Wind Action on a Shallow Sea: Some Generalizations of Ekman's Theory

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(Manuscript received October 15, revised November 26, 1956)

Abstract

In 1923 EKMAN developed a theory for the sea-level changes produced in a deep sea by the action of a steady wind. In the present paper the theory has been extended to a shallow sea, for which the Ekman "depth of frictional influence" d is comparable to the actual depth h. It is pointed out that the wind-stress divergence may be of the same importance as the wind-stress curl in this case.

The theory is furthermore extended to the transient case. It is demonstrated how the velocity profile and the flow can be expressed in terms of the local time-histories of the wind-stress and the surface slope, and a single integro-differential equation is derived for the sea-level elevation. This equation could possibly be used for prediction of meteorologic tides and storm surges.

I. Introduction

In considering the wind action on the sea Ekman used a model described by the following two equations of motion:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \zeta}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}\\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial \zeta}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}\end{aligned}\right\} \qquad (1)$$

The boundary conditions are

$$\left.\begin{aligned}\nu\left(\frac{\partial u}{\partial z}\right)_{z=0} &= \tau_x^w, \qquad \nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \tau_y^w\\ u_{z=-h} &= 0, \qquad\qquad v_{z=-h} = 0\end{aligned}\right\} \qquad (2)$$

Here x, y are horizontal locally Cartesian coordinates, and u, v are the corresponding velocities. z is the vertical coordinate counted positive upwards. The free surface is at $z = \zeta(x, y)$ and the bottom at $z = -h(x, y)$ (Fig. 1). f is the Coriolis parameter, g the acceleration of gravity and ν the coefficient of vertical friction. These are considered as constants. τ_x^w, τ_y^w, finally, are the components of the wind-stress acting on the sea surface ($\varrho = 1$). In the derivation of the equations it has been assumed that the water is homogeneous, that the pressure is hydrostatic, and that lateral friction and non-linear acceleration terms can be neglected.

The steady state velocity profile obtained from this model for given values of the wind-stress τ_x^w, τ_y^w and surface slope $\frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}$ has been discussed in detail by EKMAN in his paper of 1905. Computing the components of the total flow

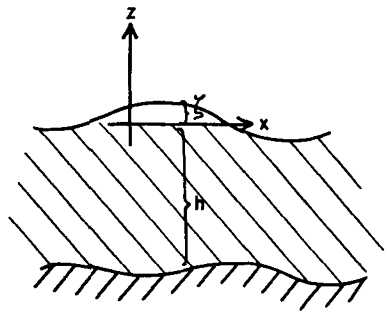

Fig. 1. Coordinate system.

$$\left.\begin{aligned} U &= \int_{-h}^{0} u\,dz \\ V &= \int_{-h}^{0} v\,dz \end{aligned}\right\} \tag{3}$$

and inserting these in the continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{4}$$

EKMAN derived furthermore, in his paper of 1923, a single steady state equation for the sea-level elevation ζ, and discussed this equation in certain simple cases. Ekman was, of course, most interested in the possible application of his theory to the wind-driven ocean circulation, and accordingly he assumed the "depth of frictional influence" $d = \pi\sqrt{\frac{2\nu}{f}}$ to be small compared with the actual depth h. As it has turned out later, however, Ekman's model is not realistic for the deep oceans, because of the neglection of such important factors as the variation of the Coriolis parameter with latitude, the decrease of the horizontal pressure gradient with depth due to the density stratification, and the large-scale lateral mixing. On the other hand, Ekman's model may depict the conditions in a small, shallow sea, with linear dimensions comparable to, say, the North Sea. In a small sea the variation of the Coriolis parameter is insignificant, and for small depths the lateral friction will probably become negligible in comparison with the vertical friction. Also, stratification should have a relatively small influence on the pressure gradient in a shallow sea. It must be pointed out, however, that the stratification may have a strong indirect effect through the coefficient of friction. For instance, in a layer where the density gradient is very large ("discontinuity layer"), the coefficient of vertical friction will fall to practically zero, and the frictional transport of momentum down to the bottom is accordingly reduced. This is of course of greatest significance in a very shallow sea, where most momentum is transported by the frictional stresses. Also in other cases, the assumption of a constant coefficient of vertical friction is not very realistic. Both experiments and theory suggest that the coefficients of turbulent friction should in turn depend on the type of flow set up, and several refined frictional laws have been presented. By introducing such refined laws in this problem one would, however, complicate the mathematical treatment considerably and no simple discussion of frictional influence on the flow would be possible. By using a constant coefficient of friction to represent the "effective friction" the mathematics becomes fairly simple and straight-forward, and the solution can be discussed in terms of a single non-dimensional number, giving the combined effect of friction, Coriolis force and the finite depth.

