$

which becomes, after multiplication by $\varrho d\psi$,

$$\left(\frac{q^2}{2} + gy \right) d\varrho + \varrho \nabla^2 \psi d\psi = F(\psi) \varrho d\psi \qquad (21)$$

But $-\nabla^2 \psi$ is the vorticity, so that

$$\nabla^2 \psi = \frac{\partial q}{\partial n} - \frac{\partial v_n}{\partial s}$$

in which s is measured along the interfacial streamline and n normal to it, v_n is the (zero) component of the velocity in the n-direction, and[1] $\partial v_n / \partial s$ is in general not zero even though v_n is zero. Since $d\psi$ is equal to $q dn$, Eq. (21) can be written as

$$\left(\frac{q^2}{2} + gy \right) d\varrho + \varrho q \frac{\partial q}{\partial n} dn + \varrho q \frac{\partial q}{\partial s} ds =$$

$$= \varrho \left[F(\psi) + \frac{\partial v_n}{\partial s} \right] \partial \psi + \varrho q \frac{\partial q}{\partial s} ds$$

[1] Strictly speaking, the absolute derivative of tensor calculus should be used. This point is not elaborated here because all one wants to say is that the quantity in question is finite.

or

$$d \left(\frac{\varrho q^2}{2} + \varrho g y \right) = \varrho \left[F(\psi) + \frac{\partial v_n}{\partial s} \right] d\psi + \varrho q \frac{\partial q}{\partial s} ds + \varrho g dy \qquad (22)$$

The multipliers of the differentials on the right-hand side of Eq. (22) are all finite, even at the interface of a discontinuous stratification, since there is no discontinuity in the direction of s. The quantity in parentheses on the left-hand side, however, undergoes a jump at such an interface. Letting $d\psi$ and ds become small indefinitely, one has, at the interface of two distinct layers,

$$\Delta \left(\frac{\varrho q^2}{2} + \varrho g y \right) = 0 \qquad (23)$$

or

$$\frac{\varrho_1 q_1^2}{2} + \varrho_1 g y = \frac{\varrho_2 q_2^2}{2} + \varrho_2 g y \qquad (24)$$

if subscripts are used to distinguish two neighbouring layers. But Eq. (24) is nothing but the usual boundary condition at the interface in accordance with the Bernoulli equation.

Thus one has established a link between flows with continuous density variations and those with discontinuous ones, and shown that the latter can be properly considered as limiting cases of the former. In this manner one gains an insight into the structure of Long's equation. Although it is not possible to obtain Long's equation directly from the differential equation and interfacial conditions governing the flow of a fluid system of many layers, it must nevertheless be remembered that, since streamlines are isopcynic lines in Long's derivation, the resulting equation governs the flow of infinitely many homogeneous layers of infinitesimally small thickness. From this point of view flows with continuous density variation can be considered as limiting cases of flows of discrete layers. Thus one can hardly question the general validity of the approximation of continuously stratified flows by flows of distinct layers. The accuracy of the approximation depends, of course, on the number of layers used.

2. *Equations of axisymmetric flows with continuous stratification*

Long's equation will now be extended to steady axisymmetric flows, under the same assumptions in regard to the properties of the fluid. If subscripts indicate partial differentiation, the equation of continuity is, in cylindrical coordinates

$$(ru\varrho)_r + (rw\varrho)_z = 0 \tag{25}$$

in which u is the velocity component in the r-direction, w is that in the z-direction, and ϱ is the density. But the persistence of ϱ along any path line, which is a consequence of the incompressibility and nondiffusiveness of the fluid, demands that

$$\frac{D\varrho}{Dt} = 0 \tag{26}$$

in which t denotes time, and

$$\frac{D}{Dt} \equiv u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$$

indicates the substantial differentiation. Since the motion is steady, path lines are streamlines, and the persistence of ϱ along any path line implies that along any streamline. With the aid of Eq. (26), Eq. (25) is now reduced to the simple form

$$(ru)_r + (rw)_z = 0 \tag{27}$$

which permits the use of Stokes' stream function ψ such that

$$u = -\psi_z/r, \qquad w = \psi_r/r \tag{28}$$

Euler's equations of motion are

$$\varrho\frac{Du}{Dt} = -p_r \tag{29}$$

$$\varrho\frac{Dw}{Dt} = -p_z - g\varrho \tag{30}$$

in which g is the gravitational acceleration in the negative z-direction, and p is the pressure.

