A Heuristic Theory of Large-Scale Turbulence and Long-Period Velocity Variations in Barotropic Flow

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(Manuscript received March 23, 1956)

Abstract

By combining an equation for the conservation of zonal momentum with the eddy vorticity equation for nondivergent barotropic flow through a channel, it has been found possible to express the time variations of the longitude-averaged zonal wind in terms of the mean zonal wind itself and certain statistics of the eddy velocity field. In large-scale turbulence—where the characteristic dimension of the eddies is comparable with the entire "width" of the flow—these eddy statistics take on the properties of autocorrelation functions; thus, they depend primarily on the overall scale of the eddies and intensity of turbulence, rather than on details of the flow pattern. Changes in the scale and intensity of large-scale turbulence are related to changes in the mean zonal flow through the principles of conservation of total kinetic energy and variance of vorticity. Under a general hypothesis regarding the "stability" of the eddy statistics, the equations for the zonally-averaged motion comprise a complete system of non-linear equations, which can be integrated by finite-sum and -difference methods. Accordingly, these equations constitute a mathematical theory of large-scale "free" turbulence, and provide the basis for a numerical method of predicting the zonally-averaged motion.

With the assumption that the eddy scale is invariant, the equation governing the mean zonal wind takes a form related to that of the classical wave equation. Numerical integrations of this equation verify that extremes of mean zonal wind (e.g., pronounced jets) maintain their identity over long periods of time, and migrate either northward or southward (or in both directions at once) in a very regular way—a phenomenon that has been observed by Riehl et al. (1950). The results also show that a strong jet in the mean westerlies tends to split into two weaker jets, which then move in opposite directions to produce the separated double jet structure typical of widespread blocking. The latter has been described as a characteristic denouement from strong zonal flow by Cressman (1950). The theoretical speed of migration of extremes in the mean zonal wind is found to agree closely with the observed speed in a case where published data are available.

Since these features of long-period velocity variations—the most striking and regular discovered to date—have a characteristic time scale of a week or two, numerical prediction methods based on the present theory show some promise of aiding in the preparation of extended-range forecasts.

1. Introduction and General Remarks

During the past year or so, increasing attention has been focussed on the possibility of applying hydrodynamical theory to the problem of long range weather prediction. The modest and rather recent successes of dynamical methods in *short* range prediction have shown that even the simple theory underlying present methods is a fair description of the atmosphere's behavior on a large scale and over short periods of time. In a manner of speaking, the theory of adiabatic nonviscous flow appears to describe the "internal" dynamics of the atmosphere—i.e., the manner in which the state of the atmosphere would evolve from a given initial state, without absorption of heat energy from external sources, and without the forces of viscosity whereby kinetic energy is dissipated into heat energy and (in the latter form) ultimately emitted to space. Accordingly, since the general physical laws that govern the atmosphere's internal dynamics are also those which control its reaction to energy input and output, one should expect that some more general formulation of hydrodynamical theory would provide a sound basis for methods of long range prediction.

Considering the problem in extreme, it is evident that a special theory of adiabatic nonviscous flow is not general enough to account for *very long* period changes—as, for instance, the change of seasons. In this case, it is clear that the change does not depend as much on the state of the atmosphere at some arbitrarily specified time in the remote past as it does on the cumulative effect of energy input and output. In short, whether the theory of adiabatic nonviscous flow is or is not capable of predicting changes over a period of given length is probably a question of degree, depending on the length of the period. The "transition" period, which is undoubtedly related in some way to the time required to dissipate a large fixed fraction of the relative kinetic energy in the absence of heat sources (or to build it up from the state of relative rest), has been variously estimated from several days to a week or so.

Viewed in this light, the problem of forecasting over periods of a few days might reasonably be approached from the standpoint of integrating the equations for adiabatic nonviscous flow in their usual "two"- or three-dimensional form. This is a particularly attractive approach, for it involves merely extending the period over which the present equations[1] for short range prediction are to be integrated. Preliminary efforts to explore this approach have already been made, in fact, by BERGTHORSSON and collaborators (1955), and more recently by the Joint Numerical Weather Prediction Unit (JNWPU). Both groups have found that the simplest of methods (based on the equations for barotropic flow) predicts most of the change in the very large scale flow patterns over periods up to three days, without regard to the actual input or output of energy or, for that matter, the transformation of potential to kinetic energy. In the few instances where the predictions were extended over longer periods, there was a marked decrease in accuracy—partially, no doubt, due to shortcomings of the *physical* model, but possibly due to the accumulation of errors that are purely *mathematical* in origin.

This same general approach to the problem of long range forecasting is also illustrated in the recent work of PHILLIPS (1955), who has integrated equations for a model whose *internal* dynamics are expressed in the usual short range prediction equations, but in which the intensity of heat sources varies with latitude and in which kinetic energy is dissipated by simple diffusion of vorticity. Although Phillips' computations were carried out for random initial conditions and were not directly aimed at long range forecasting per se, they still point up the merits and disadvantages of an approach centered around the integration of the hydrodynamical equations in two- or three-dimensional form. In general, Phillips' model displays the main features of behavior of the atmosphere, taken over a long period of time and on a large scale. As in previous attempts at extended numerical integration, the quality of the predictions began to deteriorate rapidly within a few days after the disturbances had grown to appreciable amplitude. In this case, however, the predictions were incorrect only in the sense that they did not reflect some mathematical property of the *differential* equations for the model. Thus, the "errors" of Phillips' computations were due entirely to approximations of the numerical method by which the equations were solved.

Apart from incomplete or inaccurate initial data, the errors of predictions based on hydrodynamical theory may be lumped under the following two main headings:

(1) Errors arising from the introduction of approximations into the general hydrodynamical equations. Since such approximations may radically alter the range and types of phenomena describable by the equations, this kind of error is due essentially to defects of the *physical model*.

[1] Based on the so-called theory of quasi-geostrophic flow.

(2) Errors which are inherent in the numerical methods by which the differential equations are solved—e.g., errors due to finite-difference approximations and roundoff errors. Errors of this class are purely mathematical in character, in the sense that they would be present in numerical integrations of the *general* hydrodynamical equations.

With errors of the first type we shall not be concerned here, except to remark that the *physical* problem confronting us is not so much to discover the general mechanisms of long period changes, as it is to give simple and interpretable mathematical expression to the operation of more or less known mechanisms, in terms of meteorological observables.

With regard to purely mathematical errors, it should be noted that the difficulties of the approach outlined earlier are essentially those of integrating the usual "two"- or three-dimensional form of the equations for quasi-geostrophic flow—difficulties that are enhanced by the fact that the equations are now to be integrated over extended periods of time. Since the quasi-geostrophic equations are nonlinear, they are usually replaced by a corresponding set of finite-difference equations, which are solvable by a sequence of algebraic operations. Now, the condition for (computational) stability of solutions of the finite-difference equations is quite severe, requiring that the integrations be carried out over a long succession of short time intervals, each of the order of an hour. For a forecast period of *fixed* duration, the cumulative effect of roundoff error, truncation error, and nonlinear interactions with those errors does not decrease to zero with an increasing number of submultiple time increments. Thus, if the forecast period is long enough, this effect will obscure all features of meteorological interest.

The situation is further aggravated by the fact that first and higher order finite-differences of the unaveraged isobaric height do not vary smoothly from one point to the next, producing large truncation errors in the approximation of derivatives. Taken together, errors of the type described above generally manifest themselves in increasingly "ragged" contour and vorticity patterns, a symptom of computational distress that is recognizable in the extended predictions of PHILLIPS (1955), BERGTHORSSON ET AL. (1955), and the JNWPU. The *symptoms* may be alleviated rather arbitrarily by any of several methods of intermittent smoothing,[1] but none of these methods guarantees to preserve the correct interaction between components of different scale.

A second difficulty arises from the inaccuracy with which the phase speed of individual circulation centers can be predicted. Owing to the uncertainties of initial state, systematic truncation error and defects of the physical models, the probability that the predicted phase angle is *more* than 90 degrees away from the correct value approaches the probability that it is *less*, *when the period of the forecast is much in excess of the typical period of the transient fluctuations.* If this is indeed the case, as appears likely, one might as well abandon any attempt to predict zonal variations over long periods, and deal only with zonal averages. This argument alone strongly suggests that the fundamental equations for long range weather prediction should be made to apply to zonally-averaged variables, rather than to their point values.

Returning temporarily to questions of cumulative roundoff and truncation error, one would expect that equations for zonally-averaged motion would involve smaller phase velocities and correspondingly less stringent conditions for computational stability. Further, latitudinal and time variations of zonally averaged variables are quite smooth, so that truncation errors should not accumulate as rapidly in this case as they do in numerical integration of the full "two-" or three-dimensional form of the quasi-geostrophic equations. This surmise is borne out by the results of later sections.

In summary, it appears desirable from the standpoint of reducing purely mathematical errors, and sufficient (in view of ultimate limitations on the detail of accurate long range forecasts) to deal with a new set of derived hydrodynamical equations, which applies to the zonally averaged state of the atmosphere.

There is, in retrospect, another reason for developing a theory of mean motions. Aside from pragmatic questions of forecasting, it would be desirable to interpret the behavior of the atmosphere over long periods of time in terms of relatively simple geometrical and physical concepts. On the face of it, the three-dimensional form of the equations for even

[1] e.g., the methods proposed by PHILLIPS (1955) and SHUMAN (1955).

quasi-geostrophic flow does not lead to such simple concepts. The equations of mean motion, however, do admit of a rather clearcut geometrical interpretation.

Having discussed at length some of the difficulties in applying hydrodynamical theory to the problem of long and "extended" range forecasting, and having discussed several strong motives for dealing with equations for the zonally averaged motions of the atmosphere, we can now state briefly the principal objectives of the remainder of this paper. They are simply to derive a complete system of equations that applies to "zonally averaged" motions, with particular emphasis on an equation for predicting the mean zonal wind profile, and to present a numerical method by which these equations can be integrated. It should also be made clear that the main purpose of this work is not to develop a method of prediction that is immediately applicable to long range forecasting, but to outline and explore a methodology or approach that may lead to greater insight into less special problems of long range prediction.

For simplicity, and to clarify general features of method, we shall deal with the motions of a barotropic, nonviscous fluid, flowing through an infinitely long channel bounded by rigid parallel walls. Our problem, then, is to derive a complete system of equations that applies to the mean motions of this fluid, averaged along lines parallel to the walls and at variable distance from one of the boundaries. In particular, we shall seek to derive and solve an equation for the changes in the mean component of velocity parallel to the walls.