In the time-dependent case, FREDHOLM computed the velocity profile developing under the action of a suddenly introduced constant wind-stress, when the depth is infinite (EKMAN 1905). This investigation has later on been extended to the case of a sea of finite depth by FJELDSTAD (1930) and HIDAKA (1933).

Knowing the above mentioned responses, one may find the velocity profile and flow at any place and time expressed in terms of the local time-histories of the wind-stress and surface slope. The continuity equation

$$\frac{\partial \zeta}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{5}$$

will then provide a relation between the rate of change of the sea-level and the local time-histories of the wind-stress, the surface slope, and the surface curvature. This equation can be integrated step by step to give the changes in sea-level of a given sea from the knowledge of the wind-stress values. The boundary condition is that the normal flow vanishes at the coast. In special, the equation could be used for numerical prediction of storm-surges. Such predictions have been attempted recently for the North Sea by HANSEN (1956) with good success. Hansen's model is based on the vertically integrated equations, where except U, V and ζ also the bottom stress components τ_x^b, τ_y^b appear as unknown variables. The bottom stress has then to be expressed in terms of these other variables through an extra equation, and Hansen uses here a relation based on certain formulae from steady channel flow. According to the present investigation one can, however, give an exact relation between the bottom stress and the integrated variables, and one might hope that the prediction equation (34, 35), which utilizes this exact relation, would improve Hansen's result. Practical experiments with the equation have accordingly been planned.

The author wants to thank Dr. Fischer, now at the International Meteorological Institute in Stockholm, for valuable discussions on the subject.

II. Sea-level in steady state

In the steady state the equations of motion reduce to

$$\left.\begin{aligned} \nu \frac{\partial^2 u}{\partial z^2} + fv &= g\frac{\partial \zeta}{\partial x} \\ \nu \frac{\partial^2 v}{\partial z^2} - fu &= g\frac{\partial \zeta}{\partial y} \end{aligned}\right\} \tag{6}$$

with unchanged boundary conditions (2).

It is found convenient to introduce a complex notation as follows:

$$\left.\begin{aligned} w &= u + iv \\ \frac{\partial \zeta}{\partial n} &= \frac{\partial \zeta}{\partial x} + i\frac{\partial \zeta}{\partial y} \\ \tau^w &= \tau_x^w + i\tau_y^w \end{aligned}\right\} \tag{7}$$

Equation (6) can then be written

$$\nu \frac{\partial^2 w}{\partial z^2} - ifw = g\frac{\partial \zeta}{\partial n} \tag{8}$$

with boundary conditions

$$\left.\begin{aligned} \nu\left(\frac{\partial w}{\partial z}\right)_{z=0} &= \tau^w \\ w_{z=-h} &= 0 \end{aligned}\right\} \tag{9}$$

Consider for the moment both the wind-stress τ^w and the surface slope $\frac{\partial \zeta}{\partial n}$ as prescribed, they are functions of x and y only. The general solution to equation (8) is

$$w = C_1 e^{\sqrt{\frac{if}{\nu}}\cdot z} + C_2 e^{-\sqrt{\frac{if}{\nu}}\cdot z} + \frac{ig}{f}\frac{\partial \zeta}{\partial n}$$

The boundary conditions (9) determine C_1 and C_2 uniquely and one gets

$$w = \frac{1}{\sqrt{if\nu}} \frac{\sinh\sqrt{\frac{if}{\nu}}\,(h+z)}{\cosh\sqrt{\frac{if}{\nu}}\,h}\tau^w - \frac{ig}{f}\left(\frac{\cosh\sqrt{\frac{if}{\nu}}\,z}{\cosh\sqrt{\frac{if}{\nu}}\,h} - 1\right)\frac{\partial \zeta}{\partial n} \tag{10}$$