Eliminating p from Eqs. (29) and (30) by cross differentiation, one has

$$\left(\varrho\frac{Du}{Dt}\right)_z - \left(\varrho\frac{Dw}{Dt}\right)_r = g\varrho_r$$

or

$$\varrho\frac{D(u_z - w_r)}{Dt} + \varrho(u_z - w_r)(u_r + w_z) + \\ + \varrho_z\frac{Du}{Dt} - \varrho_r\frac{Dw}{Dt} - g\varrho_r = 0 \tag{31}$$

But since ϱ is a function of ψ alone, one has, with the aid of Eq. (28),

$$\varrho_z\frac{Du}{Dt} - \varrho_r\frac{Dw}{Dt} - g\varrho_r = \\ = \frac{d\varrho}{d\psi}\left(\psi_z\frac{Du}{Dt} - \psi_r\frac{Dw}{Dt} - \psi_r g\right) = \\ = r\frac{d\varrho}{d\psi}\left(-u\frac{Du}{Dt} - w\frac{Dw}{Dt} - wg\right) = \\ = -r\frac{d\varrho}{d\psi}\frac{D}{Dt}\left(\frac{q^2}{2} + gz\right) \tag{32}$$

in which

$$q^2 = u^2 + w^2$$

On the other hand

$$u_r + w_z = \psi_z/r^2 = -u/r \tag{33}$$

$$u_z - w_r = -\frac{1}{r}\nabla'^2\psi \tag{34}$$

in which

$$\nabla'^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

is the Stokian operator.

If Eqs. (32) and (33) are substituted in Eq. (31), one obtains, after dividing through by r,

$$\frac{\varrho}{r}\frac{D(u_z - w_r)}{Dt} - \frac{\varrho u}{r^2}(u_z - w_r) - \\ - \frac{d\varrho}{d\psi}\frac{D}{Dt}\left(\frac{q^2}{2} + gz\right) = 0$$

or

$$\varrho\frac{D}{Dt}\frac{u_z - w_r}{r} - \frac{d\varrho}{d\psi}\frac{D}{Dt}\left(\frac{q^2}{2} + gz\right) = 0$$

Because the substantial derivatives of ϱ and ψ or of their functions are zero, one has

$$\frac{D}{Dt}\left[\varrho\left(\frac{u_z - w_r}{r}\right) - \frac{d\varrho}{d\psi}\left(\frac{q^2}{2} + gz\right)\right] = 0$$

or

$$\varrho\left(\frac{u_z - w_r}{r}\right) - \frac{d\varrho}{d\psi}\left(\frac{g^2}{2} + gz\right) = f(\psi) \tag{35}$$

If Eqs. (28) and (34) are substituted into Eq. (35) and the result divided throughout by ϱ, it follows, finally, that

$$\frac{\nabla'^2\psi}{r^2} + \frac{d\ln\varrho}{d\psi}\left(\frac{\psi_r^2 + \psi_z^2}{2r^2} + gz\right) = F(\psi) \tag{36}$$

which is the desired equation.

The corresponding equation for spherical coordinates (r, Θ, φ) is

$$\frac{\nabla'^2\psi}{r^2 \sin\Theta} + \frac{d\ln\varrho}{d\psi}\left(\frac{\psi_\Theta^2 + r^2\psi_r^2}{2r^4\sin^2\Theta} + gr\right) = F(\psi) \quad (37)$$

where now

$$\nabla'^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{\cot\Theta}{r^2}\frac{\partial}{\partial\Theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\Theta^2}$$

and ψ is Stokes' stream function in terms of which the velocity components u and v in the r- and Θ-directions are given, respectively, by

$$u = -\psi_\Theta/r^2 \sin\Theta, \qquad v = \psi_r/r \sin\Theta$$

It should be noted that gravitation is assumed to act in the negative r-direction, and the motion is assumed to be independent of the third coordinate φ.

IV. Stability of Flows with Continuous Stratification

In a well-known paper RAYLEIGH (1880) showed that two-dimensional flows of a homogeneous fluid are always neutrally stable if the effect of viscosity on the disturbance is neglected and if the curvature of the velocity-distribution curve for the primary flows is of the same sign throughout. In this section two-dimensional parallel flows with continuous density variation between two parallel fixed boundaries will be considered, and an upper bound for the amplification factor will be given, assuming that the flow is indeed unstable. The inequality giving this bound includes Rayleigh's theorem as a special case.

An early paper of SQUIRE (1933) contains the result that the stability or instability of a three-dimensional disturbance in two-dimensional flows of a homogeneous viscous fluid can be predicted from that of a two-dimensional one at a lower Reynolds number. The proof of this result was later considerably simplified by LIN (1952) who used a more physical approach. The extension of Squire's result to flows of a non-homogeneous fluid with a free surface and interfaces was given by YIH (1955), who applied Lin's approach. Because of this extension, only two-dimensional disturbances need be considered here.

If the direction of the primary flow is taken to be the x-direction, and the y-direction is taken opposite to that of gravitation, the velocity U, the density r, and the pressure P of the primary flow are functions of y only. Derivatives of these quantities with respect to y will be indicated by primes. The velocity components of the disturbance in the x- and y-directions will be denoted by u and v, respectively, and deviations of density and pressure (due to the disturbance) from the mean will be denoted simply by ϱ and p.