Since the flow under consideration is barotropic and essentially nondivergent, one can state at the outset that it cannot exhibit any feature of behavior which depends in a vital way on energy input or output—simply because potential energy, the intermediary form through which heat energy must pass to be converted into kinetic energy, cannot be transformed into kinetic energy in this model. Nevertheless, it is still conceivable that this model will display some important and typical aspects of long period velocity variations in the atmosphere, *when taken in the mean*. It is generally observed, in fact, that the mechanism of baroclinic instability (by which potential energy is transformed into kinetic energy of the large scale disturbances) operates only sporadically and in sudden bursts, and over only a small fraction of a hemisphere at any given time. Further, sudden increases in the total zonal momentum of the atmosphere over a whole hemisphere occur only once every week or two (vide RIEHL ET AL., 1950), and are percentagewise not very great. For these reasons, one might expect that the variations of *mean* zonal wind in the barotropic model (beginning with equivalent initial conditions) would parallel those of the atmosphere for periods approaching or perhaps slightly exceeding the "transition" period, particularly during the stage immediately following an almost impulsive increase of total kinetic energy. It has already been pointed out that forecasts based on the equation for barotropic flow are fairly accurate *in detail* for periods up to three days. Finally, having excluded the transformation of potential to kinetic energy, it is only consistent to exclude also the dissipation of kinetic energy by viscosity. These conjectures will be confirmed qualitatively and semiquantitatively by reference to data published by CRESSMAN (1950) and RIEHL, YEH and LASEUR (1950).

Turning to the ingredients of any theory of mean motion, one soon sees that a *complete* system of equations for the zonally-averaged motion cannot be derived from the equations of barotropic flow alone. Owing to the fact that the configuration of *unaveraged* flow is not uniquely determined by a given configuration of *averaged* flow, some information is irretrievably lost in the process of averaging. To make up for this loss, it is necessary to state some hypothesis regarding the statistical properties of the "eddies"[1]—i.e., deviations from zonal averages—and/or their interaction with the mean flow itself. The hypothesis to be introduced here capitalizes on the geometry of very large scale eddies, and strictly applies only when the characteristic horizontal dimension of the eddies is comparable with the distance between the "walls" that contain the flow. Interpreted in terms of the atmosphere's dimensions, the distance between "walls" corresponds roughly to the distance from equator to pole; thus, in view of the tre-

[1] It will frequently be convenient to adopt the language of turbulence, so that "eddy", "fluctuation", "disturbance", "deviation", and "perturbation" will be used more or less interchangeably.

mendous scale of atmospheric motions, the "hypothesis of large scale" is superficially a reasonable one. In this same connection, it is important to note that the most effective agencies of mean zonal momentum transport are the disturbances of largest scale.

The aims and viewpoints expressed above are, of course, far from new. In essence, they are the aims and viewponts of a theory of large-scale turbulence, in which the large-scale disturbances are regarded as eddies in a nonlaminar flow; a complete theory of mean motions would, in fact, be a bona fide theory of turbulence. The type of "turbulence" in question is quite different, to be sure, from the usual variety—in that the mean state probably does not approach a quasi-steady asymptotic state. The mean state of the atmosphere's boundary layer, on the other hand, can and does approach such a steady state, simply because small scale eddies are continuously generated by mechanical means and continuously dissipated by viscosity. For this reason, one might characterize large-scale atmospheric disturbances as "free turbulence". Other distinctions between these two types of turbulence —as, for instance, the nondiffusive nature of large-scale eddies—have been pointed out by STARR (1953) and others.

The development of the mathematical theory will center around an equation that expresses the way in which changes in the mean flow depend on the eddy velocity. By introducing the way in which changes in the eddy velocity are related in turn to the mean flow, it is then possible to state how changes in the mean zonal wind depend on the mean flow itself. The information required to arrive at a complete system of equations is the form of certain cross-correlations between turbulent fluctuations. The key point in the development is that, with the introduction of the "hypothesis of large-scale", those cross-correlations take on the properties of normalized auto-correlations—properties that vary only slightly from one day to the next.

The resulting prognostic equation for the mean zonal wind can be integrated by finite-difference methods, beginning with the initial profile of mean zonal wind and the initial transport of zonal momentum. By-products of the integration are indices of the intensity of meridional flow, the characteristic scale of the disturbances, and the latitude of maximum eddy circulation.

To make sure that the behavior of solutions of the mean zonal wind equation is not enforced by a fortuitous choice of initial conditions, the method of integration mentioned above will be applied in a case when the initial transport of zonal momentum vanishes everywhere. In this case, it is easy to see that the fundamental equation has the mathematical form of a wave equation, and that a pronounced "jet" in the mean zonal wind profile tend to move northward or southward at a definite speed, depending partly on the relative dimensions of the disturbances and the jet itself, and partly on the RMS value of the meridional wind. This effect is strongly reminiscent of the phenomenon described by RIEHL (1950). Finally, according to this theory, there is also a tendency for a sharp maximum of mean zonal wind (such as would correspond to strong zonal flow around a large sector of a hemisphere) to split into two distinct maxima, which move apart to produce the "double-jet" structure typical of widespread blocking. The latter phenomenon has been reported by CRESSMAN (1950) as being a fairly characteristic development from strong zonal flow.

2. Fundamental Equations and Boundary Conditions for Barotropic Flow between Parallel Walls

The equations of motion for a nonviscous barotropic fluid may be written in the form:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2+v^2}{2}\right)-v(\zeta+f)+\frac{\partial P}{\partial x}=0 \quad (1)$$

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial y}\left(\frac{u^2+v^2}{2}\right)+u(\zeta+f)+\frac{\partial P}{\partial y}=0 \quad (2)$$

where t is time; x and y are coordinate distances parallel and normal to the walls, respectively; u and v are the corresponding components of the horizontal velocity vector; ζ is $(\partial v/\partial x-\partial u/\partial y)$, the vertical component of relative vorticity; and f (assumed to be a linear function of y only) is the Coriolis parameter or vertical component of the earth's vorticity. The function P is

$$\int_{p_0}^{p}\alpha(p')dp'$$

where p is pressure, p_0 is a fixed reference pressure, p' is a dummy variable corresponding to p and α is the specific volume, which in this model is a function of pressure only. Partial differentiations with respect to x, y, and t are carried out holding height fixed.

Since the percentage variations of pressure experienced by individual fluid elements next to the surface are very small over most of the globe, large-scale barotropic flow is essentially nondivergent. Thus, the equation of mass continuity reduces to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

Taken together, Eqs. (1), (2), and (3) comprise a complete system of equations involving the three dependent variables u, v, and P.

One of the fundamental equations in a theory of mean motion may be derived simply by applying the zonal average to each term of Eq. (1). Thus, noting that the averaging operation may be commuted with differentiations with respect to y and t,

$$\frac{\partial \bar{u}}{dt}-\overline{v\zeta}-f\bar{v}=0 \tag{4}$$

where the "bar" above a variable denotes its zonal average, defined as

$$\bar{\phi}\equiv \lim_{M\to\infty}\frac{1}{2M}\int_{-M}^{M}\phi dx$$

The averages of the second and fourth terms in Eq. (1) vanish, since

$$\overline{\frac{\partial \phi}{\partial x}}\equiv \lim_{M\to\infty}\frac{1}{2M}\left[\phi(M,y,t)-\phi(-M,y,t)\right]$$

In general, therefore, *the zonal average of the x-derivative* of any continuous variable which is always and everywhere finite *must vanish.*[1] This property will be invoked frequently throughout the mathematical development. Eq. (4) may be simplified further by introducing the condition for nondivergent flow—that is, since no fluid passes through either of the walls, the net volume transport across any line parallel to the walls must vanish. Thus, $\bar{v}=0$ and Eq. (4) becomes

$$\frac{\partial \bar{u}}{\partial t}=\overline{v\zeta}=\overline{v'\zeta'} \tag{5}$$

in which the primes denote deviations from zonal means. The equation above simply states that the rate at which the mean zonal wind changes at any "latitude" (line of constant y) is equal to the mean "northward" transport of vorticity at that latitude. In particular, since v vanishes at the walls, it implies that $\bar{u}$ at the walls does not change with time. This boundary condition, as it turns out, is necessary to determine a unique solution of the equations of mean motion.

Eq. (5) may be put into a more familiar form by introducing the definition of ζ and integrating by parts.

$$\overline{v\zeta}=\overline{\frac{\partial}{\partial x}\left(\frac{v^2}{2}-\frac{u^2}{2}\right)}-\frac{\partial}{\partial y}\overline{uv}=-\frac{\partial}{\partial y}\overline{uv}$$

so that Eq. (5) becomes

$$\frac{\partial \bar{u}}{\partial t}=-\frac{\partial}{\partial y}\overline{uv} \tag{6}$$

or, since $\bar{v}=0$

$$\frac{\partial \bar{u}}{\partial t}=-\frac{\partial}{\partial y}\overline{u'v'} \tag{7}$$

Eq. (7) states the well-known fact that the rate at which the mean zonal momentum increases is equal to the convergence of the mean eddy flux of zonal momentum. Integrating both sides of Eq. (6) with respect to y from one wall (at $y=0$) to the other (at $y=W$), we find that

$$\frac{\partial}{\partial t}\int_0^W \bar{u}\,dy=-\left[uv\right]_0^W=0$$

Thus, the total zonal momentum does not vary with time, another fact that will be of considerable use later in the development.

Eq. (5) or Eq. (7) states how changes in the mean zonal wind depend on deviations from purely zonal flow. Our next concern is to express the manner in which changes of the eddy velocity depend on the mean zonal flow itself. The required relationship is found in the

[1] This property can be retained in averages over a finite interval if the flow is periodic—e.g., if the fluid covers the surface of a sphere. In this case, the appropriate modification of the zonal average is to fix the interval of integration at 2π radians of longitude.

vorticity equation, obtained by differentiating Eq. (1) with respect to y, differentiating Eq. (2) with respect to x, and subtracting the first from the second. Since the flow is nondivergent, the vorticity equation takes the form

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v = 0 \qquad (8)$$

where $\beta = df/dy$. Writing each variable as the sum of its zonal mean and the deviation from the mean, applying the averaging operator to Eq. (8), and subtracting the *averaged* equation from Eq. (8), we then find that

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + v'\frac{\partial \bar{\zeta}}{\partial y} + \beta v' + (\mathbf{V}' \cdot \nabla \zeta')' = 0 \quad (9)$$

in which $\mathbf{V}$ is the horizontal velocity vector, and ∇ is the horizontal vector gradient. It will be noted that the fifth term in Eq. (9) is the only one that contains products of deviations, a property that distinguishes its statistics from those of terms in which the deviations enter linearly.

As it stands, Eq. (9) does not alone comprise a complete system of equations, for it involves the two dependent variables u and v. The missing information is contained in Eq. (3)—which, since $\bar{v} = 0$, may be written as

$$\frac{\partial u'}{dx} + \frac{\partial v'}{\partial y} = 0$$

This equation implies that there exists an "eddy streamfunction" ψ, such that

$$u' = -\frac{\partial \psi}{\partial y} \qquad v' = \frac{\partial \psi}{\partial x} \qquad \zeta' = \nabla^2 \psi$$

where ∇^2 is the horizontal Laplacian operator. Substituting the expressions above into Eq. (9),

$$\frac{\partial}{\partial t}\nabla^2\psi + \bar{u}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial^2 \bar{u}}{\partial y^2} + \beta\frac{\partial \psi}{\partial x} + J'(\psi, \nabla^2\psi) = 0 \quad (10)$$

Eq. (10) states how changes in the eddy streamfunction or eddy velocity depend on $\bar{u}$, the mean zonal flow.