This solution represents the sum of the Ekman "drift-current" and "gradient current" for finite depth (EKMAN 1905). Integrating the velocity profile (10) gives for the total complex flow,

$$W = U + iV = \int_{-h}^{0} w\,dz = \frac{A}{f}\tau^w + \frac{ghB}{f}\frac{\partial \zeta}{\partial n}$$

where A and B are functions only of $\frac{h}{d}$,

$$\text{where } d = \pi\sqrt{\frac{2\nu}{f}}.$$

Writing $A = C + iD$, $B = E + iF$, the flow components become

$$\left.\begin{aligned} U &= \frac{1}{f}(C\tau_x^w - D\tau_y^w) + \frac{gh}{f}\left(E\frac{\partial \zeta}{\partial x} - F\frac{\partial \zeta}{\partial y}\right) \\ V &= \frac{1}{f}(D\tau_x^w + C\tau_y^w) + \frac{gh}{f}\left(F\frac{\partial \zeta}{\partial x} + E\frac{\partial \zeta}{\partial y}\right) \end{aligned}\right\} \tag{11}$$

and the complete expressions for C, D, E and F are

$$\left.\begin{aligned}
C &= \frac{2\sin\pi\frac{h}{d}\sinh\pi\frac{h}{d}}{\cos 2\pi\frac{h}{d}+\cosh 2\pi\frac{h}{d}} \\
D &= \frac{2\cos\pi\frac{h}{d}\cosh\pi\frac{h}{d}}{\cos 2\pi\frac{h}{d}+\cosh 2\pi\frac{h}{d}} - 1 \\
E &= \frac{1}{2\pi\frac{h}{d}}\,\frac{\sin 2\pi\frac{h}{d}-\sinh 2\pi\frac{h}{d}}{\cos 2\pi\frac{h}{d}+\cosh 2\pi\frac{h}{d}} \\
F &= -\frac{1}{2\pi\frac{h}{d}}\,\frac{\sin 2\pi\frac{h}{d}+\sinh 2\pi\frac{h}{d}}{\cos 2\pi\frac{h}{d}+\cosh 2\pi\frac{h}{d}} + 1
\end{aligned}\right\} \quad (12)$$

These are shown graphically as functions of $\frac{h}{d}$ in Fig. 2 a—d.

Inserting the values for U and V in the continuity equation (4) one obtains the wanted equation for the sea-level elevation. It becomes

$$ghE\nabla^2\zeta + \gamma_1\frac{\partial\zeta}{\partial x} + \gamma_2\frac{\partial\zeta}{\partial y} =$$
$$= -C\,\mathrm{div}\,\tau^w + D\,\mathrm{curl}\,\tau^w + \delta_1\tau_x^w + \delta_2\tau_y^w \quad (13)$$

where

$$\left.\begin{aligned}
\gamma_1 &= \frac{\partial}{\partial x}(ghE) + \frac{\partial}{\partial y}(ghF) \\
\gamma_2 &= -\frac{\partial}{\partial x}(ghF) + \frac{\partial}{\partial y}(ghE) \\
\delta_1 &= -\left(\frac{\partial C}{\partial x} + \frac{\partial D}{\partial y}\right) \\
\delta_2 &= \frac{\partial D}{\partial x} - \frac{\partial C}{\partial y}
\end{aligned}\right\} \quad (14)$$

and

$$\left.\begin{aligned}
\mathrm{div}\,\tau^w &= \frac{\partial\tau_x^w}{\partial x} + \frac{\partial\tau_y^w}{\partial y} \\
\mathrm{curl}\,\tau^w &= \frac{\partial\tau_y^w}{\partial x} - \frac{\partial\tau_x^w}{\partial y} \\
\nabla^2\zeta &= \frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}
\end{aligned}\right\} \quad (15)$$

The equation is of the elliptic type and can be solved numerically for any closed basin, using the boundary condition that the normal flow vanishes at the coasts. In terms of ζ this condition becomes

$$gh\left(E\frac{\partial\zeta}{\partial r} - F\frac{\partial\zeta}{\partial s}\right) = -C\tau_r^w + D\tau_s^w \quad (16)$$

where r is the normal and s the tangential coordinate counted positive inwards and *cum sole*, respectively. In the special case of constant depth (13) reduces to

$$ghE\nabla^2\zeta = -C\,\mathrm{div}\,\tau^w + D\,\mathrm{curl}\,\tau^w \quad (17)$$

and in the case of infinite depth it simplifies further to

$$\frac{gd}{2\pi}\nabla^2\zeta = -\mathrm{curl}\,\tau^w \quad (18)$$

In Ekman's derivation of the steady-state equation for ζ it was assumed that $\frac{h}{d} \gg 1$, and consequently the terms C div τ^w, $\delta_1\tau_x^w$, and $\delta_2\tau_y^w$ were missing.

