Some Remarks About Nonlinear Oscillations In Tidal Channels

By H. KREISS, International Institute of Meteorology in Stockholm

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Abstract

In tidal rivers one observes an unsymmetry between flood and ebbcurrent. This phenomenon is examined for channels with constant mean depth and mean cross-section area. The hydrodynamic equations are integrated over the vertical and it is shown that one is not allowed to introduce the usual simplifications (linearisation, neglection of friction) if one wants to describe this phenemenon. With the aid of a perturbation method the equations are integrated and it is shown that for sufficiently small ratio maximum of the flow/mean depth one can approximate the solution of by two harmonics. This approximation is examined in detail.

1. Introduction

In tidal estuaries one often observes that the floodcurrent u_F is of shorter duration than the ebb-current u_E and that the maximum value of u_F is greater than the maximum value of u_E. These facts are of importance for the transport of sediments and do explain why a river sometimes gets filled with sand. (Figure 1.)

Of late effective numerical methods have been developed for the computation of such tidal currents (HANSEN, 1956). These methods are the only ones that permit us to compute tides in rivers with varying cross-section and varying mean depth. On the other hand they do not readily give an insight into the physical structure of the motions since each computation gives just one special numerical solution and does not tell anything about what would happen if the controlling parameters were changed. For this reason it may be useful to develop analytical theories for simple examples and to study their physical structure in detail. In this paper we shall discuss such a theory for the simple case of tidal motions in channels with constant mean depth and constant cross-section.

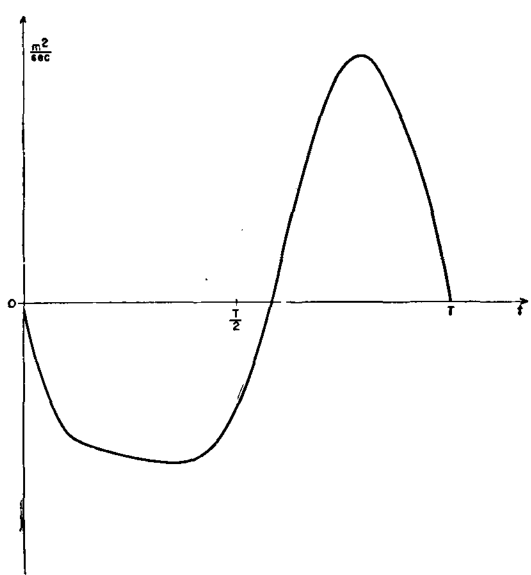

Fig. 1. Flow-values in a fixed point as a function of time. The positive values represent the flood current u_F the negative the ebb current u_E.

2. Basic equations

In the theoretical study of tidal motions in channels, one usually assumes that the channel is connected with the open sea at one end and closed at the other. Using the symbols u, ζ, h for velocity, the surface elevation and the mean depth respectively, one starts with the vertically integrated hydrodynamical equations in the form:

$$\left.\begin{aligned} &\bar{u}_t + \frac{\lambda}{h}\bar{u}\bar{u}_x + R(\bar{u}) + g(h+\zeta)\zeta_x = 0 \\ &\bar{u}_x + \zeta_t = 0 \\ &\bar{u} = \int_{-h}^{\zeta} u\,dz \end{aligned}\right\} \quad 2.1$$

f_t, f_x, f_{xt} etc. being defined by

$$f_t = \frac{\partial f}{\partial t};\; f_x = \frac{\partial f}{\partial x};\; f_{xt} = \frac{\partial^2 f}{\partial x \partial t}$$

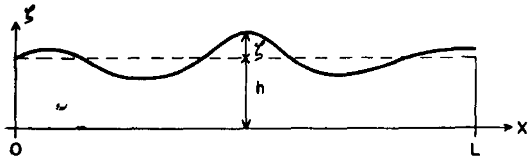

Fig. 2.

The boundary conditions are:

$$\text{for } x = 0 \quad \zeta(0, t) = f(t) = A \cos \frac{2\pi}{T} t$$

$$\text{and for } x = L, \quad \bar{u} = 0 \quad \text{(Figure 2)}$$

In 2.1 $R(\bar{u})$ is the friction-term and one generally assumes:

$$R(\bar{u}) = -R(-\bar{u})$$

This assumption is only true when the friction is independent of the history of the motion; it is well known that in some tidal rivers one has to choose different friction-laws for ebb- and flood-currents.

The factor λ derives from the fact that the velocity is a function of depth. This factor most likely varies from section to section and from time to time. One can write $\lambda = 1 + \varepsilon(x, t)$. In many cases, however, ε is small compared with 1. Later it will be shown that the entire term $\frac{\lambda}{h}\bar{u}\bar{u}_x$ normally does not influence the solution of 2.1 strongly. Under those circumstances the assumption $\lambda = 1$ is a reasonable one.

In the study of tides one is not interested in the transient solution of 2.1 and we shall therefore restrict this investigation to the T-periodical solution.

In all earlier analytical investigations the equations 2.1 have been simplified in one or both of the following two ways:

a) One neglects the nonlinear terms $\frac{\lambda}{h}\bar{u}\bar{u}_x$ and $\zeta\zeta_x$ and investigates:

$$\left.\begin{aligned} &\begin{cases} \bar{u}_t + R(\bar{u}) + gh\zeta_x = 0 \\ \bar{u}_x + \zeta_t = 0 \end{cases} \\ &\begin{cases} \zeta(0, t) = A \cos \frac{2\pi}{T} t \\ \bar{u}(L, t) = 0 \end{cases} \end{aligned}\right\} \quad 2.1\text{ a}$$

There is no possibility to describe the previously mentioned lack of symmetry between ebb and flood currents by this system, since the transformation:

$$x = x^*;\; t = t^* + \frac{T}{2};\; \zeta = -\zeta^*;\; u = -u^*$$

does not change the system 2.1 a. In fact:

$$\begin{cases} \bar{u}\left(t + \frac{T}{2}\right) = -\bar{u}(t) \\ \zeta\left(t + \frac{T}{2}\right) = -\zeta(t) \end{cases}$$

b) One neglects the friction term and investigates:

$$\left.\begin{aligned} &\begin{cases} \bar{u}_t + g(h+\zeta)\zeta_x + \frac{\lambda}{h}\bar{u}\bar{u}_x = 0 \\ \bar{u}_x + \zeta_t = 0 \end{cases} \\ &\begin{cases} \zeta(0, t) = A \cos \frac{2\pi}{T} t \\ \bar{u}(L, t) = 0 \end{cases} \end{aligned}\right\} \quad 2.1\text{ b}$$

(H. Lamb 1932; J. Stoker 1948.)

But also these equations do not lead further since they remain unchanged by the transformation:

$$x = x^*;\quad t = -t^*;\quad \bar{u} = -\bar{u}^*;\quad \zeta = \zeta^*$$

Therefore

$$\bar{u}(-t) = -\bar{u}(t)$$
$$\zeta(-t) = \zeta(t)$$

and we get again symmetry between ebb and flood current.

It follows that we must take into account both friction and the nonlinear terms. The simplest way to do this is to choose

$$R(\bar{u}) = \mathrm{const} \cdot \bar{u} = \sigma \bar{u}$$

In accordance with earlier investigations we put

$$\sigma = \frac{\varrho}{h};\quad \varrho = 2 \cdot 10^{-3} \left[\frac{\text{meter}}{\text{sec.}}\right]$$

Thus we shall investigate the system

$$\left.\begin{array}{l} \left\{\begin{array}{l} \bar{u}_t + \dfrac{\lambda}{h}\bar{u}\bar{u}_x + \varrho\,\dfrac{\bar{u}}{h} + g(h+\zeta)\zeta_x = 0 \\ \bar{u}_x + \zeta_t = 0 \end{array}\right. \\ \left\{\begin{array}{l} \zeta(0, t) = A\cos\dfrac{2\pi}{T}t \\ \bar{u}(L, t) \equiv 0 \end{array}\right. \end{array}\right\} \quad 2.2$$

It is convenient to write 2.2 in nondimensional form:

Putting

$$\left.\begin{array}{ll} \bar{u} = \dfrac{2\pi hL}{T}\,\bar{u}'; & \xi = h\zeta' \\ t = \dfrac{T}{2\pi}\,t'; & x = Lx' \end{array}\right\} \quad 2.3$$

we get

$$\left\{\begin{array}{ll} \alpha^2(\bar{u}'_{t'} + \beta\bar{u}' + \lambda\bar{u}'\bar{u}'_{x'}) + (1+\zeta')\zeta'_{x'} = 0 & 2.4\text{ a} \\ \bar{u}'_{x'} + \zeta'_{t'} = 0 & 2.4\text{ b} \end{array}\right.$$

with

$$\alpha = \frac{2\pi L}{\sqrt{gh}\,T};\quad \beta = \frac{\varrho T}{2\pi h}$$

From the continuity equation 2.4 b it follows, that a function φ exists with the properties:

$$\bar{u}' = -\varphi_{t'}\qquad \zeta' = \varphi_{x'} \qquad 2.5$$

Dropping the $'$ we get from 2.4 a and 2.5

$$\left.\begin{array}{l} -\alpha^2(\varphi_{tt} + \beta\varphi_t) + \varphi_{xx} + \alpha^2\lambda\varphi_t\varphi_{xt} + \\ + \varphi_{xx}\varphi_x = 0 \\ \left\{\begin{array}{l} \varphi(1, t) = 0 \\ \varphi_x(0, t) = \dfrac{A}{h}\cos t \end{array}\right. \end{array}\right\} \quad 2.6$$

As far as I know, is it not possible to represent the solution of 2.6 by a finit number of elementary functions.