Owing to the elliptic character of Eq. (10), regarded as a linear equation in which $\partial\psi/\partial t$ is the unknown, it is also necessary to specify the values of $\partial\psi/\partial t$ along the walls. To begin with, ψ must be constant along the walls at any given time, for v (or v') must vanish there. Now, the streamfunction contains an additive and arbitrary function of time—i.e., if the function Ψ is a solution of Eq. (10), then so also is the function $\psi = \Psi + f(t)$, simply because ψ does not enter undifferentiated with respect to x or y. Thus, we may choose the function $f(t)$ in such a way that ψ vanishes along $y = 0$ at all times. The value of ψ along $y = W$ may now be calculated from the relation between ψ and u'.

$$\psi(x, W, t) = -\int_0^W u' dy$$

But ψ along $y = W$ is constant at any given time, so that

$$\psi(x, W, t) = \bar{\psi}(W, t) = -\int_0^W \bar{u'} dy = 0$$

Finally, since ψ vanishes along both walls at all times, it follows that $\partial\psi/\partial t$ must also vanish along both walls at all times.

We may now regard Eqs. (5) and (10) as a complete system of equations involving the variables $\bar{u}$ and ψ. Beginning with initial values of ψ and $\bar{u}$ at every point in the flow, one can (in principle) compute $\partial\bar{u}/\partial t$ at all points in the flow from Eq. (5) and extrapolate $\bar{u}$ over a very short interval of time, *except along the walls*. It has already been shown, however, that $\bar{u}$ cannot vary along the walls, so that the distribution of $\bar{u}$ can be regenerated completely at a time slightly later than the initial instant. Similarly, it is possible to calculate $\partial\psi/\partial t$ from the initial distributions of ψ and $\bar{u}$ by solving Eq. (10), subject to the conditions that $\partial\psi/\partial t$ vanish on $y = 0$ and $y = W$. Thus ψ can also be extrapolated from its initial distribution. Finally, since the information available at the initial instant can be regenerated completely at a slightly later time, the process above may be repeated again and again, to compute the distributions of ψ and $\bar{u}$ over a period of any length desired.

This view of the problem of predicting the mean zonal wind speed is essentially the one that will be adopted in the remainder of the mathematical development.

3. The Equation for the Mean Zonal Flow

A method for predicting the mean zonal momentum, proposed several years ago by LORENZ (1952), will serve to point up one of the difficulties in synthesizing a general hypothesis. This method is based on an equation related to Eq. (7), and rests on an assumption whose validity depends on a rather loose empirical correlation between the net eddy transport of zonal momentum and the mean zonal momentum itself. From Eq. (5), however, we infer that the net eddy transport of zonal momentum is equivalent to the mean northward transport of vorticity or, in other words, the cross-correlation between the vorticity ζ and the "northward" component of velocity v. In general, since the fields of ζ and v tend to be almost exactly 90 degrees out of phase, the cross-correlation between them is usually quite small, and is extremely sensitive to small changes in the phase difference or to small changes in the general "shape" of the flow pattern. This suggests that the cross-correlation between v and ζ should not be specified by hypothesis, for the very reason that it is a cross-correlation between quantities which (at any fixed time) bear no direct geometrical relation to each other. Our next concern is to derive an equation for the mean zonal flow, in which the only statistics of the eddy velocity to appear are autocorrelations or cross-correlations between variables whose contemporaneous fields are geometrically related.

An equation of the desired form may be obtained by differentiating Eq. (5) with respect to time.

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \overline{v' \frac{\partial \zeta'}{\partial t} + \zeta' \frac{\partial v'}{dt}}$$

Thus, expressing v' and ζ' in terms of ψ, integrating by parts, and noting that $v' = v$,

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \overline{v \nabla^2 \left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial \psi}{\partial t} \nabla^2 v} \qquad (11)$$

This is one of the key equations in the present theory. An important variant of this same equation is derived by rewriting the right hand side of Eq. (11) as

$$\overline{\nabla \cdot \left(v \nabla \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{dt} \nabla v\right)} = \frac{\partial}{\partial y} \overline{v \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial \psi}{\partial t}\frac{\partial v}{\partial y}} \qquad (12)$$

The expression above verifies that the total zonal momentum is conserved, if its initial time derivative is zero.

The next step of the development is to eliminate the time derivatives from the right hand side of Eq. (11). This will be done by solving Eq. (10) for $\partial\psi/\partial t$. We first note that Eq. (10) can be put in the form

$$\nabla^2 \left(\frac{\partial \psi}{\partial t}\right) + \bar{u} \nabla^2 v - v \nabla^2 \bar{u} + \beta v + E = 0 \qquad (13)$$

where $E = J'(\psi, \nabla^2 \psi)$. To simplify matters still further, we shall rewrite Eq. (13) as

$$\nabla^2 \left(\frac{\partial \psi}{\partial t} + \bar{u} v\right) = 2 \nabla \cdot v \nabla \bar{u} - \beta v - E \qquad (14)$$

The equation above will now be multiplied by the function

$$G = -\frac{1}{4\pi} \ln \left[\frac{\sinh^2 \frac{\pi}{2W}(\xi - x) + \sin^2 \frac{\pi}{2W}(\eta - y)}{\sinh^2 \frac{\pi}{2W}(\xi - x) + \sin^2 \frac{\pi}{2W}(\eta + y)}\right]$$

in which ξ and η are dummy variables corresponding to x and y, respectively. This function is the so-called Green's function for an infinitely long strip of width W.[1] The relevant properties of G are that:

(1) $\nabla^2 G = \dfrac{\partial^2 G}{\partial \xi^2} + \dfrac{\partial G}{\partial \eta^2} = 0$ except at the point (x, y)

(2) G vanishes when $\eta = 0$, $\eta = W$, or when $\xi \to \pm \infty$

(3) $\dfrac{\partial G}{\partial \xi}$ vanishes when $\xi \to \pm \infty$

(4) G behaves like $-\dfrac{1}{2\pi} \ln r$, where $r^2 = (\xi - x)^2 + (\eta - y)^2$, for very small r.

[1] The Green's function may be visualized as the displacement at the point (ξ, η) of an elastic membrane rigidly fixed to two parallel rods a distance W apart, when a steady concentrated force is applied at the single point (x, y).

A consequence of this property is that

$$\lim_{\varrho \to 0} \int_0^{2\pi} \left(\frac{\partial G}{\partial r}\right)_C \varrho d\Theta = -1$$

in which ϱ is the radius of a very small circle C with center at (x, y), and Θ is the angle between the x-axis and a line from the fixed point (x, y) to the variable point (ξ, η) on the circle C. Making use of standard vector identities and property (1) of the function G, we may write Eq. (14) in the form

$$\nabla \cdot \left[G \nabla \left(\frac{\partial \psi}{\partial t} + \bar{u} v\right) - \left(\frac{\partial \psi}{\partial t} + \bar{u} v\right) \nabla G\right] =$$
$$= 2 \nabla \cdot G v \nabla \bar{u} - 2 v \nabla G \cdot \nabla \bar{u} - \beta G v - G E \quad (15)$$

Each term of the equation above is now integrated over the area A, bounded by the walls and a small circle C of radius ϱ with center at (x, y), as shown in Fig. 1. By the application

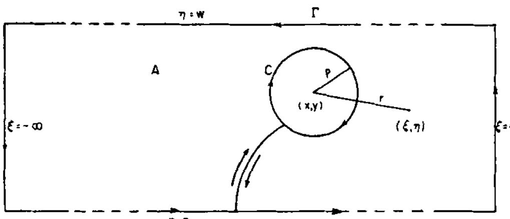

Fig. 1. Path of integration by Green's method.

of Gauss' theorem, the area integrals of the left hand side and first term on the right hand side of Eq. (15) can be transformed immediately into line integrals taken around the path Γ indicated by arrows on Fig. 1, with the result that

$$\oint_\Gamma \left[G \frac{\partial}{\partial n}\left(\frac{\partial \psi}{\partial t} + \bar{u} v\right) - \left(\frac{\partial \psi}{\partial t} + \bar{u} v\right) \frac{\partial G}{\partial n}\right] dl =$$
$$= 2 \oint_\Gamma G v \frac{\partial \bar{u}}{\partial n} dl - \int_A (2 v \nabla G \cdot \nabla \bar{u} + \beta G v + G E) dA \quad (16)$$

where a derivative with respect to n is the component of the vector gradient normal to the right of the path Γ, dl is an increment of length along Γ and dA is an area increment of A.

We now investigate the line integrals that enter into Eq. (16). To begin with, the contributions from integrating down one side and back along the other side of the "cut" adjoining the line $\eta = 0$ and the circle C exactly cancel. Moreover, owing to properties (2) and (3) of the function G, and because v and $\partial \psi / \partial t$ vanish on the walls, the line integrals around the outer boundary of A vanish. It remains to evaluate the limit of the line integrals around the circle C, as the radius ϱ approaches zero. Now, since $\bar{u}$, v, and $\partial \psi / \partial t$ are continuous at (x, y), the first term on the right hand side of Eq. (16) vanishes entirely in the limit, and the remainder of the equation reduces to:

$$\lim_{\varrho \to 0} \int_0^{2\pi} \left(\frac{\partial \psi}{\partial t} + \bar{u} v\right)_C \left(\frac{\partial G}{\partial r}\right)_C \varrho d\Theta$$
$$= - \int_A (2 v \nabla G \cdot \nabla \bar{u} + \beta G v + G E) dA$$

It is now understood that the region A covers the *entire* area between the walls. Making use of property (4) of the function G,

$$\frac{\partial \psi}{\partial t} + \bar{u} v = \int_0^W \int_{-\infty}^{\infty} (2 v \nabla G \cdot \nabla \bar{u} + \beta G v + G E) d\xi d\eta \quad (17)$$

where v, $\bar{u}$, and E are to be regarded as functions of ξ and η, wherever they appear under the integral signs. Finally, the expressions for $\nabla^2 (\partial \psi / \partial t)$ and $\partial \psi / \partial t$, given by Eqs. (13) and (17), will now be substituted into Eq. (11).

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \overline{v^2} \frac{\partial^2 \bar{u}}{\partial y^2} - 2 \overline{\nabla^2 v \int_0^W \int_{-\infty}^{\infty} v \nabla G \cdot \nabla \bar{u} d\xi d\eta}$$
$$- \beta \left(\overline{v^2} + \overline{\nabla^2 v \int_0^W \int_{-\infty}^{\infty} G v d\xi d\eta}\right)$$
$$- \left(\overline{vE} + \overline{\nabla^2 v \int_0^W \int_{-\infty}^{\infty} G E d\xi dn}\right) \quad (18)$$

This equation forms the basis for most of the remaining discussion.