The appearance of the wind-stress divergence in the more general equation (13) is of special interest. The wind-stress divergence will have its maximum effect in a sea of depth $h \sim \frac{d}{2}$, in which case the effect is about half of that from a wind-stress curl of the same magnitude.

One may think that the wind-stress divergence should be a small quantity compared to the wind-stress curl, but this seems not to be the case. Assuming that the wind-stress is proportional to the square of the surface wind speed, one can put

$$\tau^w = kW\mathbf{W} \quad (19)$$

where $\mathbf{W}$ is the wind-vector and $W = |\mathbf{W}|$, and the wind-stress divergence becomes

$$\mathrm{div}\,\tau^w = kW\,\mathrm{div}\,\mathbf{W} + kW\frac{\partial W}{\partial s} \quad (20)$$

where s is a coordinate in the wind-direction. The first one of the two terms appearing in the above expression is generally negligible since the divergence of the wind is quite small, but the second one may well be comparable with the wind-stress curl in regions

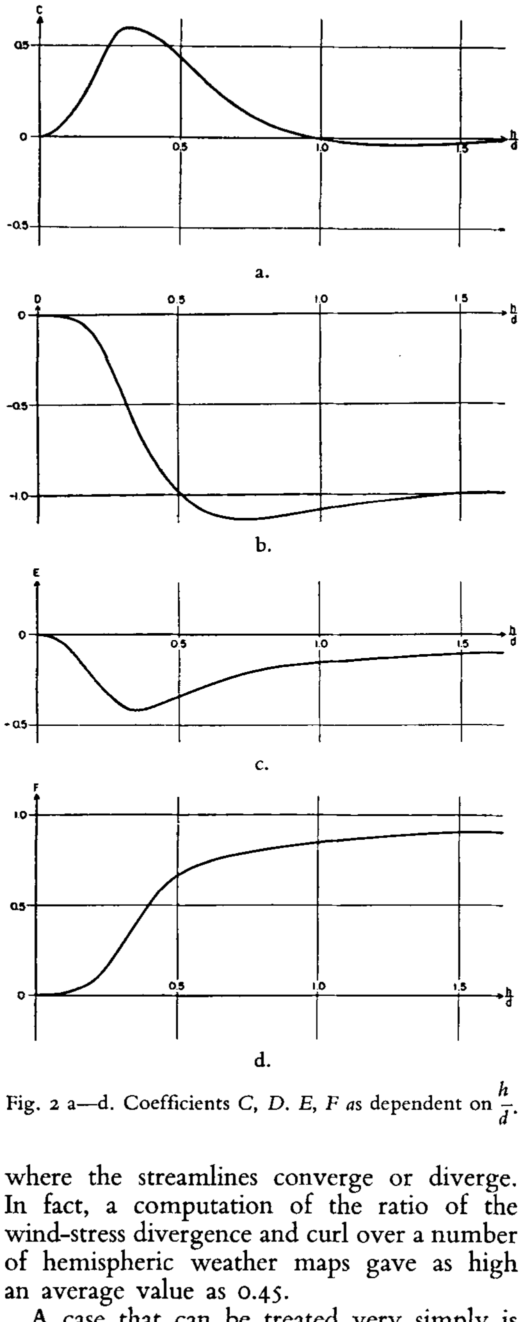

Fig. 2 a—d. Coefficients C, D. E, F as dependent on $\frac{h}{d}$.

where the streamlines converge or diverge. In fact, a computation of the ratio of the wind-stress divergence and curl over a number of hemispheric weather maps gave as high an average value as 0.45.

A case that can be treated very simply is the one of a uniform wind-stress acting over a sea of uniform depth. This case has been discussed by EKMAN (1923) and SCHALKWIJK (1947). Introducing the angle φ for the deviation of the "stau-direction" $\left(\text{direction of } \frac{\partial \zeta}{\partial n}\right)$ to the right of the wind-direction and the angle $\gamma = \left|\frac{\partial \zeta}{\partial n}\right|$ for the magnitude of the slope, one has

$$\operatorname{tg} \varphi = \frac{FC - ED}{FD + EC} \tag{21}$$

$$\frac{gh}{\tau^w} \cdot \gamma = \frac{\sqrt{(FC - ED)^2 + (FD + EC)^2}}{E^2 + F^2} \tag{22}$$