But if the ratio amplitude A/mean depth h is small compared with unity we shall see that the solution of 2.6 is "situated in the neighbourhood" of the solution of the linearised equation 2.1 a. With the aid of the perturbation method we can then approximate the solution of 2.6 by means of a number of solutions of linear equations.

We can write 2.6 in the form:

$$\left.\begin{array}{l} -\alpha^2(\varphi_{tt} + \beta\varphi_t) + \varphi_{xx} + \\ + \varepsilon(\alpha^2\lambda\varphi_t\varphi_{tx} + \varphi_{xx}\varphi_x) = 0;\ \varepsilon = 1 \\ \left\{\begin{array}{l} \varphi(1, t) = 0 \\ \varphi_x(0, t) = \dfrac{A}{h}\cos t \end{array}\right. \end{array}\right\} \quad 2.7$$

Writing formally $\varphi = \Sigma\varphi_\nu\varepsilon^\nu$, introducing this expression into 2.7 and arranging after powers of ε we get:

$$\begin{array}{l} -\alpha^2(\varphi_{\nu tt} + \beta\varphi_{\nu t}) + \varphi_{\nu xx} + \\ + (\varphi_{0xx}\varphi_{\nu-1,\,x} \cdot \cdot \cdot \cdot\ \varphi_{\nu-1,\,xx}\varphi_{0x}) + \\ + \alpha^2\lambda(\varphi_{0t}\varphi_{\nu-1,\,tx} \cdot \cdot \cdot \cdot\ \varphi_{\nu-1,\,t}\varphi_{0tx}) = 0 \quad 2.8 \end{array}$$

The boundary conditions are fulfilled by the demand

$$\left.\begin{array}{ll} \varphi_0(1, t) = 0 & \varphi_{0x}(0, t) = \dfrac{A}{h}\cos t \\ \varphi_\nu(1, t) = 0 & \varphi_{\nu x}(0, t) = 0\quad \nu > 0 \end{array}\right\} \quad 2.9$$

Then

$$\varphi = \Sigma\varphi_\nu \qquad 2.10$$

is a formal solution of 2.6 and by the aid of 2.5 we get formal expressions for $\bar{u}'$ and ζ'.

This whole development is of any use only if already $\varphi_0 + \varphi_1$ gives a sufficiently good approximation to the solution φ of 2.6. If one has to take additional φ_ν into account the mathematical apparatus increases immensely and becomes unwieldy.

Therefore we shall restrict us to the case that $\varphi_0 + \varphi_1$ and its first derivatives gives a sufficiently good approximation to φ and its first derivatives. To investigate under which conditions this is true we shall derive a differential-equation for $\psi = \varphi - (\varphi_0 + \varphi_1)$. From equation 2.8 we get the equations for φ_0 and φ_1.

$$\left.\begin{aligned} & -\alpha^2(\varphi_{0tt} + \beta\varphi_{0t}) + \varphi_{0xx} = 0 \\ & \varphi_0(1, t) = 0 \\ & \varphi_{0x}(0, t) = \frac{A}{h}\cos t \end{aligned}\right\} \quad 2.11$$

$$\left.\begin{aligned} & -\alpha^2(\varphi_{1tt} + \beta\varphi_{1t}) + \varphi_{1xx} = \\ & -\frac{1}{2}\frac{\partial}{\partial x}(\varphi_{0x}^2 + \lambda\alpha^2\varphi_{0t}) \\ & \varphi_1(1, t) = 0; \quad \varphi_{1x}(0, t) = 0 \end{aligned}\right\} \quad 2.12$$

Subtracting these two equations from equation 2.7 we get for ψ

$$\left.\begin{aligned} & -\alpha^2(\psi_{tt} + \beta\psi_t) + \psi_{xx} + \\ & + \frac{1}{2}\frac{\partial}{\partial x}\left\{\alpha^2\lambda\left[\psi_t^2 + 2(\varphi_{0t} + \varphi_{1t})\psi_t\right] + \right. \\ & \left. + 2(\varphi_{0x} + \varphi_{1x})\psi_x + \psi_x^2\right\} = \\ & = -\frac{1}{2}\frac{\partial}{\partial x}\{\alpha^2\lambda(2\varphi_{0t}\cdot\varphi_{1t} + \varphi_{1t}^2) + \\ & + 2\varphi_{0x}\varphi_{1x} + \varphi_{1x}^2\} \\ & \psi_0(1, t) = \psi_{1x}(0, t) = 0 \end{aligned}\right\} \quad 2.13$$

Now we assume that the ratio of the amplitude A to the mean depth h is small compared with unity. It then follows, as we shall see in the next section, that also $|\varphi_{0t}|$, $|\varphi_{0x}| \ll 1$ and $|\varphi_{1t}| \ll |\varphi_{0t}|$; $|\varphi_{1x}| \ll |\varphi_{0x}|$ and we can approximate ψ by $\overline{\psi}$. $\overline{\psi}$ is defined by the equation (all "small terms" in equation 2.13 are neglected)

$$\left.\begin{aligned} & -\alpha^2(\overline{\psi}_{tt} + \beta\overline{\psi}_t) + \overline{\psi}_{xx} = \\ & = -\frac{\partial}{\partial x}(\alpha^2\lambda\varphi_{0t}\varphi_{1t} + \varphi_{0x}\varphi_{1x}) \\ & \overline{\psi}_0(1, t) = \overline{\psi}_{1,x}(0, t) \end{aligned}\right\} \quad 2.14$$

This simplification of 2.13 is permissible, because we are only interested in the order of magnitude of ψ and its first derivatives. One can also justify it by numerical examples. I have solved both equations 2.13 and 2.14 for some examples in which $\frac{A}{h} = 0.2$ and found that maximum $|\psi| < 1.5$ maximum $|\overline{\psi}|$ and that this inequality also is valid for the first derivatives.

We shall discuss equation 2.14 at the end of section 3 and give some estimates of $\overline{\psi}$, when we have determined φ_0 and φ_1. But we can point out already here the analogy between the equation 2.12 for φ_1 and 2.14 for $\overline{\psi}$. We have mentioned, that if $|\varphi_0|$ is small compared with unity, $|\varphi_1|$ is small compared with $|\varphi_0|$. From 2.12 and 2.14 it is then at least plausible that $|\overline{\psi}|$ is small compared with $|\varphi_1|$.

We shall now introduce some notations which will be useful in the following.

Putting $-\varphi_{it} = \overline{u}_i'$ $\varphi_{ix} = \zeta_i'$ $(i = 0, 1)$ we can write

$$\left.\begin{aligned} \overline{u}' &= \overline{u}_0' + \overline{u}_1' \\ \zeta &= \zeta_0' + \zeta_1' \end{aligned}\right\} \quad 2.15$$

$\overline{u}_0'$, ζ_0' is the solution of the linearized equation 2.1 a in nondimensional form.

$\overline{u}_1'$, ζ_1' gives us the deviation of the solution of the nonlinear equation 2.2 from $(\overline{u}_0', \zeta_0')$.

Therefore we shall often use the phrases

the linear part for $(\overline{u}_0', \zeta_0')$
the nonlinear part for $(\overline{u}_1', \zeta_1')$

3. Determination of φ_0

a) φ_0

We know from 2.11 that for φ_0

$$\left.\begin{aligned} & -\alpha^2(\varphi_{0tt} + \beta\varphi_{0t}) + \varphi_{0xx} = 0 \\ & \varphi_{0x}(0, t) = \frac{A}{h}\cos t \quad \varphi_0(1, t) = 0 \end{aligned}\right\} \quad 3.1$$

holds.