It was pointed out earlier that E is the only term of Eq. (13) that involves products of deviations. One may, therefore, think of E

as representing the interaction of the eddies with themselves, and regard $\overline{vE}$ as an effect of the auto-interaction of eddies on the mean zonal flow. By definition,

$$\overline{vE} = \overline{v(\mathbf{V}' \cdot \nabla \zeta' - \overline{\mathbf{V}' \cdot \nabla \zeta'})}$$

or, since $\bar{v} = 0$,

$$\overline{vE} = \overline{v'\left(u'\frac{\partial \zeta'}{\partial x} + v'\frac{\partial \zeta'}{\partial y}\right)}$$

Integrating by parts, we find that the equation above can be rewritten in the form

$$\overline{vE} = \frac{\partial}{\partial y}\overline{\zeta'(v'^2)'} - \overline{u'(\zeta'^2)'} - \overline{\zeta'\frac{\partial}{\partial y}\left(\frac{u'^2+v'^2}{2}\right)'}$$

Thus, $\overline{vE}$ depends essentially on correlations between the deviation of one quantity and the deviation of the *square of the deviation* of another quantity. Now, in very large scale eddies the deviation fields u', v', and ζ' are of about the same scale, whereas the scale of the deviations of the *squares* of u', v', and ζ' is almost exactly *half* that scale. Accordingly, the fields of u', v', and ζ' tend to be uncorrelated with the fields of $(u'^2)'$, $(v'^2)'$, and $(\zeta'^2)'$, *without regard to the relative phases of those fields.* For this and similar reasons, the terms in Eq. (18) that involve E will be omitted. This simplification implies that the changes in the *mean* zonal flow are brought about primarily through the interaction of the eddies with the mean flow itself, rather than by the auto-interaction of eddies.

Our next concern is to isolate the effect of the eddies on the mean zonal flow. To do so, we shall investigate the correlation between $\nabla^2 v$ and the integrals on the right hand side of Eq. (18), a typical example of which is the expression

$$I = \overline{L(x, y)\int_0^W\int_{-\infty}^{\infty} G(\xi - x, y, \eta)v(\xi, \eta)d\xi d\eta}$$

where the symbol $L(x, y)$ stands for $\nabla^2 v$ at the point (x, y). Noting that x and ξ enter into G only in the combination $(\xi - x)$, and introducing a new variable of integration τ, equal to $(\xi - x)$,

$$I = \overline{L(x, y)\int_0^W\int_{-\infty}^{\infty} G(\tau, y, \eta)v(x + \tau, \eta)d\tau d\eta}$$

or, rearranging the order of integration,

$$I = \int_0^W\int_{-\infty}^{\infty} G(\tau, y, \eta)\overline{L(x, y)v(x + \tau, \eta)}d\tau d\eta$$

Finally, since $\overline{L(x, y)v(x, y)}$ is not a function of τ or η,

$$I = \overline{v\nabla^2 v}\int_0^W\int_{-\infty}^{\infty} G(\tau, y, \eta)R(\tau, y, \eta)d\tau d\eta$$

in which $R(\tau, y, \eta)$

$$= \overline{L(x, y)v(x+\tau, \eta)}/\overline{L(x, y)v(x, y)}.$$

Analogous procedures may be applied to transform the second term on the right hand side of Eq. (18), with the result that

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \overline{v^2}\left[\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{2}{\lambda^2}\int_0^W\int_{-\infty}^{\infty} \mathrm{R}\nabla \mathrm{G}\cdot\nabla \bar{u}d\tau dn - \beta\left(1 - \frac{1}{\lambda^2}\int_0^W\int_{-\infty}^{\infty} RGd\tau d\eta\right)\right] \quad (19)$$

where $\lambda(y)$ is given by $\lambda^{-2} = -\overline{v\nabla^2 v}/\overline{v^2}$.

To simplify Eq. (19) still further, we note that the area integral in the second term in brackets may be transformed as follows:

$$J = \int_0^W\int_{-\infty}^{\infty} \mathrm{R}\nabla \mathrm{G}\cdot\nabla \bar{u}d\tau d\eta$$

$$= \int_0^W\int_{-\infty}^{\infty}(\nabla\cdot R\bar{u}\nabla G - \bar{u}\nabla R\cdot\nabla G - \bar{u}R\nabla^2 G)d\tau d\eta$$

$$= \bar{u}(y)R(0, y, y) - \int_0^W\int_{-\infty}^{\infty}\bar{u}\nabla R\cdot\nabla Gd\tau d\eta$$

But, by definition, $R(0, y, y) = 1$. Moreover,

$$\int_0^W\int_{-\infty}^{\infty}\nabla R\cdot\nabla Gd\tau d\eta$$

$$= \int_0^W\int_{-\infty}^{\infty}(\nabla\cdot R\nabla G - R\nabla^2 G)d\tau d\eta = 1 \quad (20)$$

Thus, letting $\bar{u} = U + \frac{1}{W}\int\limits_0^W \bar{u}dy$, we may re-write the integral J as

$$J = \bar{u} - \frac{1}{W}\int\limits_0^W \bar{u}dy - \int\limits_0^W\int\limits_{-\infty}^{\infty} U(\eta)\nabla R\cdot\nabla G d\tau d\eta$$

$$= U - \int\limits_0^W U(\eta)F(y,\eta)d\eta$$

where $F(y,\eta) = \int\limits_{-\infty}^{\infty} \nabla R\cdot\nabla G d\tau$. Finally, since it has already been shown that $\int\limits_0^W \bar{u}\, dy$ does not vary with time, Eq. (19) can be put in the form

$$\frac{\partial^2 U}{\partial t^2} = \overline{v^2}\left[\frac{\partial^2 U}{\partial y^2} + \frac{2}{\lambda^2}\left(U - \int\limits_0^W U(\eta)F(y,\eta)d\eta\right)\right.$$

$$\left. - \beta\left(1 - \frac{1}{\lambda^2}\int\limits_0^W\int\limits_{-\infty}^{\infty} RGd\tau d\eta\right)\right] \quad (21)$$

This is the general prognostic equation for deviations of the mean zonal wind from an unchanging grand average (taken over the entire flow), and is the fundamental physical basis for the present theory of mean motions.

Taken together, Eqs. (10) and (21) comprise a complete system of equations, which can be integrated by a modification of the procedure outlined at the end of Section 2. Beginning with initial values of U and ψ, it is possible to compute the eddy statistics $\overline{v^2}$, λ, F and R, and to calculate $\partial^2 U/\partial t^2$ from Eq. (21). Since the initial values of $\partial U/\partial t$ (i.e., $\partial\bar{u}/\partial t$) are given in terms of ψ by Eq. (5), one can then predict U and $\partial U/\partial t$ at a time slightly later than the initial moment. As before, ψ may be predicted by solving Eq. (10) and extrapolating, in order to regenerate the eddy statistics $\overline{v^2}$, λ, F and R at a time later than the initial moment, and so on.

In effect, however, the scheme outlined above is equivalent to integrating the equations for the *unaveraged* motion. The main objective, therefore, is to introduce a hypothesis by which the eddy statistics can be specified or predicted, without integrating the general equations in full detail.

4. The Statistics of Large-Scale Turbulence

According to Eq. (21), changes in the mean zonal wind profile depend partly on the shape of the mean zonal wind profile itself, and partly on certain statistics of the meridional wind component—namely $\overline{v^2}$, λ, R, and F. Our present purpose is to discuss the character and interpretation of these statistics, and the extent to which they depend on details of the flow.

The quantity R is defined as

$$\overline{v(x+\tau,\eta)\nabla^2 v(x,y)}/\overline{v(x,y)\nabla^2 v(x,y)}$$

Thus, R is essentially a normalized cross-correlation function for a variable "lag" or "shift", depending on τ, y, and η. Its value at $\tau = 0$ and $\eta = y$ is necessarily unity for all y, as shown in Fig. 2. Moreover, since v vanishes

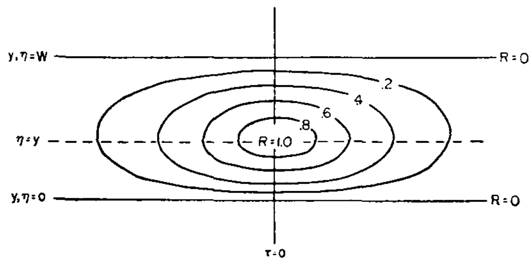

Fig. 2. Schematic pattern of $R(\tau, y, \eta)$ for $y = .4W$.

at the walls, R vanishes at $\eta = 0$ and $\eta = W$, for all y and τ. Now, the geometrical connection between v and $\nabla^2 v$ is given by the identity

$$\int\limits_0^W\int\limits_{-\infty}^{\infty} G\nabla^2 v d\xi d\eta = \int\limits_0^W\int\limits_{-\infty}^{\infty}[\nabla\cdot(G\nabla v - v\nabla G)$$

$$+ v\nabla^2 G]d\xi d\eta = -v(x,y)$$

Thus, since G has a singularity at (x, y) and decreases monotonically to zero as η goes to zero or W and as τ approaches $\pm\infty$, one may think of v as a weighted area average of $\nabla^2 v$, with greatest weight attached to the point (x, y) where the average applies. Accordingly, if the typical dimension of the eddies were much smaller than W, one would expect little

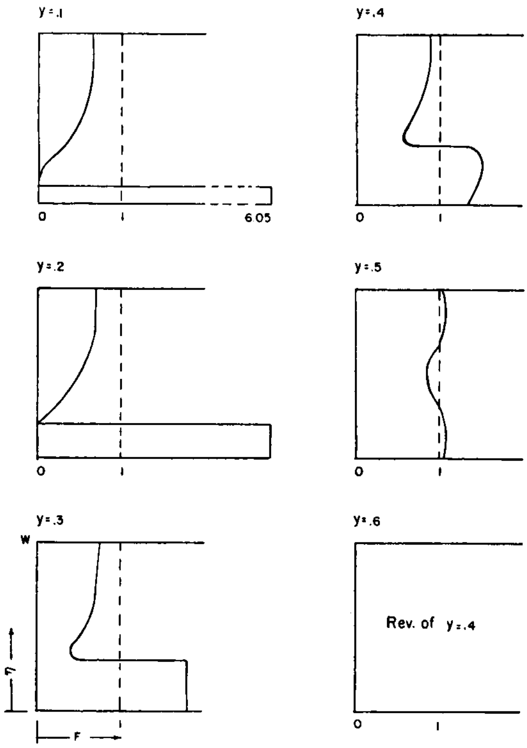

Fig. 3. The weighting-function $F(y, \eta)$ for $v = V \sin \frac{\pi x}{L} \sin \frac{\pi y}{W}$.

or no correlation between v and $\nabla^2 v$. On the other hand, if the scale of the eddies were comparable with W, maxima and minima of v would very nearly coincide with minima and maxima (respectively) of $\nabla^2 v$, so that there would be a high negative correlation between v and $\nabla^2 v$. In this case, $R(\tau, y, \eta)$ takes on the properties of an *auto*correlation function, attaining its maximum value near the point (x, y) and decreasing monotonically to zero as η goes to zero or W. Moreover, since the eddies are generally aperiodic, $R(\tau, y, \eta)$ will tend to approach zero in a regular way as τ approaches $\pm\infty$, the rate at which R decreases being determined by the scale of the eddies.