With this notation, Euler's equations of motion are

$$(r + \varrho)\,[u_t + (U + u)\,u_x + v(U' + u_y)] = -p_x \quad (38)$$

$$(r + \varrho)\,[v_t + (U + u)\,v_x + vv_y] = -P' - p_y - g(r + \varrho) = -p_y - g\varrho \quad (39)$$

since $P' = -gr$. The equation of continuity

$$u_x + v_y = 0$$

permits the use of the stream function ψ such that

$$u = -\psi_y, \qquad w = \psi_x \quad (40)$$

If Eqs. (40) are substituted in Eqs. (38) and (39), and only terms of the first order are taken, one has

$$r(-\psi_{yt} - U\psi_{xy} + U'\psi_x) = -p_x \quad (41)$$

$$r(\psi_{xt} + U\psi_{xx}) = -p_y - g\varrho \quad (42)$$

The persistence of the density along a path line is stated to the first order by the equation

$$\varrho_t + U\varrho_x + vr' = 0$$

or

$$\varrho_t + U\varrho_x + \psi_x r' = 0 \quad (43)$$

For a two-dimensional spatially periodic disturbance, one may take

$$\left.\begin{aligned}\psi &= f(y)\ \exp mi(x - ct)\\ \varrho &= \Theta(y)\ \exp mi(x - ct)\end{aligned}\right\} \quad (44)$$

in which

$$c = c_r + ic_i$$

with c_r being the celerity of the disturbance, and c_i the amplification (or damping) factor. If c_i is not zero, a positive c_i is always associated with a negative one with the same magnitude. With the aid of Eqs. (43), Eq. (44) yields the relationship

$$\Theta = r'f/(c - U) \tag{45}$$

Now if p is eliminated from Eqs. (41) and (42) by cross differentiation, and Eqs. (44) and (45) are substituted into the result, one obtains, after simplification,

$$(rf')' + \left[\frac{(rU')'}{c - U} - m^2 r - \frac{gr'}{(c - U)^2}\right] f = 0 \tag{46}$$

which is associated with the boundary conditions

$$f(a) = 0, \qquad f(b) = 0 \tag{47}$$

in which a and b are the ordinates of the solid boundaries. The complex conjugate of Eq. (46) is (with the asterisk denoting complex conjugate)

$$(rf'^*)' + \left[\frac{(cU')'}{c^* - U} - m^2 r - \frac{gr'}{(c^* - U)^2}\right] f^* = 0 \tag{48}$$

with the corresponding boundary conditions

$$f^*(a) = 0, \qquad f^*(b) = 0 \tag{49}$$

If now Eq. (46) is multiplied by f^* and integrated from a to b, Eq. (47) is multiplied by f and integrated in the same interval, and the difference is taken, one has

$$\int_a^b (rf')'f^* dy - \int_a^b (rf^{*\prime})' f dy =$$
$$= \int_a^b \left[(rU')' \left(\frac{1}{U - c} - \frac{1}{U - c^*}\right) + \right.$$
$$\left. + gr' \left(\frac{1}{(U - c)^2} - \frac{1}{(U - c^*)^2}\right)\right] ff^* dy \tag{50}$$

The left-hand side of this equation can be shown to be zero by integration by parts and by utilization of the boundary conditions given by Eqs. (47) and (49). After a straightforward calculation of the right-hand side and division by 2, Eq. (50) assumes the following form

$$c_i \int_a^b [(U - c_r)^2 + c_i^2]^{-2} \{(rU')'[(U - c_r)^2 + c_i^2] +$$
$$+ (U - c_r) gr'\} ff^* dy = 0 \tag{51}$$

From Eq. (51), one obtains the result that, if $(rU')'$ is of the same sign between a and b,

$$\operatorname{Max} \left|\frac{gr'}{2(rU')'}\right| \geqslant |c_i| \tag{52}$$

For if not, the integrand in Eq. (51) will be positive or negative definite depending on whether $(rU')'$ is positive or negative, and c_i will be zero, leading to a contradiction. The inequality (52) provides an upper bound for the amplification factor if instability occurs – under the restriction that $(rU')'$ does not change sign. If r is constant, one sees that it leads to neutral stability ($c_i = 0$) under the condition that U'' is of the same sign from a to b. Thus Rayleigh's theorem is an immediate consequence of (52).

From Eq. (51) two more deductions can be readily made. If $gr'/(rU')'$ is positive throughout the field of flow, a periodic disturbance travelling in the negative x-direction with a celerity exceeding the maximum value of U in that direction must necessarily be neutrally stable. Similarly, if the same quantity is negative throughout, the celerity of an unstable disturbance travelling in the positive direction of x cannot exceed the algebraic maximum of the mean velocity.

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