Writing the boundary conditions of 3.1 in complex form, we get that $\varphi_0 =$ realpart ω_0 and that for ω_0

$$\left.\begin{aligned} & -\alpha^2(\omega_{0tt} + \beta\omega_{0t}) + \omega_{0xx} = 0 \\ & \omega_{0x}(0, t) = \frac{A}{h}e^{it} \quad \omega_0(1, t) = 0 \end{aligned}\right\} \quad 3.2$$

holds. As usual, we get the 2π-periodic solution ω_0 of 3.2 by assuming

$$\omega_0 = f(x)\, e_{it}$$

Putting this expression into 3.2 we get for $f(x)$

$$\left.\begin{aligned} &\alpha^2(1 - i\beta)\cdot(\times) + f''(x) = 0 \\ &f'(0) = \frac{A}{h}, \quad f(1) = 0 \end{aligned}\right\} \qquad 3.3$$

Therefore

$$f(x) = \frac{A}{hK_1\left(e^{K_1} + e^{-K_1}\right)}\left(e^{K_1(x-1)} - e^{-K_1(x-1)}\right)$$

with $K_1^2 = -\alpha^2(1 - i\beta)$ thus $K_1 = K_{1r} + iK_{1i} =$

$$= \frac{\alpha}{\sqrt{2}}\left(\sqrt{\sqrt{1+\beta^2} - 1} + \frac{i\beta}{\sqrt{\sqrt{1+\beta^2} - 1}}\right) \qquad 3.4$$

and we get

$$\omega_0 = \frac{Ae^{it}}{hK_1\left(e^{K_1} + e^{-K_1}\right)}\left(e^{K_1(x-1)} - e^{-K_1(x-1)}\right) =$$

$$\frac{Ae^{it}}{hK_1\left(1 + e^{-2K_1}\right)}\left(e^{K_1(x-2)} - e^{-K_1 x}\right) \qquad 3.5$$

$$\text{with } \varphi_0 = \text{realpart } \omega_0 = \frac{1}{2}\left(\omega_0 + \overline{\overline{\omega}}_0\right)$$

$\overline{\overline{\omega}}_0$ being defined as the conjugate complex value of ω_0.

b) φ_1

From the equations 2.12 and 3.6 it follows that for φ_1

$$-\alpha^2(\varphi_{1tt} + \beta\varphi_{1t}) + \varphi_{1xx} = -\frac{1}{2}\frac{\partial}{\partial x}\left(\varphi_{0x}^2 + \lambda\alpha^2\varphi_{0t}^2\right) =$$

$$= -\frac{1}{8}\frac{\partial}{\partial x}\left(\omega_{0x}^2 + \overline{\omega}_{0x}^2 + 2\,\omega_{0x}\overline{\omega}_{0x} + \right.$$

$$\left. + \lambda\alpha^2\left(\omega_{0t}^2 + \overline{\overline{\omega}}_{0t}^2 + 2\,\omega_{0t}\overline{\overline{\omega}}_{0t}\right)\right) \qquad 3.7$$

holds. 3.7 is a linear equation and therefore we can split it up into

$$-\alpha^2\left(\varphi_{1tt}^{(1)} + \beta\varphi_{1t}^{(1)}\right) + \varphi_{1xx}^{(1)} =$$

$$= -\frac{1}{4}\frac{\partial}{\partial x}\left(\omega_{0x}\overline{\overline{\omega}}_{0x} + \lambda\alpha^2\omega_{0t}\overline{\overline{\omega}}_{0t}\right) \qquad 3.8\text{ a}$$

$$-\alpha^2\left(\varphi_{1tt}^{(2)} + \beta\varphi_{1t}^{(2)}\right) + \varphi_{1xx}^{(2)} =$$

$$= -\frac{1}{8}\frac{\partial}{\partial x}\left(\omega_{0x}^2 + \overline{\overline{\omega}}_{0x}^2 + \lambda\alpha^2\left(\omega_{0t}^2 + \overline{\overline{\omega}}_{0t}^2\right)\right) \qquad 3.8\text{ b}$$

$$\varphi_{1x}^{(i)}(0, t) = \varphi_1^{(i)}(1, t); \quad (i = 1.\ 2)$$

Thus $\varphi_1 = \varphi_1^{(1)} + \varphi_1^{(2)}$

The right side of 3.8 a is independent of t, therefore also $\varphi_1^{(1)}$ is independent of t and we get by simple integration

$$\varphi_{1x}^{(1)} = \frac{1}{4}\int_0^x \frac{\partial}{\partial x}\left(\omega_{0x}\overline{\omega}_{0x} + \lambda\alpha^2\omega_{0t}\overline{\overline{\omega}}_{0t}\right)dx \qquad 3.9$$

From 3.8 b we see that we can write $\varphi_1^{(2)}$ as the realpart of a function ω_1 for which

$$\left.\begin{aligned} &-\alpha^2(\omega_{1tt} + \beta\omega_{1t}) + \omega_{1xx} = \\ &= -\frac{1}{4}\frac{\partial}{\partial x}\left(\omega_{0x}^2 + \lambda\alpha^2\omega_{0t}^2\right) \\ &\omega_{1x}(0, t) = \omega_1(1, t) = 0 \end{aligned}\right\} \qquad 3.10$$

holds. From 3.5 we have

$$\frac{\partial}{\partial x}\omega_{0x}^2 = \frac{2A^2K_1e^{it}}{h^2\left(1 + e^{-2K_1}\right)^2}\left(e^{2K_1(x-2)} - e^{-2K_1 x}\right)$$

$$\lambda\alpha^2\frac{\partial}{\partial x}\omega_{0t}^2 =$$

$$= -\frac{2A^2\lambda\alpha^2e^{2it}}{Kh^2\left(1 + e^{-2K_1}\right)^2}\left(e^{2K_1(x-2)} - e^{-2K_1 x}\right) =$$

$$= -\frac{\lambda\alpha^2}{K_1^2}\frac{\partial}{\partial x}\omega_{0x}^2$$

Therefore we get

$$-\alpha^2(\omega_{1tt} + \beta\omega_{1t}) + \omega_{1xx} = -\frac{1}{2}\left(1 - \frac{\lambda\alpha^2}{K_1^2}\right)\cdot$$

$$\cdot\frac{A^2K_1e^{2it}}{h^2\left(1 + e^{-2K_1}\right)}\left(e^{2K_1(x-2)} - e^{-2K_1 x}\right) \qquad 3.11$$

$$\omega_{1x}(0, t) = \omega_1(1, t) = 0 \qquad 3.11\text{ a}$$

A particular solution of 3.11 we get in the form

$$\omega_{1P} = \left(1 - \frac{\lambda\alpha^2}{K_1^2}\right)\frac{iA^2}{h^2}\cdot$$

$$\cdot\frac{K_1e^{2it}}{4\alpha^2\beta\left(1 + e^{-2K_1}\right)^2}\left\{e^{2K_1(x-2)} - e^{-2K_1 x}\right\} \qquad 3.12$$

Now we must find a solution ω_{1H} of the homogenous equation 3.11 so that $\omega_1 = \omega_{1P} + \omega_{1H}$ fulfills the boundary conditions 3.11 a.

Because ω_{1P} is of the form $f(x)e^{2it}$, ω_{1H} must be of the same form:

$$\omega_{1H} = g(x)\, e^{2it}$$

From the homogenous equation 3.11 we get for $g(x)$

$$4\alpha^2\left(1-\frac{i\beta}{2}\right)g(x)+g''(x)=0 \qquad 3.13$$

and therefore

$$\left.\begin{aligned} &g(x)=De^{K_2(x-2)}+Ee^{-K_2x}\\ &\text{with}\quad K_2^2=-4\alpha^2\left(1-\frac{i\beta}{2}\right)\end{aligned}\right\} \qquad 3.14$$

$$\left.\begin{aligned} &\text{thus}\quad K_2=K_{2r}+iK_{2i}=\frac{\alpha}{\sqrt{2}}\cdot\\ &\cdot\left(2\sqrt{\sqrt{1+\frac{\beta^2}{4}}-1}+\frac{i\beta}{\sqrt{\sqrt{1+\frac{\beta^2}{4}}-1}}\right)\end{aligned}\right\} \qquad 3.14$$

D and E we can determine in such a way, that ω_1 fulfills the boundary conditions 3.11 a and we find finally:

$$\omega_1=\omega_{1P}+\omega_{1H}=$$

$$=\frac{iA^2K_1e^{2it}}{h^2 4\beta\alpha^2}\,\frac{\left(1-\frac{\lambda\alpha^2}{K_1^2}\right)}{(1+e^{-2K_1})^2}\left\{e^{2K_1(x-2)}-e^{-2K_1x}-\frac{2K_1}{K_2}\frac{(1+e^{-4K_1})}{(1+e^{-2K_2})}\left(e^{K_2(x-2)}-e^{-K_2x}\right)\right\}=$$

$$=\left(1-\frac{\lambda\alpha^2}{K_1^2}\right)\frac{iA^2K_1e^{2it}}{h^2 4\beta\alpha^2(e^{-K_1}+e^{K_1})}$$

$$\left\{e^{2K_1(x-1)}-e^{-2K_1(x-1)}-\frac{2K_1}{K_2}\frac{(e^{-2K_1}+e^{2K_1})}{(e^{K_2}+e^{-K_2})}\left(e^{K_2(x-1)}-e^{-K_2(x-1)}\right)\right\} \qquad 3.15$$

$$\text{with}\quad \alpha=\frac{2\pi L}{\sqrt{gh}\,T};\quad \beta=\frac{\varrho T}{2\pi h};\quad K_\nu=\frac{\alpha}{\sqrt{2}}\left(\nu\sqrt{\sqrt{1+\frac{\beta^2}{\nu^2}}-1}+\frac{i\beta}{\sqrt{\sqrt{1+\frac{\beta^2}{\nu^2}}-1}}\right)$$

and

$$\varphi_1^{(2)}=\frac{1}{2}\left(\omega_1+\bar{\bar{\omega}}_1\right) \qquad 3.16$$