The mere fact that the general character of $R(\tau, y, \eta)$ can be described so completely in the case of large scale turbulence means that, it the typical scale of the eddies is comparable with the distance between walls, then $R(\tau, y, \eta)$ depends very little on the details of the flow. In this case, R depends primarily (but not crucially) on the typical scale of the eddies, and its day-to-day variation is mainly a reflection of changes in scale.

The quantity $F(y, \eta)$ is related to $R(\tau, y, \eta)$ as follows:

$$F(y, \eta) = \int_{-\infty}^{\infty} \nabla R \cdot \nabla G d\tau$$

In general, both R and G are positive, attain their maximum values at or near the point (x, y), and vanish as η goes to zero or W and as τ approaches $\pm\infty$. Thus, $F(y, \eta)$ is generally positive, as shown in Fig. 3. Like $R(\tau, y, \eta)$, $F(y, \eta)$ depends very little on the details of the flow and, even less than R, depends on the scale of the eddies.

An important property of F is derivable from Eq. (20), which, together with the definition of F, implies that

$$\int_0^W F(y, \eta) d\eta = 1$$

for all y. We may, therefore, regard the expression

$$U^*(y) = \int_0^W U(\eta) F(y, \eta) d\eta$$

as a *normalized* weighted average of U, in which the weighting factor is $F(y, \eta)$. As shown in Fig. 3, $F(y, \eta)$ does not vary appreciably with η when y lies near $W/2$. Thus, when y is near $W/2$, the weighted average $U^*(y)$ is very nearly the unweighted average

$$\frac{1}{W} \int_0^W U(\eta) d\eta$$

which, by the definition of U, vanishes. To see the consequences of this property, let us consider the behavior of U^* in the case of a very narrow, strong "jet" in the mean zonal flow, located near the middle of the channel and superimposed on a roughly *uniform* mean zonal flow. In this case, the mean zonal wind profile is characterized by large positive values of U in a narrow band around $y = W/2$, and by small negative values over two rather broad regions adjacent to the walls. Now, as we have already seen, $U^*(y)$ is very small when y lies near $W/2$—certainly much smaller in magnitude than the *unaveraged* values of U

near $y = W/2$. But $U^*(y)$ is also small elsewhere, simply because U is small in the broad regions of uniform mean zonal flow, and because least relative weight is attached to conditions in the middle of the channel when y is near zero or W. Thus, if the mean zonal wind profile is of the general type described above, U^* is negligible in comparison with U for purposes of solving Eq. (21). This conclusion will be verified by the results of a later section.

A further consequence: not only does $F(y, \eta)$ depend very little on the details of the flow (or even on the overall scale of the eddies), but the solution of Eq. (21) is still *less* affected by the small extent to which F *does* depend on scale.

The quantity $\lambda(y)$ is defined by the equation

$$\frac{1}{\lambda^2} = -\frac{\overline{v\nabla^2 v}}{\overline{v^2}}$$

Thus, λ has the units of a length, and is essentially a statistical measure of the scale or typical dimension of the eddies. Owing to the previously discussed relation between v and $\nabla^2 v$ in large-scale turbulence, λ has the properties of an autocorrelation function, and depends very little on the details of the v-field. For the same reason, moreover, one might expect that $\lambda(y)$ would vary only slowly across the width of the channel.[1]

Since the general behavior of solutions of Eq. (21) depends critically on λ, and because both R and F depend on the scale of the eddies to some extent, it is natural to ask if the *scale* of the turbulence can be specified or predicted with sufficient accuracy. Judging from the manner in which λ enters Eq. (21), the relevant question concerns the comparative magnitudes of the percentage time variations of scale and the maximum percentage variations of U. In order to estimate variations of scale, we shall consider another and equally legitimate statistical measure of the characteristic dimension of the eddies, denoted by S.[2]

$$S = \frac{V}{K}$$

where

$$V = \int_0^W \overline{(\nabla^2\psi)^2}\, dy \qquad K = \int_0^W \overline{\nabla\psi \cdot \nabla\psi}\, dy$$

Thus, S is essentially the inverse of the square of the characteristic scale. Differentiating S with respect to t, we find that

$$\frac{\partial S}{\partial t} = \frac{1}{K}\left(\frac{\partial V}{\partial t} - S\frac{\partial K}{\partial t}\right) \tag{22}$$

The next concern is to express the changes of V and K—which are the total variance of eddy vorticity and total eddy kinetic energy, respectively—in terms of changes in U. Turning to the former, we first multiply Eq. (8) by ζ and apply the bar-operator. Simultaneously making use of Eq. (5) to eliminate $\overline{v\zeta}$,

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\zeta^2} + \frac{1}{2}\frac{\partial}{\partial y}\overline{v\zeta^2} + \beta\frac{\partial \bar{u}}{\partial t} = 0$$

Next, integrating the equation above with respect to y from $y = 0$ to $y = W$, noting that v vanishes at the walls, and recalling that the total zonal momentum is conserved,

$$\frac{\partial}{\partial t}\int_0^W \overline{\zeta^2}\, dy = 0$$

Finally, we decompose ζ into the eddy vorticity and mean vorticity, with the result that

$$\frac{\partial}{\partial t}\int_0^W \overline{(\nabla^2\psi)^2}\, dy = -\frac{\partial}{\partial t}\int_0^W \left(\frac{\partial U}{\partial y}\right)^2 dy \tag{23}$$

The change in eddy kinetic energy may be calculated by multiplying Eq. (1) by u, multiplying Eq. (2) by v, and adding one equation to the other. At the same time applying the bar-operator,

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\mathbf{V}\cdot\mathbf{V}} + \frac{1}{2}\frac{\partial}{\partial y}\overline{v\mathbf{V}\cdot\mathbf{V}} + \frac{\partial}{\partial y}\overline{Pv} = 0$$

Thus, integrating this equation with respect to y from $y = 0$ to $y = W$ (where v vanishes),

$$\frac{\partial}{\partial t}\int_0^W \overline{\mathbf{V}\cdot\mathbf{V}}\, dy = 0$$

[1] In a fairly typical case, λ was found to vary by no more than 15 pct of its average value, taken across the entire width of the flow.

[2] For the present purpose, it is sufficient to deal with a "lump" measure of scale, since λ does not vary radically with y.

This equation, of course, expresses the well-known fact that the total kinetic energy of an enclosed barotropic fluid is invariant. Finally, splitting up the total kinetic energy into eddy kinetic energy and kinetic energy of the mean zonal flow,

$$\frac{\partial}{\partial t}\int_0^W \overline{\nabla\psi\cdot\nabla\psi}\,dy = -\frac{\partial}{\partial t}\int_0^W U^2 dy \qquad (24)$$

Substituting from Eqs. (23) and (24) into Eq. (22), integrating by parts, and noting that $\partial U/\partial t$ vanishes at $y = 0$ and $y = W$,

$$\frac{\partial S}{\partial t} = \frac{2}{K}\int_0^W \frac{\partial U}{\partial t}\left(\frac{\partial^2 U}{\partial y^2} + SU\right) dy \qquad (25)$$

The formula above is a general relationship between variations in the overall scale of the eddies and changes in the mean zonal flow.

The question as to the importance of variations of scale may now be dealt with by stating Eq. (25) as an inequality.

$$\left|\frac{\partial S}{\partial t}\right| < \frac{2}{K}\int_0^W \left|\left(\frac{\partial U}{\partial t}\right)\right| \left|\frac{\partial^2 U}{\partial y^2} + SU\right| dy$$

whence

$$\left|\frac{\partial S}{\partial t}\right| << \frac{2}{K}\left|\frac{\partial U}{\partial t}\right|_{\max}\int_0^W \left|\frac{\partial^2 U}{\partial y^2} + SU\right| dy$$

in which a repeated inequality sign indicates the number of intervening "links" in the chain of inequalities. Now, where U reaches a real extreme, its sign is necessarily opposite to that of $\partial^2 U/\partial y^2$. Thus, since $\partial^2 U/\partial y^2$ and SU are of the same order of magnitude,

$$\left|\frac{\partial S}{\partial t}\right| <<< \frac{2}{K}\left|\frac{\partial U}{\partial t}\right|_{\max}\cdot S\int_0^W |U|\, dy$$

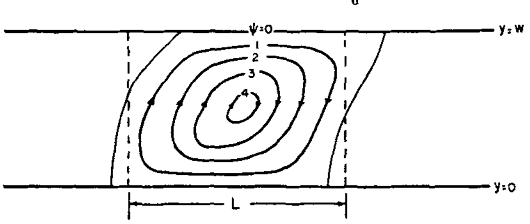

Fig. 4. Schematic pattern of the eddy streamfunction ψ in a large scale eddy. The eddy velocity is directed along the eddy streamlines, and is proportional to the gradient of the eddy streamfunction.

In well-developed turbulence, moreover, the average eddy speed is fully as great as the maximum value of $|U|$, so that

$$\frac{1}{K}\int_0^W |U|\, dy < \frac{1}{|U|_{\max}}$$

and finally,

$$\frac{1}{S}\left|\frac{\partial S}{\partial t}\right| <<<< \frac{1}{|U|_{\max}}\left|\frac{\partial U}{\partial t}\right|_{\max}$$

Despite its semi-quantitative character, the argument above shows that variations in the scale of the eddies are probably much smaller (taken percentagewise) than the maximum variations of U. This conclusion is amply supported, in fact, by the observed behavior of the atmosphere; changes in the overall scale of the transient disturbances are observed to be of the order of tens of percent over the course of a week, whereas the deviation of the mean zonal wind from its grand average may vary by 100 percent or even change sign. Thus, one might be justified in assuming that R and F (which depend primarily on the scale of the eddies) and λ are independent of time—a hypothesis that will be explored in later sections.

It remains to discuss $\overline{v^2}(y)$, a statistical measure of the intensity of turbulence. Since $\overline{v^2}$ is an autocorrelation function, one should expect that its general form would depend very little on details of the flow. Moreover, because $\overline{v^2}$ must vanish at the walls, geometrical constraints alone demand that $\overline{v^2}$ reach its maximum somewhere near the middle of the channel at all times. This, in fact, is borne out by the data published by RIEHL AND COLLABORATORS (1950).