In this case the surface slope will be independent of the horizontal extensions of the sea, it depends only on the wind-stress value and the ratio $\frac{h}{d}$. The angle φ is small and positive, with a maximum value $\sim 12°$ occurring when $\frac{h}{d} \sim 1$, and it tends to zero both for very small and very large depths. $\frac{gh}{\tau^w} \cdot \gamma$ approaches $\frac{3}{2}$ for small values of $\frac{h}{d}$ and 1 for large values of $\frac{h}{d}$, and the sea-surface slope is thus in both cases inversely proportional to the depth. The actual variations of φ and γ with $\frac{h}{d}$ according to (21) and (22) are seen in Figs. 3—4.

The qualitative effects of an added wind-stress curl and wind-stress divergence can be seen from an inspection of equation (13) and Figs. 2 a—c. It is found that a positive wind-stress curl contributes with a positive $\nabla^2 \zeta$, i.e. a sea-surface concavity, while a negative wind-stress contributes with a negative $\nabla^2 \zeta$, i.e. a sea-surface convexity. This corresponds to a sea-level rising and sea-level sinking, respectively, at the coast, if the sea-level is assumed to be unchanged out in the open sea. The effect will increase numerically with the depth of the sea up to the limiting value given by (18).

In the same way it is found that a positive wind-stress divergence contributes with a sea-surface convexity, a negative wind-stress divergence with a sea-surface concavity, in the range $0 < \frac{h}{d} < 1$. For larger values of $\frac{h}{d}$ the sign may change (in fact it changes each time $\frac{h}{d}$ passes an integer), but the effect is then very small.

If the depth is variable a uniform wind will also give rise to sea-surface concavities and convexities, because of the terms $\delta_1 \tau_x^w$, $\delta_2 \tau_y^w$, $\gamma_1 \frac{\partial \zeta}{\partial x}$, and $\gamma_2 \frac{\partial \zeta}{\partial y}$. From an inspection of these terms it is found that a decrease of the depth in the direction of the wind or to the right of the wind gives rise to a sea-surface concavity, an increase of the depth in these directions to a sea-surface convexity, provided that the sea is sufficiently shallow $\left(\frac{h}{d} < 0.5\right)$. For larger depths the effects will be reversed but then also their magnitudes become small. Finally, a decrease of the depth in the "stau-direction" or to the right of the stau-direction gives rise to a sea-surface convexity, an increase of the depth in these directions to a sea-surface concavity. When the depth h becomes very large the effect of a variation of the depth in the stau-direction disappears, while the effect of a variation perpendicular to the "stau-direction" remains finite.

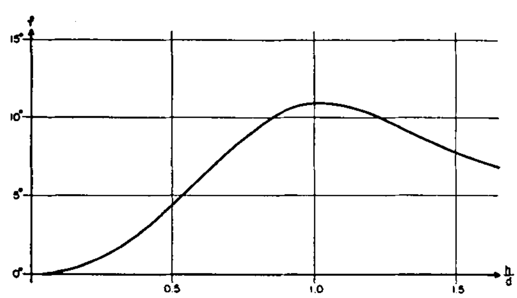

Fig. 3. The angle φ as dependent on $\frac{h}{d}$.

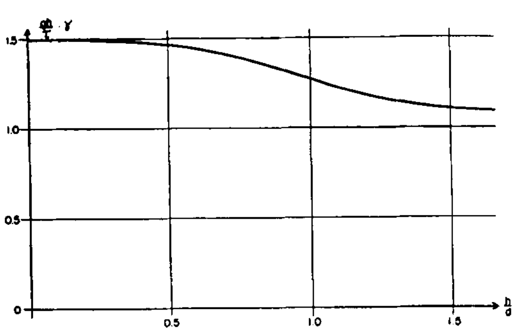

Fig. 4. The angle γ as dependent on $\frac{h}{d}$.

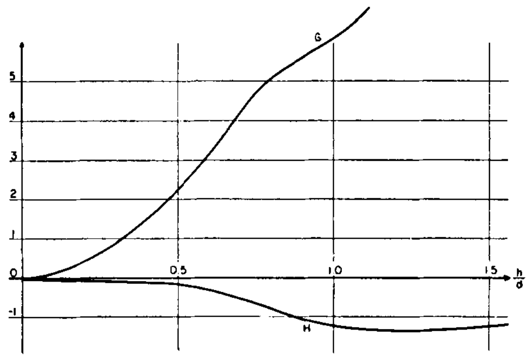

Fig. 5. Appearance of $G\left(\frac{h}{d}\right)$ and $H\left(\frac{h}{d}\right)$.