4. Validity of approximating φ by $\varphi_0+\varphi_1$

It was shown in section 3 that we can approximate $\psi=\varphi-(\varphi_0+\varphi_1)$ by $\bar{\psi}$, where $\bar{\psi}$ is determined by

$$\left.\begin{aligned} &-\alpha^2(\bar{\psi}_{tt}+\beta\bar{\psi}_t)+\bar{\psi}_{xx}=\\ &=-\frac{\partial}{\partial x}(\varphi_{0x}\varphi_{1x}+\lambda\alpha^2\varphi_{0t}\varphi_{1t})\\ &\bar{\psi}_{0x}(0,t)=\bar{\psi}_0(1,t)=0\end{aligned}\right\} \qquad 4.1$$

There are no mathematical difficulties to solve this equation. We shall only integrate it for the case of a long channel for which we may neglect the reflected wave and merely state the result for the general case. From 3.5, 3.9 and 3.15 we get:

$$-\frac{\partial}{\partial x}(\varphi_{0x}\varphi_{1x}+\lambda\alpha^2\varphi_{0t}\varphi_{1t})=$$

$$=-\frac{1}{4}\frac{\partial}{\partial x}\left[(\omega_{0x}+\bar{\bar{\omega}}_{0x})(\omega_{1x}+\bar{\bar{\omega}}_{1x})+\right.$$

$$\left.+\lambda\alpha^2(\omega_{0t}+\bar{\bar{\omega}}_{0t})(\omega_{1t}+\bar{\bar{\omega}}_{1t})\right]-$$

$$-\frac{1}{2}\frac{\partial}{\partial x}\left[(\omega_{0x}+\bar{\omega}_{0x})\varphi_{1x}^{(2)}\right]$$

Therefore we can introduce functions $\bar{\psi}_\nu$ ($\nu=1,2,3$) for which

$$\left.\begin{aligned} &-\alpha^2(\bar{\psi}_{\nu tt}+\beta\bar{\psi}_{\nu t})+\bar{\psi}_{\nu xx}=\\ &=\begin{cases}-\frac{1}{2}\frac{\partial}{\partial x}(\omega_{0x}\omega_{1x}+\lambda\alpha^2\omega_{0t}\omega_{1t}) & \nu=1\\ -\frac{1}{2}\frac{\partial}{\partial x}(\bar{\bar{\omega}}_{0x}\omega_{1x}+\lambda\alpha^2\bar{\bar{\omega}}_{0t}\omega_{1t}) & \nu=2\\ -\frac{\partial}{\partial x}\varphi_{1x}^{(2)}\omega_{0x} & \nu=3\end{cases}\end{aligned}\right\} \qquad 4.2$$

holds, and we get then

$$\bar{\psi}=\text{realpart}\,(\bar{\psi}_1+\bar{\psi}_2+\bar{\psi}_3) \qquad 4.3$$

For long channels we may neglect e^{-K_ν} compared with one and we get from 3.5, 3.9 and 3.15:

$$\omega_0 = -\frac{A}{hK_1} e^{it} e^{-K_1 x}$$

$$\omega_1 = -\frac{iA^2 K_1}{h^2 4\beta\alpha^2}\left(1 - \frac{\lambda\alpha^2}{K_1^2}\right)\left(e^{-2K_1 x} - \frac{2K_1}{K_2} e^{-K_2 x}\right) e^{2it}$$

Without introducing appreciable errors we may put $\frac{2K_1}{K_2} = 1$ and get

$$f(x) = -\frac{1}{2}\frac{\partial}{\partial x}(\omega_{0x}\omega_{1x} + \lambda\alpha^2 \omega_{0t}\omega_{1t}) =$$

$$= Ce^{3it}\frac{\partial}{\partial x}\left[2K_1^2 e^{-K_1 x}(e^{-2K_1 x} - e^{-K_2 x}) - \right.$$

$$\left. - 2\lambda\alpha^2 e^{-K_1 x}\left(e^{-2K_1 x} - \frac{2K_1}{K_2} e^{-K_2 x}\right)\right] =$$

$$= C^*[3K_1 e^{-3K_1 x} - (K_1 + K_2)e^{-(K_1+K_2)x}]$$

with

$$C^* = \left(1 - \frac{\lambda\alpha^2}{K_1^2}\right)^2 \frac{A^3 K_1^2}{h^3 4\beta\alpha^2}$$

Then we can write the equation for $\overline{\psi}$ in the form:

$$\left.\begin{aligned} -\alpha^2(\overline{\psi}_{1tt} + \beta\overline{\psi}_{1t}) + \overline{\psi}_{xx} = f(x)e^{3it} \\ \overline{\psi}_{1x}(0, t) = \overline{\psi}_1(1, t) = 0 \end{aligned}\right\} \quad 4.4$$

assuming $\overline{\psi}_1 = g(x)e^{3it}$ we get for $g(x)$

$$\left.\begin{aligned} 9\alpha^2\left(1 - \frac{i\beta}{3}\right)g(x) + g''(x) = \\ = C^*\{3K_1 e^{-3K_1 x} - (K_1 + K_2)e^{-(K_1+K_2)x}\} \\ g'(0) = g(1) = 0 \end{aligned}\right\} \quad 4.5$$

Putting $K_3^2 = -9\alpha^2\left(1 - \frac{i\beta}{3}\right)$; realpart $K_3 \geq 0$ we have

$$\frac{g(x)}{C^*} = \frac{3K_1}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + 9K_1^2}(e^{-3K_1 x} - e^{-(K_1+K_2)x}) -$$

$$-\left[\frac{K_1 + K_2}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + (K_1 + K_2)^2} - \frac{3K_1}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + 9K_1^2}\right](e^{-(K_1+K_2)x} - e^{-K_3 x})$$

We only introduce small errors if assuming $K_\nu = K_{1r} + i\nu K_{1i}$. Thus

$$\left|\frac{3K_1}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + 9K_1^2}(e^{-3K_1 x} - e^{-(K_1+K_2)x})\right| \leq \frac{3|K_1|}{6\beta\alpha^2}\left|e^{-3K_1 x} - e^{-(K_1+K_2)x}\right| \leq \frac{2}{27}\cdot\frac{|K_1|}{\beta\alpha^2}$$

$$\left|\frac{K_1 + K_2}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + (K_1 + K_2)^2} - \frac{3K_1}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + 9K_1^2}(e^{-(K_1+K_2)x} - e^{-K_3 x})\right| \leq$$

$$< \frac{1}{4}\left|\frac{K_1 + K_2}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + (K_1 + K_2)^2} - \frac{3K_1}{9\alpha^2\left(1 - \frac{i\beta}{3}\right) + gK_1^2}\right|$$

and from the last expression one can show that it is smaller than

$\frac{|K_1|}{8\beta\alpha^2}$ for $0 \leq \beta \leq 2$. Therefore

$\overline{\psi}_1 = g(x)e^{3it}$ with $|g(x)| \leq C^*\left|\frac{|K_1|}{\alpha^2\beta}\left(\frac{1}{8} + \frac{2}{27}\right)\right.$

In the same way one can show that the same inequality holds for $\overline{\psi}_2$ and that $\overline{\psi}_3$ is small compared with $\overline{\psi}_1$ and $\overline{\psi}_2$. Now we can decide under which conditions our approximation is valid.

We accept the approximation if

$$\frac{\text{Maximum } |\bar{\psi}_t|}{\text{Maximum } |\varphi_{0t}|} = \frac{\text{maximum } |\text{flow } \varphi_t - \text{approximation } \varphi_{0t} + \varphi_{1t}|}{\text{maximum } |\text{“linear flow” } \varphi_{0t}|} \leq$$

Now

$$\frac{\text{maximum } |\bar{\psi}_t|}{\text{maximum } |\varphi_{0t}|} = \frac{h}{A} \frac{|K_1|^2}{\alpha^2 \beta} 4 \left(\frac{1}{8} + \frac{2}{27}\right) \left| C^* \right| \leq 0.8 \left| 1 - \frac{\lambda \alpha^2}{K_1^2} \right| \frac{A^2}{4\,h^2} \cdot \frac{1+\beta^2}{\beta^2}$$

therefore the approximation is valid if

$$\frac{A}{h} \leq \sqrt{0.15 \frac{\beta^2}{1+\beta^2} \cdot \frac{1}{\left| 1 - \frac{\lambda \alpha^2}{K_1^2} \right|^2}} \qquad 4.7 \qquad \left(\text{Example } \beta = 2;\ \frac{A}{h} \leq 0.275\right)$$

In the general case we get a similar result

$$\frac{\text{maximum } |\bar{\psi}_t|}{\text{maximum } |\varphi_{0t}|} \leq 0.6 \left| 1 - \frac{\lambda \alpha^2}{K_1^2} \right| \left\{ \frac{1}{|1+e^{-2K_1}|} \left(\frac{1}{|1+e^{-2K_2}|} + \frac{1}{3} \frac{1}{|1+e^{-2K_3}|} \right) \frac{A^2}{4h^2} \frac{1+\beta^2}{\beta^2} \right\} \qquad 4.8$$

About the factor $\left| \frac{1}{1+e^{-2K_1}} \right|$ see the next section. $\left| \frac{1}{1+e^{-2K_3}} \right|$ takes its maximum if the channel is so long that the corresponding free oscillation has the period $\frac{T}{3}$. Then $\left| \frac{1}{1+e^{-2K_3}} \right|$ can become quite big if β is small and it may happen that the approximation for this length is not useful. But $\left| \frac{1}{1+e^{-2K_3}} \right|$ can only become as big as 2 or 1.5 for values of β = 2 and 3 repectively.