The time variations of the intensity of turbulence may be investigated by considering the total kinetic energy of the meridional motion. Owing to the geometry of very large scale eddies (see Fig. 4), in which there is only one pronounced maximum or minimum of the eddy streamfunction in any section across the channel, the kinetic energy of zonal eddy motion is related to the kinetic energy of the meridional motion by a factor depending on the relative dimensions of the eddies in the x and y directions. That is, approximately,

$$\int_0^W \overline{u'^2} \cdot dy = \frac{L^2}{W^2} \int_0^W \overline{v'^2}\, dy$$

in which L is the typical dimension of the eddies in the x-direction. Thus, the total eddy kinetic energy is

$$\int_0^W \overline{\nabla \psi \cdot \nabla \psi}\, dy = \left(1 + \frac{L^2}{W^2}\right) \int_0^W \overline{v^2}\, dy$$

Now, the connection between the eddy kinetic energy and the kinetic energy of the mean zonal flow may be found by integrating Eq. (24) with respect to time.

$$\int_0^W U^2 dy + \int_0^W \overline{\nabla \psi \cdot \nabla \psi} dy = \int_0^W U_0^2 dy + \int_0^W \overline{\nabla \psi_0 \cdot \nabla \psi_0}\, dy$$

where the subscript "zero" denotes conditions at some fixed "initial" time. Finally, combining the two preceding equations,

$$\int_0^W \overline{v^2} dy = \int_0^W \overline{v_0^2} dy + \frac{W^2}{W^2 + L^2} \int_0^W (U_0^2 - U^2)\, dy \tag{27}$$

This formula expresses the total kinetic energy of the meridional motion at any time in terms of U at that same time, and automatically takes account of the decrease (or increase) in the intensity of turbulence as the kinetic energy of the mean zonal flow increases (or decreases).

Having related the time variations of both the *scale* and *intensity* of turbulence to changes in the mean zonal flow, we now possess the main ingredients of a theory of large-scale mean motions. At this point, however, one should note that the eddy statistics $\overline{v^2}$, λ, R and F are not actually independent, and not all hypotheses lead to physically consistent models —e.g., models in which the total zonal momentum is conserved. Let us now inquire whether or not Eq. (21) implies conservation of total zonal momentum at all times, if $\overline{v^2}$, λ, R, and F are specified independently and arbitrarily, but U is allowed to vary according to Eq. (21). The relevant fact is that Eq. (21) is derivable from Eq. (12) by *identities only*, with the consistent substitution of the solution of Eq. (13)[1] for $\partial \psi / \partial t$. Thus, no matter what distribution of U (or $\bar{u}$) prevails at any given time, Eq. (21) must be reducible to the *form* of Eq. (12), *provided the eddy statistics* $\overline{v^2}$, λ, R *and* F *are all computed from the same distribution of* v. But this, in turn, implies that the second derivative of the total zonal momentum vanishes at all times, whence the total zonal momentum is always conserved if it was unchanging initially.

With the assurance that the total zonal momentum will be conserved, we now state the following general hypotheses:

1) The most effective agencies of momentum transport are eddies of very large scale—i.e., the characteristic dimensions of the eddies are comparable with the entire "width" of the flow.

2) The statistical measures of the *scale* and *intensity* of turbulence *at any time* may be identified with the statistics of a v-field whose amplitude is a constant multiple of the amplitude of the *initial* v-field, and whose scale in the x-direction is a constant multiple of the scale of the *same* initial v-field.

In other words, for purposes of computing the eddy statistics $\overline{v^2}$, λ, R and F and for those purposes only, we suppose that

$$v(x, y, t) = k v_0(mx, y, 0)$$

in which $v_0\ (x, y, 0)$ is the *initial* distribution of v, and k and m are at most functions of time. With this assumption, the value of k at any time is determined by Eq. (27). That is,

$$k^2 = 1 + \frac{W^2}{W^2 + L^2} \cdot \frac{\int_0^W (U_0^2 - U^2) dy}{\int_0^W \overline{v_0^2} dy} \tag{27a}$$

The value of the "scaling factor" m at any time is chosen to be consistent with Eq. (25), which relates changes of eddy scale to time variations of the mean zonal flow. Thus, under the hypothesis stated above, Eqs. (21), (25), and (27) comprise a complete system of equations involving the mean zonal wind, the eddy

[1] With E omitted throughout.

scale, and the intensity of turbulence. Accordingly, they provide the basis for a general theory of large-scale mean motions.

It was pointed out earlier that, on both theoretical and observational grounds, it may be legitimate to assume that the scale of the eddies is independent of time, in which case $m=1$ at all times. The next and subsequent sections will deal entirely with this special hypothesis, which appears to embody the essential ingredients of the theory and which has the considerable virtue of simplicity.

5. A Numerical Method for Predicting the Mean Zonal Wind

With the introduction of the special hypothesis just discussed, Eq. (21) takes the form:

$$\frac{\partial^2 U}{\partial t^2} = k^2\overline{v_0^2}\left[\frac{\partial^2 U}{\partial y^2} + \frac{2}{\lambda_0^2}\left(U - \int_0^W U(\eta)F_0(y,\eta)d\eta\right) + \beta\left(1 - \frac{1}{\lambda_0^2}\int_0^W\int_{-\infty}^{\infty} R_0 G d\tau d\eta\right)\right] \tag{28}$$

where $\overline{v_0^2}$, λ_0, R_0, and F_0 are the values of $\overline{v^2}$, λ, R, and F computed from the initial distribution of v, and are regarded as independent of time after the initial moment. The quantity k^2 is related to U by Eq. (27a). Thus, Eq. (28) involves only one dependent variable U, the deviation of the mean zonal wind (at a variable latitude y) from an unchanging grand average taken over all latitudes. Accordingly, Eq. (28) provides the complete theoretical basis for a method of predicting the mean zonal wind profile.

The purpose of this section is to outline a numerical method for solving Eq. (28)—which, since k depends on U, is nonlinear and generally cannot be solved by exact analytic methods. To start with, we assume that the eddy statistics $\overline{v_0^2}$, λ_0, R_0 and F_0 have been computed from the distribution of v at the beginning of the forecast period. We suppose, further, that the initial values of U are known, and that the initial values of $\partial U/\partial t$ have been calculated from the initial flow pattern by means of Eq. (5) or (7). In addition to the initial conditions specified above, it is also necessary to state conditions on U at the walls. These are contained in Eq. (5), which implies that U remains unchanged at $y=0$ and $y=W$.

Beginning with the initial values of U and $\partial U/\partial t$, one can immediately predict U at a time $\triangle t$ later than the initial moment ($t=0$). It is then possible to compute the entire right hand side of Eq. (28) by finite-sums and finite-differences from the predicted values of U at $t=\triangle t$ in the interior of the channel, taken together with the specified values of U at the walls. Thus, Eq. (28) enables us to calculate $\partial^2 U/\partial t^2$ at $t=\triangle t$. But with the values of U known at $t=0$ and $t=\triangle t$, and with $\partial^2 U/\partial t^2$ known at $t=\triangle t$, one can compute the values of U at $t=2\triangle t$. Finally, since the information available at $t=2\triangle t$ and $t=\triangle t$ is as complete as it was at $t=\triangle t$ and $t=0$, it is possible to recompute $\partial^2 U/\partial t^2$ at $t=2\triangle t$, and so to predict U at $t=3\triangle t$, and so on. This scheme is essentially the one by which the computations to be described in the next section were carried out.

6. The Case of No Initial Momentum Transport

In order to study the general behavior of solutions of Eq. (28), the method of integration described in the preceding section was applied to two types of initial mean zonal wind profiles—both distinguished by the presence of a narrow intense jet, located in the middle of the channel in one case, and near one edge of the channel in the other. To make sure that the essential properties of the solutions would not be obscured by a fortuitous choice of initial values of $\partial U/\partial t$, the *initial* rate of change of U was constrained to vanish at all latitudes, by specifying an *initial* eddy velocity field in which the mean eddy transport of zonal momentum vanishes at all latitudes. This does not, of course, imply that there is no momentum transport at *later* times; it does, however, insure that the changes of mean zonal wind (or momentum) will reflect only the intrinsic dynamical properties of the system.

According to Eq. (5), the case of no mean transport of momentum is one in which v and $\nabla^2\psi$ are completely uncorrelated—i.e., in which the fields of v and $\nabla^2\psi$ are exactly 90 degrees out of phase. But this is also the case when v and $\nabla^2 v$ (or $\partial\nabla^2\psi/\partial x$) are almost exactly in phase, in which case v and $\nabla^2 v$ are almost perfectly correlated. We therefore consider an initial eddy velocity field in which $\nabla^2 v_0 = -v_0/\mu^2$, an equation which has the simple solution

$$v_0 = V \sin\frac{\pi y}{W} \sin\frac{\pi x}{L} \tag{29}$$

where V is the maximum meridional wind speed and

$$\frac{1}{\mu^2} = \left(\frac{\pi}{W}\right)^2 + \left(\frac{\pi}{L}\right)^2$$

It is readily verified that the mean zonal momentum transport implied by Eq. (29) vanishes everywhere. The eddy statistics are computed by substituting the expression for v_0 given by Eq. (29) into the definitions of $\overline{v_0^2}$, λ_0, R_0, and F_0. They are:

$$\overline{v_0^2} = \frac{V^2}{2} \sin^2 \frac{\pi y}{W}$$

$$\lambda_0 = \mu$$

$$R_0 = \frac{\sin\frac{\pi\eta}{W}\cos\frac{\pi\tau}{L}}{\sin\frac{\pi y}{W}}$$

$$F_0 \simeq \frac{\pi^2}{2L^2}\frac{\sin\frac{\pi\eta}{W}}{\sin\frac{\pi y}{W}}\cdot g(y,\eta) + \frac{\pi}{W}\frac{\cos\frac{\pi\eta}{W}}{\sin\frac{\pi y}{W}}\cdot\frac{\partial g}{\partial\eta}$$

$$\text{where } g(y,\eta) = \begin{cases} \eta\left(1-\frac{y}{W}\right) & \text{if } \eta < y \\ y\left(1-\frac{\eta}{W}\right) & \text{if } \eta > y \end{cases}$$

$F_0(y, \eta)$ is plotted as a function of η in Fig. 3, for nine values of y.[1] In the numerical integration of Eq. (28), L was taken equal to W, $\beta = 3/W$ day, and $V = .2\ W$/day. Lengths were expressed in units of W, time in units of days, and speeds in units of W/day.

The results of the numerical integration of Eq. (28) for the case of a narrow intense jet in the middle of the channel are summarized in Fig. 5 (a), which shows the meridional profiles

[1] The values of F_0 given here are not exact. Certain approximations have been introduced to facilitate integration.