This last result is of course not very realistic. It is due to the fact that the horizontal divergence disappears outside the frictional layers in this model, forcing all vertical velocities that are induced at the bottom to penetrate up to the surface layer. In a real sea there exist compensating divergencies outside these layers due to stratification, variation of the Coriolis parameter and non-linear terms.

It may be of interest to derive the expression for the vorticity of the flow $\eta = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$ in terms of the wind-stress field. This is made here only for the case of a uniform depth. From (11) is obtained

$$\eta = \frac{1}{f}(C \operatorname{curl} \tau^w + D \operatorname{div} \tau^w) + \frac{gh}{f} F \nabla^2 \zeta \quad (23)$$

and eliminating $\nabla^2 \zeta$ with the aid of equation (17) one gets

$$\eta = \frac{1}{f}(G \operatorname{curl} \tau^w + H \operatorname{div} \tau^w) \quad (24)$$

where

$$\left.\begin{aligned} G &= C + \frac{F}{E} \cdot D \\ H &= D - \frac{F}{E} \cdot C \end{aligned}\right\} \quad (25)$$

G and H are given as function of $\frac{h}{d}$ in Fig. 5. G and H are of comparable magnitude for shallow water, and the wind-stress divergence will then contribute to the horizontal circulation. For deep water the divergence term is small compared to the curl-term. Introducing a stream-function Ψ through

$U = -\dfrac{\partial \Psi}{\partial y}$, $V = \dfrac{\partial \Psi}{\partial x}$ (24) can be written

$$\nabla^2 \Psi = \frac{1}{f} \left(G \operatorname{curl} \tau^w + H \operatorname{div} \tau^w \right) \tag{26}$$

and this can be integrated, numerically at least, over any closed basin to give the stream-lines. The boundary condition is $\Psi = 0$. This equation should be compared with the one obtained by introducing a frictional resistance proportional to the flow,

$$\tau^b = rW \tag{27}$$

which law is often used in discussions of wind-driven circulations. In this latter case one gets

$$\nabla^2 \Psi = \frac{1}{r} \operatorname{curl} \tau^w \tag{28}$$

and the effect of wind-stress divergence is completely lost. For the curl-term the expressions agree if $r = \dfrac{1}{G} f$.

III. Sea-level in transient state

In the steady state it was possible to find the velocity profile at each point as expressed in terms of the local wind-stress τ^w, the surface slope $\dfrac{\partial \zeta}{\partial n}$, and the number $\dfrac{h}{d}$. Substitution of the expressions for the flow components in the continuity equation (4) then gave a differential equation for ζ only.

In the time-dependent case the velocity profile cannot of course be determined by the local, instantaneous values of the wind-stress and the surface slope. It is true, however, that *the velocity profile is uniquely determined by the local time-histories of the wind-stress and the surface slope at each point and instant.* The bottom-stress and the flow are thus also determined by the time-histories.

From a physical point of view it may seem peculiar that only the *local* wind-stress and surface slope should enter in the determination, since this would allow us to create any wanted local velocity field without considering the development at other neighbouring points. One must remember, however, that the surface slope is not a "forcing function" at our disposal (forcing the surface by any outer action to take on a certain wanted form must give rise to extra pressure gradients and equations (1) will then be violated), but it is in turn related to the spatial variations of the flow through the continuity equation. Hereby one gets a coupling between different regions of the fluid. Nevertheless, in the mathematical discussion of the local velocity profile, one may formally look at the surface slope as a forcing function prescribed from the outside.