5. Estimates of the linear part $\bar{u}_0'$ and the nonlinear part $\bar{u}_1'$

In the first four sections it has been shown that the flow distribution of tidal motions can be approximated—under certain conditions—as the sum of two terms $\bar{u}_0'$ and $\bar{u}_1'$, the first of which represents the solution of the linearized equation 2.4. We can interprete this representation in the following way:

The tides in the open sea (boundary conditions $\zeta'(0, t) = \frac{A}{h} \cos \frac{2\pi}{T} t$) induces the T-periodical oscillation $\bar{u}_0'$. This oscillation itself induces the $1/2 \cdot T$-periodical oscillation $\bar{u}_1'$.

In this section we shall establish some rough measures for $\bar{u}_0'$ and the ratio between the linear part $\bar{u}_0'$ and the nonlinear part $\bar{u}_1'$ in order to get estimates for these quantities. Besides we shall also obtain some important information about the oscillations.

a) measure for $\bar{u}_0'$

From 2.14, 2.15, 3.5 and 3.6 we get

$$\bar{u}_0' = -\text{realpart}$$

$$\left\{ \frac{Aie^{it}}{(1+e^{-2K_1})} \cdot \frac{1}{hK_1} \left(e^{K_1(x-2)} - e^{-K_1 x}\right) \right\} \qquad 5.1$$

Therefore we can use as a measure for $\bar{u}_0'$

$$m(\bar{u}_0') = \left| \frac{A}{hK_1} \cdot \frac{1}{(1+e^{-2K_1})} \right| = \frac{A}{\alpha h \sqrt[4]{1+\beta^2}} \cdot \frac{1}{|1+e^{-2K_1}|} \qquad 5.2$$

This measure is quite good for not too short channels. For short channels (that means short compared with the wavelength of $\bar{u}_0$)[1] it is bad because α is small compared with one and therefore also $e^{K_1(x-1)} - e^{-K_1(x-1)}$ is small.

Quite interesting is the factor

$$\frac{1}{|1+e^{-2K_1}|} = \frac{1}{\sqrt{(1+e^{-2K_{1r}} \cos 2K_{1i})^2 + e^{-4K_{1r}} \sin^2 2K_{1i}}} \qquad 5.3$$

[1] $\bar{u}_0$ is the velocity in dimensional form; we get it from the nondimensional velocity $\bar{u}_0'$ by inverting the transformation 2.3

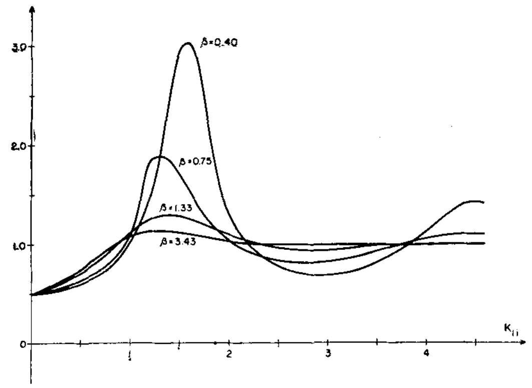

Fig. 3. $\frac{1}{|1 + e^{-2K_1}|}$ for different values of β.

This factor reaches its maximum value when the channel is just of the length, for which the corresponding free oscillation has the period T. In Figure 3 we have plotted $|1 + e^{-2K_1}|^{-1}$ as a function of the ratio $\frac{\text{channel length } L}{\text{wave length of } \bar{u}_0} = K_{1i} = \frac{\alpha\beta}{\sqrt{2}\cdot\sqrt{\sqrt{1+\beta^2}-1}}$ for different values of β that means for different values of frictional influence, and we see:

in general $|1 + e^{-2K_1}|^{-1} \sim 1$ but in the "neighbourhood of resonance" $|1 + e^{-2K_1}|^{-1}$ can be quite big if the frictional influence is small.

From 5.3 we see that $|1 + e^{-2K_1}|^{-1}$ reaches its first maximum when

$$\cos 2K_{1i} \sim -1$$

or

$$K_{1i} = \frac{\alpha\beta}{\sqrt{2}\cdot\sqrt{\sqrt{1+\beta^2}-1}} = \frac{\pi}{2}$$

now

$$\alpha = \frac{2\pi L}{\sqrt{gh}\, T}$$

and thus resonance for the linear part occurs when

$$L_T = \frac{\sqrt{gh}}{4}\cdot T\cdot\frac{\sqrt{2}\sqrt{\sqrt{1+\beta^2}-1}}{\beta} \qquad 5.4$$

As a mean value for $m(\bar{u}_0')$ we get

$$\frac{1}{\alpha h}\cdot\frac{1}{\sqrt[4]{1+\beta^2}}$$

If we invert the transformation 2.3 we get a measure for the mean velocity

$\frac{\bar{u}_0}{h} = \frac{1}{h}\int_{-h}^{\zeta} u_0 dz$ and we obtain

$$m\left(\frac{\bar{u}_0}{h}\right) = \sqrt{\frac{g}{h}}\cdot A\cdot\frac{1}{\sqrt[4]{1+\beta^2}} \qquad 5.5$$

b) measure for $r = \frac{\bar{u}_1'}{\bar{u}_0'}$

From equation 3.9, 3.14, 3.15 and 2.15 we get:

$$\bar{u}_1' = -\text{realpart}\left(1 - \frac{\lambda\alpha^2}{K_1^2}\right)\frac{A^2K_1^2}{h^2\beta K_2\alpha^2}\cdot\frac{1 + e^{-4K_1}}{(1 + e^{-2K_1})^2(1 + e^{-2K_2})}$$

$$\left\{e^{K_2(x-2)} - e^{-K_2x} - \frac{K_2}{2K_1}\frac{(1 + e^{-2K_2})}{(1 + e^{-4K_2})}\left(e^{2K_1(x-2)} - e^{-2K_1x}\right)\right\} \qquad 5.6$$

and therefore we can choose as a measure for $r = \frac{\bar{u}_1'}{\bar{u}_0'}$

$$m(r) = \frac{1}{m(\bar{u}_0')}\cdot\left|1 - \frac{\lambda\alpha^2}{K_1^2}\right|\frac{A^2|K_1^2|}{h^2\beta|K_2|\alpha^2}\cdot\frac{|1 + e^{-4K_1}|}{|1 + e^{-2K_1}|^2|1 + e^{-2K_2}|} =$$

$$= \left|1 - \frac{\lambda\alpha^2}{K_1^2}\right|\cdot\frac{A}{2h}\cdot\frac{\sqrt{1+\beta^2}}{\beta}\cdot\sqrt[4]{\frac{1+\beta^2}{1+\frac{\beta^2}{4}}}\cdot\frac{1}{|1 + e^{-2K_1}|}\cdot\left|\frac{1 + e^{-4K_1}}{1 + e^{-2K_2}}\right| \qquad 5.7$$

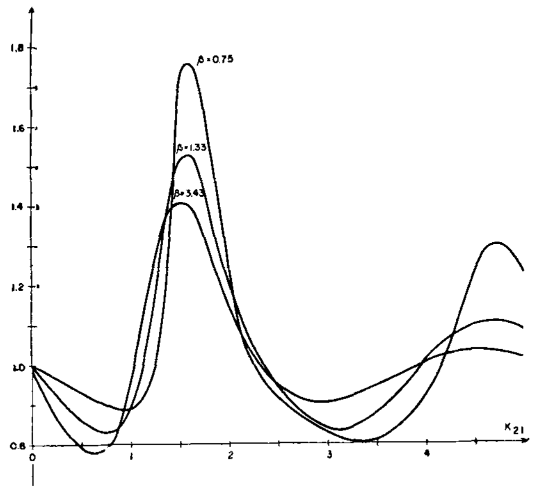

Fig. 4. $|D|^{-1} = \left|\frac{1 + e^{-4K_1}}{1 + e^{-2K_2}}\right|$ for different values of β.