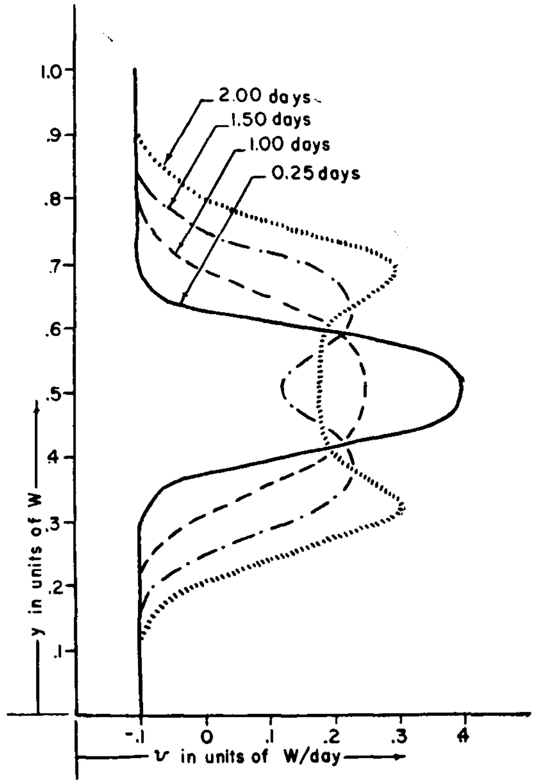

Fig. 5. Time sequences of mean zonal wind profiles for the case of zero initial momentum transport.
(a) based on numerical solution of Eq. (28).

of U at regular intervals of time after $t = 0$. The most striking features of this sequence of mean zonal wind profiles are (1) the weakening and broadening of the jet between $t = 0$ and $t = 1$, (2) the "splitting" of the jet at about $t = 5/4$ and (3) the regular progression of the two jets toward the north and south, each travelling at about .12 W/day. A similar phenomenon is reported by CRESSMAN (1950), who observed that the formation of a double-jet structure is a characteristic and rather sudden denouement from situations of strong zonal flow. Cressman, in fact, states: "This study has suggested that a synoptic pattern consisting of a *single high-level west-wind maximum*, containing long waves of small amplitude, such as is found in the 22 February (1949) chart in Fig. 3, is in itself unstable, and must necessarily break down into a more complicated pattern *by means of a splitting of the maximum.*"[1] Cressman's

[1] The italics are mine. Author.

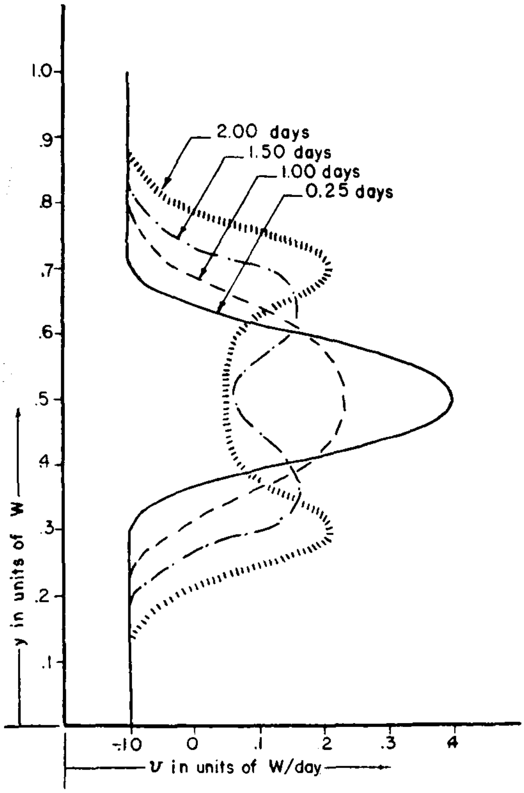

Fig. 5 (b). Same as (a) with β = 0.

conclusions, of course, were based entirely on a descriptive study of the observed behavior of the atmosphere. Since the double-jet structure is typical of well-developed "blocking" situations, it appears that the present theory of mean motions—whose sole physical content is expressed in the equations of barotropic flow—contains a mechanism for the initiation and development of "blocking".

During the course of the numerical integrations, it became apparent that the earth's rotation had very little effect on the changes of the mean zonal wind. To verify this suspicion, the computations summarized in Fig. 5(a) were repeated for conditions that were unchanged in all respects, except that β was set equal to zero. The results are shown in Fig. 5(b), which indicates that the earth's rotation had the main effect of increasing the total zonal momentum (a quantity which should be conserved), but did not significantly alter the general behavior of the mean zonal wind profile. For this reason and because the increase of total zonal momentum was directly traceable to approximations of integration, the effect of the earth's rotation was omitted from all subsequent calculations.

In section 4, during the discussion of the weighting function $F(y, \eta)$, it was pointed out that the quantity

$$U^*(y) = \int_0^W U(\eta) F(y, \eta)\, d\eta$$

is generally negligible in comparison with U, for purposes of integrating Eq. (28). To support this conclusion, the computations summarized in Fig. 5(a) were repeated for conditions that were unchanged in all respects, except that both β and U^* were set equal to zero. The results are shown in Fig. 5(c). Comparing Figs. 5(b) and 5(c), we see that the omission of U^* has little effect on the evolution of the mean zonal wind profile, even to details, and certainly does not alter its general behavior.

The evidence presented in Figs. 5(a), 5(b),

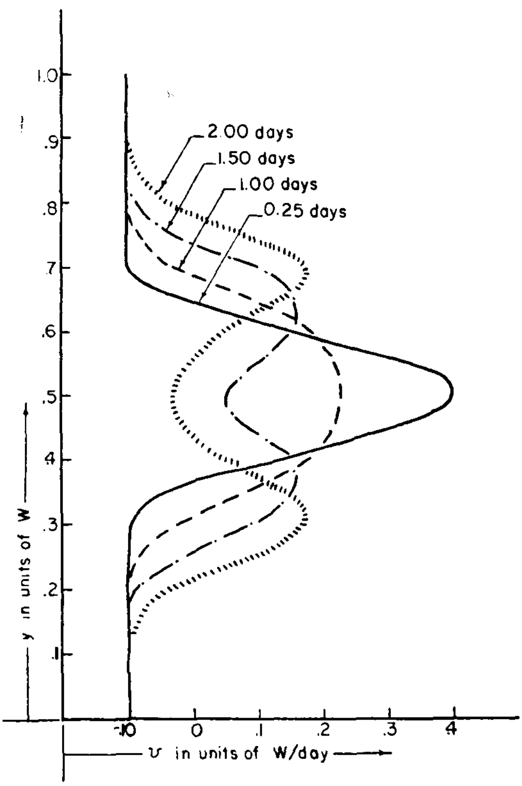

Fig. 5 (c). Same as (a) with β = 0 and U^* = 0.

and 5(c) strongly suggests that, with good approximation, the essential dynamical properties of the system are expressed in the following simplified form of Eq. (28):

$$\frac{\partial^2 U}{\partial t^2} = k^2 \overline{v_0^2} \left[\frac{\partial^2 U}{\partial y^2} + \frac{2U}{\lambda_0^2}\right] \tag{30}$$

This equation is particularly convenient from the standpoint of computation, for it obviates the necessity of evaluating U^* by the method of finite-sums.

The results of the numerical integration of Eq. (28) for the case of a narrow intense jet near one edge of the channel are illustrated in Fig. 6(a), which, as before, shows the meridional profiles of U at regular intervals of time after $t = 0$. The most striking single feature of this sequence of mean zonal wind profiles is the regular northward progression of the maximum mean zonal wind.[1] This feature stands out even more vividly in Fig. 6(b), a latitude-time diagram on which lines of constant U have been drawn. The effect described above is strongly reminiscent of the phenomenon reported by RIEHL ET AL. (1950), who observe that an extreme in the anomaly of mean zonal momentum typically migrates (either northward or southward) in a regular way, sometimes persisting over periods of several weeks. As did CRESSMAN (1949), they also find that distinct extremes of mean zonal momentum may travel in opposite directions. What is significant is that Eq. (28) contains a mechanism whereby extremes of mean zonal wind may be propagated toward the north, toward the south or in both directions at once, depending on the initial conditions.

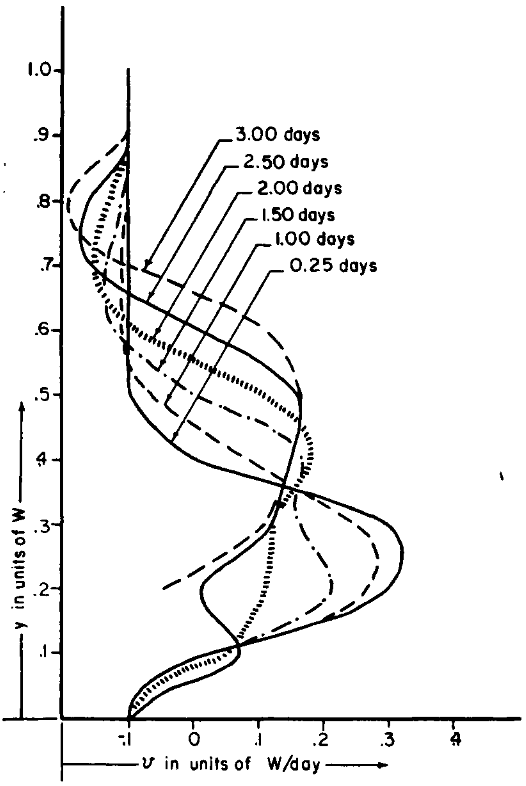

Fig. 6 (a). Time sequence of mean zonal wind profiles for the case of zero initial momentum transport.

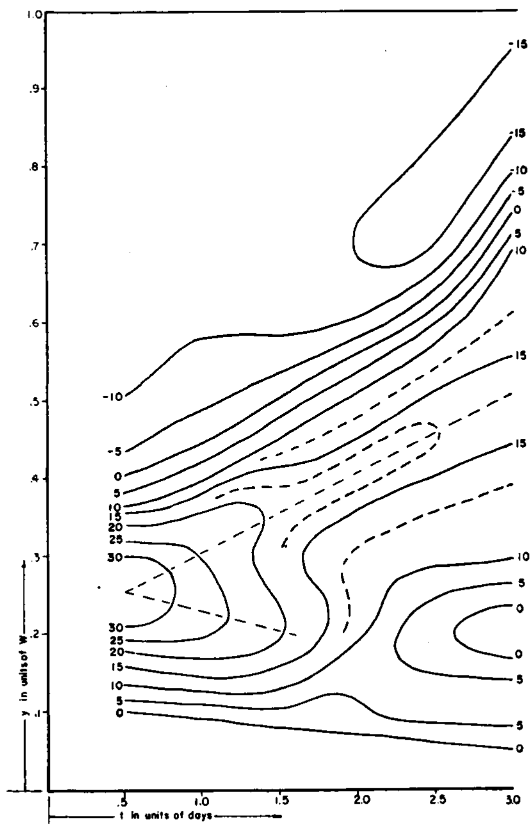

Fig. 6 (b). Lines of constant U (in units of .1 W/day) on a latitude-time diagram.

[1] The rather confused pattern of variations in the immediate vicinity of the southern wall is due to the "reflection" of a *southward* moving maximum, which reverses its sense upon being reflected at the boundary.

7. The General Nature of Long-Period Variations of Mean Zonal Wind in Barotropic Flow

The fundamental reason for the peculiar behavior of solutions of Eq. (28) is apparent when one realizes that Eq. (28) is essentially a "wave-equation"—i.e., a second order equation of hyperbolic type. Eq. (30) is, in fact, reducible to a generalized form of the so-called "telegraphic" equation by transformations of both the dependent and independent variables. It is characteristic of such second order equations that extremes which are present in the initial conditions are propagated in either or both directions in a regular way, and are preserved over long periods of time.