To find the explicit expression for the velocity profile, one should start from two special profiles $w_a(z,t)$ and $w_b(z,t)$ satisfying the following equations and boundary conditions, respectively:

$$\frac{\partial w}{\partial t} + ifw = \nu \frac{\partial^2 w}{\partial z^2},$$

$$\nu \left(\frac{\partial w}{\partial z} \right)_{z=0} = 1, \quad w_{z=-h} = 0, \quad w_{t=0} = 0 \tag{29 a}$$

w_a

$$\frac{\partial w}{\partial t} + ifw = -g + \nu \frac{\partial^2 w}{\partial z^2},$$

$$\nu \left(\frac{\partial w}{\partial z} \right)_{z=0} = 0, \quad w_{z=-h} = 0, \quad w_{t=0} = 0 \tag{29 b}$$

w_b

w_a gives the response to a suddenly introduced wind-stress of unit magnitude, w_b the response to a suddenly introduced surface slope of unit magnitude. For the special case of infinite depth the solution to the problem (29 a) was given by FREDHOLM (EKMAN 1905). The general solutions have been derived by FJELDSTAD (1930) and HIDAKA (1933). These can be written on the form

$$w_a = -\frac{1}{\pi^2} \frac{h}{\nu} \left(\frac{d}{h} \right)^2 \sum_{n=0}^{\infty} \frac{1}{\mu_n^2 + i}$$

$$\cos \frac{2n+1}{2} \pi \frac{z}{h} e^{-(\mu_n^2 + i) tf} + \overline{w}_a \tag{30 a}$$

$$w_b = -\frac{\sqrt{2}}{\pi} \frac{g}{f} \frac{d}{h} \sum_{n=0}^{\infty} \frac{(-1)^n}{\mu_n^2 + i}$$

$$\cos \frac{2n+1}{2} \pi \frac{z}{h} e^{-(\mu_n^2 + i) tf} + \overline{w}_b \tag{30 b}$$

where

$$\mu_n = \frac{1}{2\sqrt{2}}(2n+1)\frac{h}{d} \qquad (31)$$

and $\bar{w}_a$, $\bar{w}_b$ stand for the corresponding steady state solutions.

The response to a suddenly introduced differential wind-stress $\triangle \tau^w$ and sea-surface slope $\triangle \frac{\partial \zeta}{\partial n}$ is

$$w = \triangle \tau^w w_a(z,t) + \triangle \frac{\partial \zeta}{\partial n} w_b(z,t)$$

Putting

$$\triangle \tau^w = \frac{\partial \tau^w}{\partial t} \triangle t$$

$$\triangle \frac{\partial \zeta}{\partial n} = \frac{\partial^2 \zeta}{\partial t \partial n} \triangle t$$

and summing up the contributions from such differential steps in the wind-stress and surface slope for past times ($\alpha < t$) gives

$$w = \int_0^\infty \left[\frac{\partial \tau^w}{\partial t}(t-\alpha) w_a(z,\alpha) + \right.$$
$$\left. + \frac{\partial^2 \zeta}{\partial t \partial n}(t-\alpha) w_b(z,\alpha)\right] d\alpha \qquad (32)$$

It is then assumed that τ^w and $\frac{\partial \zeta}{\partial n}$ start from zero values.

After a partial integration the following more convenient form of (32) is obtained.

$$w = \int_0^\infty \left[\tau^w(t-\alpha)\frac{\partial w_a}{\partial t}(z,\alpha) + \right.$$
$$\left. + \frac{\partial \zeta}{\partial n}(t-\alpha)\frac{\partial w_b}{\partial t}(z,\alpha)\right] d\alpha \qquad (33)$$

Integrating vertically and inserting the expressions for w_a and w_b one obtains for the total flow

$$W = \frac{4}{\pi}\int_0^\infty \tau^w(t-\alpha)\sum_{n=0}^\infty \frac{(-1)^n}{2n+1} e^{-(\mu_n^2+i)\alpha f} d\alpha$$
$$- \frac{8}{\pi^2} gh \int_0^\infty \frac{\partial \zeta}{\partial n}(t-\alpha)\sum_{n=0}^\infty \frac{1}{(2n+1)^2} e^{-(\mu_n^2+i)\alpha f} d\alpha \qquad (34)$$

By using this expression in the continuity equation, written in the form

$$\frac{\partial \zeta}{\partial t} = -\operatorname{div} W \qquad (35)$$

one arrives at a single prediction equation for ζ. On the right side of the equation the time-histories of the wind-stress, the surface slope and surface curvature enter. It is thus an integro-differential equation for ζ. It may be solved numerically for a given basin and stress-field by using a step-by-step technique. The boundary condition $W_r = 0$ at the coast, can be formulated as a condition for $\frac{\partial \zeta}{\partial n}$ at each step.

It should be possible to use (34, 35) for actual predictions of meteorological tides and storm surges. The method would probably improve the results obtained by other methods in the case of fairly shallow water, where bottom friction is of importance. So far no computations have been performed, but some experiments for the North Sea basin are planned.

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