Besides the already mentioned resonance another resonance can influence $m(r)$ strongly. The length of the channel is such that the corresponding free oscillation has the period $1/2\ T$. Then $|1 + e^{-2K_2}|^{-1}$ takes its maximum.

However, the influence of $|1 + e^{-2K_2}|^{-1}$ cannot become too strong, because $|1 + e^{-4K_1}|$ then is small. In Figure 4 we have plotted $\left|\frac{1 + e^{-4K_1}}{1 + e^{-2K_2}}\right|$ as a function of

$$K_{2i} = \frac{\text{channel length } L}{\text{wavelength of the nonlinear part } u}$$

for different values of β. We see that $\left|\frac{1 + e^{-4K_1}}{1 + e^{-2K_2}}\right|$ may become as big as two.

In the same way we can compute when resonance for the nonlinear part $\bar{u}_1$ occurs. We must have

$$K_{2i} = \frac{\alpha\beta}{\sqrt{2}\cdot\sqrt{\sqrt{1 + \frac{\beta^2}{4}} - 1}} = \frac{\pi}{2}$$

and therefore

$$L_{T/2} = \frac{\sqrt{gh}}{8}\, T \cdot \frac{\sqrt{2}\cdot\sqrt{\sqrt{1 + \frac{\beta^2}{4}} - 1}}{\frac{\beta}{2}} \tag{5.8}$$

As a mean value for $m(r)$ we get:

$$m(r) = \frac{A}{2h}\,\frac{\sqrt{1 + \beta^2}}{\beta}\cdot\sqrt[4]{\frac{1 + \beta^2}{1 + \frac{\beta^2}{4}}}\cdot\left|1 - \frac{\lambda\alpha^2}{K_1^2}\right| = \frac{A}{2h}\,\frac{\sqrt{1 + \beta^2}}{\beta}\cdot\sqrt[4]{\frac{1 + \beta^2}{1 + \frac{\beta^2}{4}}}\cdot\frac{\sqrt{(\lambda + 1)^2 + \beta^2}}{\sqrt{1 + \beta^2}} \tag{5.9}$$

for small β-values we may approximate this expression by

$$m(r) = \frac{A}{h\beta} = \frac{2\pi A}{\varrho T} \text{ because } \beta = \frac{\varrho T}{2\pi h};\ \lambda = 1$$

$$\left(\text{example: } T = 12^h = 4.32\cdot 10^4 \text{ sec};\ \varrho = 2\cdot 10^{-3}\left[\frac{\text{m}}{\text{sec}}\right] \text{ and therefore } m(r) = 0.07\,\frac{A}{[\text{meter}]}\right)$$

The last equation shows that for small values of frictional influence $m(r)$ is independent of the depth and only dependent on the coefficient of friction, the period T and the amplitude A. Therefore the nonlinear part is also important, when the depth is large and its influence is particularly strong when resonance occurs, because the frictional influence is small.

But we can go one step further. In hydraulics one uses the following friction law:

$$R(\bar{u}) = \varrho^*\,\frac{|\bar{u}|\,\bar{u}}{h^2}$$

If therefore the velocities are small we may replace our friction law

$$R(\bar{u}) = \frac{\varrho\bar{u}}{h}$$

by

$$R^*(\bar{u}) = \frac{\varrho^*\,|\bar{u}_{\max}|\,\bar{u}}{h^2}$$

From 5.5 we get $|\bar{u}_{\max}| \sim \sqrt{gh}A$ and thus

$$R(\bar{u}^*) = \frac{\varrho^{**}\sqrt{gh}A}{h^2}\,\bar{u} = \varrho\,\frac{\bar{u}}{h} \text{ with } \varrho = \varrho^{**}\sqrt{\frac{g}{h}}A.$$

Using this expression for ϱ in equation 5.10 we get

$$m(r) = \frac{2\pi}{\varrho^{**} T}\sqrt{\frac{h}{g}}$$

Now $m(r)$ is also independent of the amplitude A and proportional to $\sqrt{h}$. Even if the depth is large and the amplitude is small the influence of the nonlinear part might be relatively strong. The reason why the influence of the nonlinear part does not tend to o more rapidly than the linear part when we let $h \to \infty$, $L \to \infty$ and $A \to 0$ is found from the equation 3.10.

The right side $\omega_{0x}\omega_{0xx}$ of this equation represents an oscillation which converges towards a free oscillation of 3.10 when β tends to o. Therefore also $\omega_{0x}\omega_{0xx}$ must tend to o if ω_1 is to remain finite.

Now we are also in the position to say something about the influence of the term $\frac{\lambda}{h}\bar{u}\bar{u}_x$. We can regard $\bar{u}_1' = \bar{u}_1'(\lambda)$ as a function of λ. We want to compare the influence of this term and the term $gh\zeta\zeta_x$. Therefore we examine the ratio $\left|\frac{\bar{u}_1'(\lambda=1) - \bar{u}_1'(\lambda=0)}{\bar{u}_1'(\lambda=0)}\right|$. From 5.6 we get

$$\left|\frac{\bar{u}_1'(\lambda=1) - \bar{u}_1'(\lambda=0)}{\bar{u}_1'(\lambda=0)}\right| = \frac{1}{\sqrt{1+\beta^2}}, \quad 5.11$$

if β is small compared with one $\frac{1}{\sqrt{1+\beta^2}} \sim 1$.

In this case the influence of the term $\frac{\lambda}{h}\bar{u}\bar{u}_x$ is as big as the influence of the term $gh\zeta\zeta_x$. But for $\beta = 3$ its influence is only 30 % of the influence of $gh\zeta\zeta_x$.

6. Discussion of the solution for different cases

In the previous sections we have seen that the flow $\bar{u}'$ is composed by two harmonics $\bar{u}_0'$, $\bar{u}_1'$ which have the periods T, $1/2\ T$ respectively, and that the "nonlinear influence" is particularly strong when resonance occurs. We have also seen that in general only the first resonance is pronounced, which is due to the frictional influence. Therefore we can divide all channels into different classes, with respect to their lengths.

1) The length is small compared with the "resonance length" $L_{T/2}$ (5.8), where resonance for the nonlinear part occurs.
2) The length is in the "neighbourhood" of $L_{T/2}$.
3) The length is in the "neighbourhood" of L_T, where resonance for the linear part occurs. (5.4)
4) The channel is so long that the reflected wave is negligible.

We are therefore going to discuss the following four important cases:

a) short channels
b) infinitely long channels (the reflected wave is negligible)
c) the length is equal to $L_{T/2}$
d) the length is equal to L_T

a) Short chaunels

For a short channel α is small compared with one and we may therefore approximate e^{2K_1} sufficiently well by $1 + 2K_1 + 2K_1^2$. From 5.1 and 5.6 we then get:

$$\bar{u}_0' = -\,\text{realpart}\ \frac{iA}{h}\frac{e^{it}}{\left(1+\frac{K_1^2}{2}\right)}(x-1)$$

$$\bar{u}_1' = \text{realpart}\ \frac{4A^2K_1^2(x-1)\,e^{2it}}{h^2(2+K_1)^2(2+K_2^2)}$$

if $|K_1| \le 0.3$[1] then $|K_2| \le 0.6$ and we may neglect $\frac{K_1^2}{2}$ against 1.

Therefore:

$$\left.\begin{aligned} \bar{u}_0' &= -\frac{A}{h}|x-1|\sin\frac{2\pi}{T}t \\ \frac{|\bar{u}_1'|}{\max|\bar{u}_0'|} &\le 0.06\frac{A}{h} \end{aligned}\right\} \quad 6.1$$

[1] $|K_1| = \alpha\sqrt[4]{1+\beta^2} = \frac{2\pi L}{\sqrt{gh}\,T}\cdot\sqrt[4]{1+\beta^2}$, if $|K_1| \le$ ≤ 0.3 then $L \le \frac{0.3\sqrt{gh}T}{2\pi\sqrt[4]{1+\beta^2}}$

Thus we see that we can neglect $\bar{u}_1'$ even when $\frac{A}{h}=0.3$. From the continuity-equation 2.4 b we get

$$\zeta' = \frac{A}{h}\cos\frac{2\pi}{T}t \quad \text{thus} \quad \frac{\partial \zeta'}{\partial x} = 0 \qquad 6.2$$

Thus for short rivers with varying cross-section one can put $\frac{\partial \zeta}{\partial x} = 0$ and determine $\bar{u}$ from the continuity equation.

Before discussing the other three cases we shall derive a formula for the flow $\bar{u}$ and discuss some general properties of this expression.

From 5.1 and 5.6 we get

$$\bar{u}' = \text{realpart}\left\{\frac{A}{(1+e^{-2K_1})hK_1}\left\{ie^{it}\left(e^{K_1(x-2)} - e^{-K_1x}\right) + \left(1 - \frac{\lambda\alpha^2}{K_1^2}\right)\frac{A}{h\beta}\frac{K_1^3}{K_2\alpha^2}\cdot\right.\right.$$

$$\left.\left.\cdot\frac{(1+e^{-4K_1})}{(1+e^{-2K_1}(1+e^{-2K_2})}e^{2it}\left(e^{K_2(x-2)} - e^{-K_2x} - \frac{K_2}{2K_1}\frac{(1+e^{-2K_2})}{(1+e^{-4K_1})}\left[e^{2K_1(x-2)} - e^{-2K_1x}\right]\right)\right\}\right\} \qquad 6.3$$

writing

$$\left.\begin{aligned} &K_1 = |K_1|e^{i\vartheta_1} \quad K_2 = |K_2|e^{i\vartheta_2} \quad \left(1 - \frac{\lambda}{K_1^2}\right) = |E|e^{i\vartheta_3} \\ &\frac{1}{1+e^{-2K_1}} = |B|e^{ib}; \quad \frac{(1+e^{-4K_1})}{(1+e^{-2K_1})(1+e^{-2K_2})} = |C|e^{ic}; \quad \frac{1+e^{-2K_2}}{1+e^{-4K_1}} = |D|\cdot e^{id} \end{aligned}\right\} \qquad 6.4$$

we get

$$\bar{u}' = \frac{A|B|}{h|K_1|}\,\text{realpart}\left\{l_1(-K_1x) + \frac{A|K_1|^3}{h\beta|K_2|\alpha^2}|E||C|\left[l_2(-K_2x) - \frac{|K_2|}{2|K_1|}|D|\,l_2(-2K_1x+d)\right]\right\} -$$

$$-\frac{A|B|}{h|K_1|}\,\text{realpart}\left\{l_1[K_1(x-2)] + \frac{A|K_1|^3}{h\beta|K_2|\alpha^2}|E||C|\left[l_2[K_2(x-2) - \frac{|K_2|}{2|K_1|}|D|\,l_2[2K_1(x-2)+d]\right]\right\}$$

$$\text{where } l_1(s) = e^{i\left(t+\frac{\pi}{2}-\vartheta_1+b+s\right)} \quad l_2(s) = e^{i(2t+2\vartheta_1-\vartheta_2+c+b+\vartheta_3+s)} \qquad 6.5$$

In general we may put $d=0$, $K_{1r} = K_{2r}$, $K_{2i} = 2K_{1i}$, without introducing too large errors.

Using the abbreviation $t^* = t - \vartheta_1 - K_{1i}x + b$ we get

$$\bar{u}_1' = -\frac{A}{h}\frac{B}{|K_1|}e^{-K_{1r}x}\cdot$$

$$\cdot\left\{\sin t^* + \frac{EA|K_1|^3}{h\beta|K_2|\alpha^2}|C|\cos\left[2t^* + (4\vartheta_1 - \vartheta_2 - 180) + \vartheta_3\right]\left(1 - \frac{|K_2|}{2|K_1|}\cdot|D|e^{-K_{1r}x}\right)\right\} +$$

$$+\frac{A}{h}\frac{|B|}{|K_1|}e^{K_{1r}(x-2)}\left\{\sin\left[t^* + 2K_{1i}(x-1)\right] + \frac{EA|K_1|^3}{h\beta|K_2|\alpha^2}|C|\right.$$

$$\left.\cos\left[2t^* + (4\vartheta_1 - \vartheta_2 - 180) + \vartheta_3 + 4K_1(x-1)\right]\cdot\left(1 - \frac{|K_2|}{2|K_1|}|D|e^{K_{1r}(x-2)}\right)\right\} \qquad 6.8$$

From the last equation we see that $\bar{u}'$ is composed of two waves which are of the same structure and only differ in magnitude and phase. The first part represents the entering wave while the second part represents the reflected wave. Both waves consist of a linear part and a nonlinear part and there are two extreme cases for the superposition of the nonlinear parts.

i) if $2K_{1i}(x-1) = -\frac{\pi}{2}$ both parts are of the same form with respect to time and have the same sign.

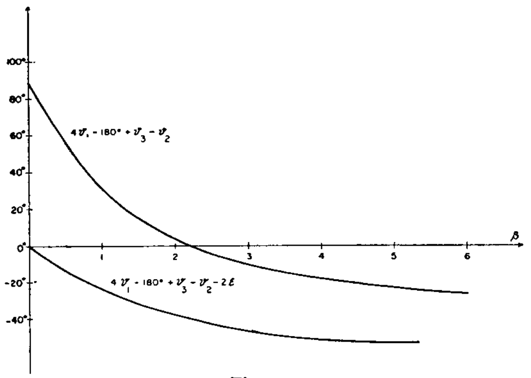

Fig. 5.

ii) if $2K_{1i}(x-1) = -\pi$ the parts are of the same form with respect to time but have different signs. We thus find that the nonlinear influence is strongest in the case i) while in case ii) it is much reduced. A closer examination follows.

i) $2K_{1i}(x-1) = -\dfrac{\pi}{2}$

From 3.4 we get

$$\frac{2K_{1r}(x-1)}{2K_{1i}(x-1)} = \frac{K_{1r}}{K_{1i}} = \frac{\sqrt{1+\beta^2}-1}{\beta}$$

$$\text{now } 2K_{1i}(x-1) = -\frac{\pi}{2}$$

and therefore $2K_{1r}(x-1) = -\dfrac{\pi}{2}\dfrac{\sqrt{1+\beta^2}-1}{\beta}$

Introducing this value into 6.8 we get

$$\bar{u}' = -\frac{A|B|}{h|K_1|}e^{-K_{1r}x}\left\{\sin\, t^* + e^{-\frac{\pi}{2}\frac{\sqrt{1+\beta^2}-1}{\beta}}\cdot\cos\, t^* + H\cos\left[2t^* + (4\vartheta_1 - \vartheta_2 - 180) + \vartheta_3\right]\right\} \qquad 6.9$$

with

$$H = \frac{EA|K_1|^3}{h\beta|K_2|\alpha^2}|D|\left\{1 - \frac{|K_2|}{2|K_1|}|D|e^{-K_{1r}x} + \left(1 - \frac{|K_2|}{2|K_1|}|D|e^{K_{1r}(x-2)}\right)e^{2K_{1r}(x-1)}\right\}$$

therefore we can write 6.9 in the form

$$\bar{u}' = -\frac{A|B|}{h|K_1|}e^{-K_{1r}x}$$

$$\{F\sin(t^*+\varepsilon) + H\cos[2t^* + (4\vartheta_1 - \vartheta_2 - 180) + \vartheta_3]\} \qquad 6.10$$

From figure 5 we see that $|\, 4\vartheta_1 - \vartheta_2 - 180° + \vartheta_3 - 2\varepsilon\,| < 90°$. Now two cases are possible: 1) $H > 0$ then we have for all β-values

$$\text{maximum } \bar{u}_F > \text{maxium } \bar{u}_E$$

One can see this from figure 6—7 where we have plotted the curves

$$\left.\begin{array}{l} -[\sin\, t + 0.2\,\cos\,(2t+90°)]; \\ -(\sin\, t + 0.2\,\cos\, 2t) \end{array}\right\} \qquad 6.11$$

and we see that if $\bar{u}' = -[\sin t^* + a\cos(2t^*+\eta)]$ with $0 \le |\,\eta\,| < 90°$, $a > 0$ the last inequality holds and that the unsymmetry is most pronounced for $\eta = 0°$ while it vanishes for $\eta = 90°$.

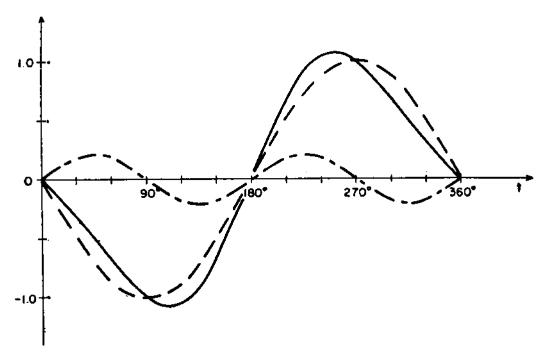

Fig. 6.

— — — — – – sint —·—·— – – 0.2 cos (2t + 90)

———— – – (sint + 0.2 cos (2t + 90))

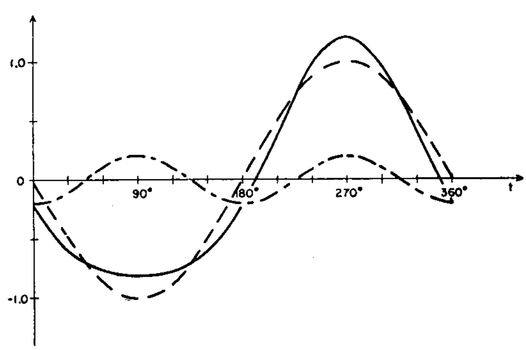

Fig 7.

— — — — – – sint —·—·— – – 0.2 cos 2t

———— – – (sint + 0.2 cos 2t)

2) $H \leq 0$

Then: maximum $\bar{u}_F \leq$ maximum $\bar{u}_E$. [This can only happen for extreme values of β. With aid of Fig. 3 one can compute that $H > 0$ for $\beta \geq 0.5$, and this condition for β is usually fulfilled.]

ii) $2K_{1i}(x-1) = -\pi$

Now

$$\bar{u}' = -\frac{A|B|}{h|K_1|} e^{-K_{1r}x} \{M \sin t^* + N \cos [2t^* + (3\vartheta_1 - 180) + \vartheta_3]\} \quad 6.12$$

with

$$M = 1 + e^{2K_{1r}(x-1)} \quad N = \frac{|K_1|^3}{h\beta|K_2|\alpha^2} |C| \left\{ 1 - \frac{|K_2|}{2|K_1|} |D| e^{-K_{1r}x} - \left(1 - \frac{|K_2|}{2|K_1|} |D| e^{K_{1r}(x-2)}\right) e^{2K_{1r}(x-1)} \right\}$$

In the previous case we found that in general

maximum $\bar{u}_F >$ maximum $\bar{u}_E$

but now

maximum $\bar{u}_F <$ maximum $\bar{u}_E$ 6.13

at the mouth of the channel. Then $2K_{1i} \sim \pi$, $e^{-K_{1r}x} \sim 1$, $|D| > 1$, and with the aid of Fig. 3, one can compute that

$$1 - \frac{|K_2|}{2|K_1|} |D| - \left(1 - \frac{|K_2|}{2|K_1|} |D| \, e^{-2K_{1r}}\right) e^{-2K_{1r}} < 0$$

for not too big values of β (for ex. $\beta = 1.3$) and therefore $N < 0$. We get the inequality 6.13 since Figure 5 shows that $4\vartheta_1 - \vartheta_2 - 180 + \vartheta_3$ is varying between $90°$ and $-20°$ while β is changing from 0 till 5.

b) Long channels

We may neglect the reflected wave and put $|B| = |C| = |D| = 1$; $b = c = d = 0$.

Then

$$\bar{u}' = -\frac{A}{h|K_1|} e^{-K_{1r}x} \left\{ \sin t^* + E \frac{A|K_1|^3}{h\beta|K_2|\alpha^2} \cos [2t^* + (4\vartheta_1 - \vartheta_2 - 180) + \vartheta_3] \left(1 - \frac{|K_2|}{2|K_1|} e^{-K_{1r}x}\right) \right\} \quad 6.14$$

From this equation and Fig. 5 we get for tidal currents in long channels

maximum $\bar{u}_F >$ maximum $\bar{u}_E$

when friction is taken into account, because $1 - \frac{|K_2|}{2|K_1|} e^{-K_{1r}x} > 0$. Furthermore we see that the relatively strongest influence of the nonlinear component occurs at the end of the channel and that this component takes its absolute biggest value W when $e^{-K_{1r}x} = 1 - \frac{|K_1|}{|K_2|}$

$$W = \frac{EA^2|K_1|^2}{\beta h^2 |K_2| \alpha^2} \left(1 - \frac{|K_1|}{|K_2|}\right) \left(\frac{3}{2} - \frac{|K_2|}{2|K_1|}\right) \cdot \frac{2\pi L}{T} \quad 6.15$$

Examples:

$h = 5$ m $\quad \beta = 2 \quad A = 1.5$ m $\quad W = 0.11$ m/sec.

$h = 5$ m $\quad \beta = 3 \quad A = 1.5$ m $\quad W = 0.2$ m/sec.

c) Channels with length equal to $L_{T/2}$

Here $2K_{1i} = \frac{\pi}{2}$ (see 5.8) and for $x = 0$ we get the case that has been discussed earlier (see 6.10). The nonlinear components of the two waves (see 6.8) are of the same form with respect to time and have the same sign. Because of resonance these components are relatively large and therefore the nonlinear influence is particularly strong. With aid of equation 6.8 one can show that also for the points $0 < x < 1$

maximum $u_F >$ maximum u_E

d) Channels with length equal to L_T

We have $2K_{1i} = \pi$. For $x = 0$, $x = 1/2$ we get the cases discussed in connection with equation 6.10, 6.12 respectively. Therefore we get for the first part of the channel

maximum $u_F <$ maximum u_E

For $x = 1/2$, the inequality changes into

maximum $u_F >$ maximum u_E

and one can show that this last inequality holds for all points with $1 > x > 1/2$.

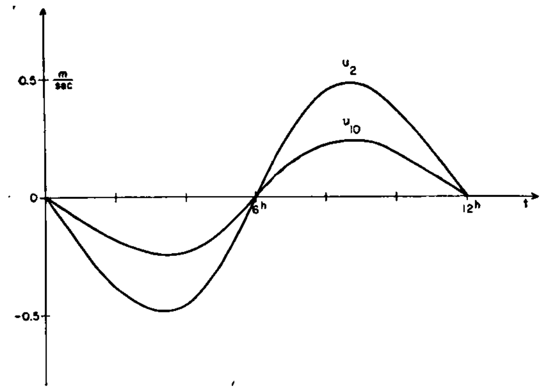

Fig. 8.

: 20 km long channel. Mean velocity $u_i = \dfrac{\bar{u}_i}{(h_i + \zeta_i)}$ in the points $x_i = i$ km. x_i = distance from the mouth. The positive values represent the flood current.

7. Some numerical computations

For comparison with our theory we shall discuss some numerical solutions of the equation 2.1 which have been obtained using the method of finite differences (cf. HANSEN, 1956.) The friction term is changed to

$$R(\bar{u}) = \frac{\varrho |\bar{u}| \bar{u}}{h^2}$$

and it is interesting to see that this does not change the structure of the solutions.

In Figure 8 we have plotted the tidal currents in a 20 km long and 10 meter deep channel.[1] Our theory is confirmed because at every point the deviation from the theoretically expected value is ≤ 0.02 m/sec (sec equations 6.1, 6.2).

In Figure 9 the tidal currents in a 50 km long and 10 meter deep channel have been plotted. According to our theory we have resonance for the nonlinear part and the numerical computation shows the same: very pronounced nonlinear influence.

In Figure 10 the tidal currents in a 98 km long and 10 meter deep channel is shown. Also here theory and computation give qualitatively the same result: resonance for the linear component occurs and we get a flow distribution as theoretically obtained for channels with length equal to L_T.

[1] In all cases is $\varrho = 1 \cdot 10^{-3}$.

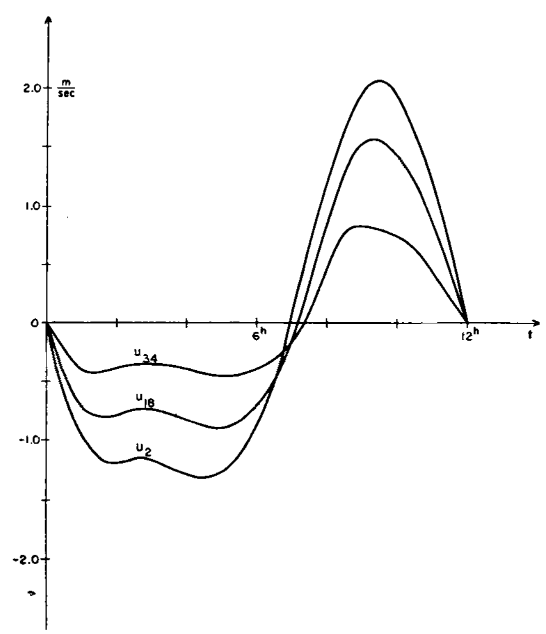

Fig. 9. : 50 km long channel. Mean velocity u_i in the points $x_i = i$ km.

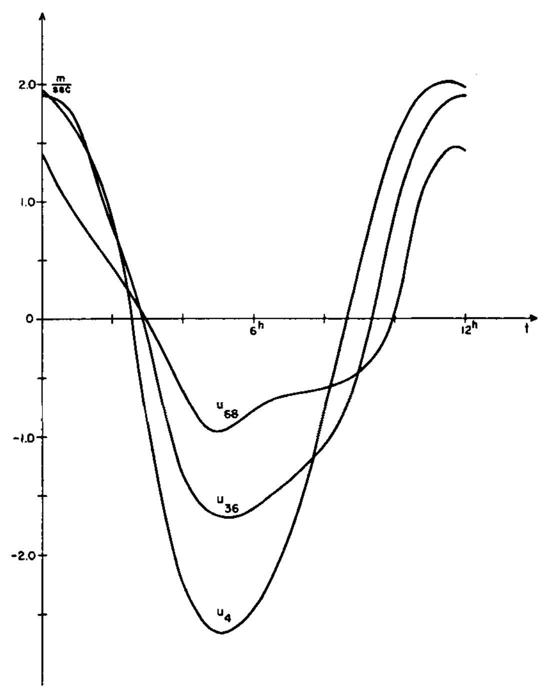

Fig. 10. 98 km long channel. Mean velocity u_i in the points $x_i = i$ km.

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