To make this point more concrete, the computations summarized in Fig. 5(a) were repeated for two special cases, in both of which the original conditions were unchanged, except that $\beta = 0$, $U^* = 0$, and $\lambda_0 = \infty$. In the first case, $\overline{v_0{}^2}$ varied from one latitude to another; in the second, $\overline{v_0{}^2}$ was regarded as a constant, equal to the average of the correct values over a narrow strip centered on $y = W/2$. The results of these computations are shown in Figs. 7(a) and 7(b).[1] Comparison of Figs. 7(a) and 7(b) with Figs. 5(a), 5(b), and 5(c) reveals that the most prominent features of the solutions of Eq. (28) also appear in the solutions of the simple equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial y^2} \tag{31}$$

where c is a constant measure of the RMS meridional wind speed. It will be recognized that the form of Eq. (31) is that of the classical wave equation, which also governs the propagation of sound waves in air, or gravity waves on the surface of a shallow body of water.

In the cases of no initial momentum transport studied here, Eq. (31) leads to a very simple geometrical interpretation: The mean zonal wind profile at any time is the superposition of two virtual mean zonal wind profiles which:

1) both have the shape of the initial mean zonal wind profile, but one half its amplitude,
2) are initially in phase, and
3) travel in opposite directions at the speed c, without change in shape.

Qualitatively, at least, this description is in accord with the main features of Figs. 5(a), 5(b), and 5(c).

It remains to see if the speed at which extremes of mean zonal wind actually migrate is in agreement with the speed predicted from theory. Now, as indicated by Figs. 7(a) and 7(b), the speed of migration at a *particular* latitude is not much affected by taking $\overline{v^2}$ to be a constant, equal to the correct value of $\overline{v^2}$ at that latitude. With this simplification, Eq. (30) has wave solutions of the form

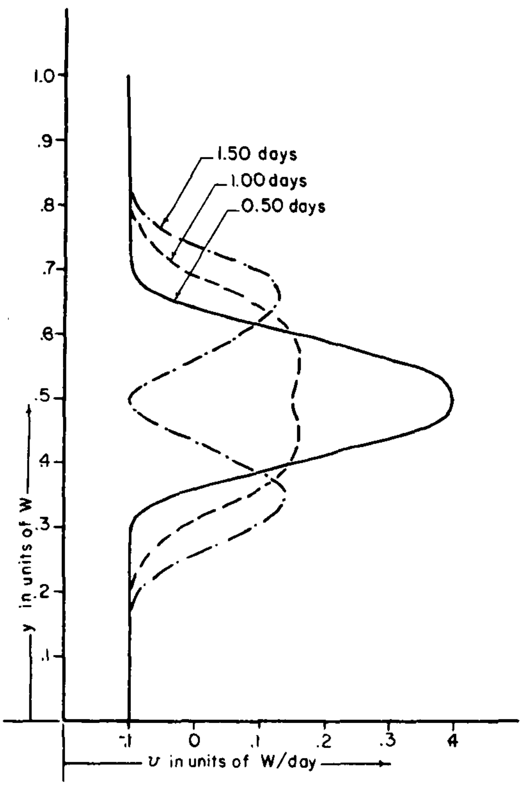

Fig. 7. Sequences of mean zonal wind profiles based on solutions of Eq. (28) with $\beta = 0$, $U^* = 0$, and $\lambda_0 = \infty$. (a) $\overline{v^2}$ variable

[1] The solution shown by the solid curves on Fig. 7 (b) was computed by an exact analytic method. The circled points are values of U for $t = 1.5$, computed by *numerical* integration. Since those points fall on the exact curve for $t = 1.5$, the numerical solutions of Eq. (28) are evidently not subject to appreciable truncation and roundoff errors.

$$U = e^{\frac{i\pi}{w}(y - ct)}$$

in which c, the characteristic speed of migration, is given by

$$c = \pm \sqrt{\overline{v^2}\left[1 - 2\left(\frac{w^2}{L^2} + \frac{w^2}{W^2}\right)\right]} \qquad (32)$$

and w is a typical "half wavelength" of latitudinal variations in U.

One of the few cases for which published data are complete enough to make a rough estimate of c is that of 14 October 1945, which has been discussed by both CRESSMAN (1950) and RIEHL ET AL. (1950). Judging from Riehl's Figure 18, the "half wavelength" of variation in U at 300 mb was about 20° latitude, the effective width of the "channel" (north-south extent of the major disturbances) was about 60° latitude, and the typical zonal dimension of the large-scale "eddies" was about 45° longitude at latitude 35° N. According to Riehl's Figure 5, the zonal average of the absolute value of the meridional wind at 300 mb was about 11 m/sec at latitude 35° N. Substituting these values for w, W, L and $\overline{|v|}$ in Eq. (32), we find that c is about 220 miles/day. The actual speed of migration may be estimated from Fig. 5 of CRESSMAN (1950), which shows that the main jet drifted southward about 20° latitude in a seven day period centered on 14 October 1945. This displacement corresponds to a speed of about 197 miles/day. Owing to the crudity of these estimates, one should probably not be impressed by the *closeness* of agreement between the predicted and actual speeds of migration. It is certainly safe, however, to say that the predicted speed of migration is of the correct order of magnitude.

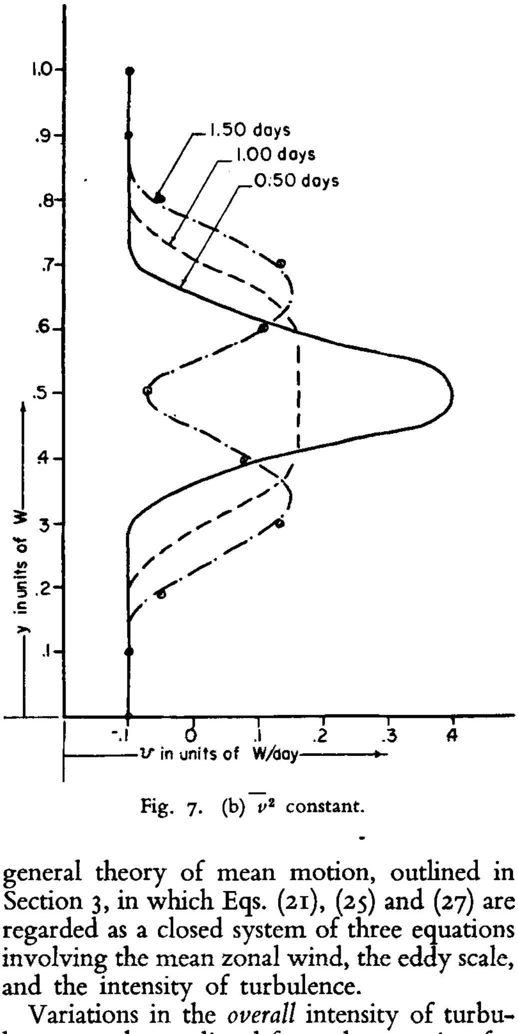

Fig. 7. (b) $\overline{v^2}$ constant.

8. Prediction of the Scale and Intensity of Turbulence

Once the sequence of mean zonal wind profiles has been tentatively predicted from Eq. (28), it is a simple matter to predict changes in some of the eddy statistics. Variations in the overall scale of the eddies, for example, can be computed from the predicted variations of U by means of Eq. (25). This, in fact, leads to the general theory of mean motion, outlined in Section 3, in which Eqs. (21), (25) and (27) are regarded as a closed system of three equations involving the mean zonal wind, the eddy scale, and the intensity of turbulence.

Variations in the *overall* intensity of turbulence may be predicted from the equation for conservation of total kinetic energy, which relates the eddy kinetic energy to the predicted kinetic energy of the mean zonal flow. The changes in another and more informative index of the intensity of turbulence may also be expressed in terms of the variations of mean zonal wind. Multiplying Eq. (9) by ζ', and applying the bar operator,

$$\frac{\partial}{\partial t}\overline{\zeta'^2} + 2\left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2}\right)\overline{v'\zeta'} + \frac{\partial}{\partial y}\overline{v'\zeta'^2} = 0$$

It was pointed out in section 3, during discussion of the auto-interaction of eddies, that correlations between one deviation field and the square of another tend to be uniformly small. Thus, simultaneously eliminating $\overline{v'\zeta'}$ by means of Eq. (5), we may rewrite the equation above as

$$\frac{\partial}{\partial t}\overline{\zeta'^2} + 2\left(\beta - \frac{\partial^2 U}{\partial y^2}\right)\frac{\partial U}{\partial t} = 0 \qquad (33)$$

This equation relates changes in the variance of eddy vorticity *at any latitude* to the changes in the mean zonal wind at that latitude. Accordingly, Eq. (33) provides the basis for a method of predicting the *latitude of maximum eddy circulation*.

Interpreted by an experienced synoptic meteorologist, a time sequence of zonal wind profiles—together with accurate predictions of eddy scale, eddy kinetic energy, and latitude of maximum eddy circulation—would probably be of considerable use in forecasting changes of regime, periods of widespread cyclonic activity, and perhaps shifts in the major storm tracks. Thus, if the present theory can be extended to account for long-period velocity variations in the atmosphere, it may prove to be a practical aid in extended-range forecasting.

9. Summary and Conclusions

The mathematical development of the present theory of mean motions hinges on two fundamental equations—one of which relates changes in the mean zonal flow to the eddy velocity, and the second of which states how the changes of eddy velocity, in turn, are related to the mean flow. By combining these equations properly, it has been found possible to derive an equation that expresses the time variations of the mean zonal wind in terms of the mean zonal wind itself and certain statistics of the eddy velocity. If the characteristic dimension or scale of the eddies is comparable with the "width" of the flow, those eddy statistics take on the properties of autocorrelation functions and depend very little on the details of the flow pattern. One of these statistics is essentially a measure of the intensity of turbulence, another is a measure of the eddy scale, and the remainder depend primarily (but not critically) on eddy scale.

A brief theoretical investigation indicates that percentage variations of eddy scale are much less than the maximum percentage variations of mean zonal wind, a conclusion that is in general agreement with observation. Supported by this argument, we have assumed that those eddy statistics which depend only on eddy scale are invariant and retain their initial values over periods of several days to a week or so. With this hypothesis, the equations of mean motion comprise a complete system that is solvable by numerical methods.

The equation for the mean zonal wind—which has the form of a "wave-equation"—has been integrated numerically for two types of initial mean zonal wind profiles, beginning with no initial momentum transport. The results of those computations show that, as a result of the intrinsic dynamical properties of the system, extremes of the mean zonal wind maintain their identity over considerable periods of time, and migrate either northward or southward in a very regular way, the direction of motion depending on initial conditions. This feature is strongly reminiscent of the regular long period variations of mean zonal momentum reported by RIEHL ET AL. (1950). The computations showed, further, that a narrow intense "jet" near the middle of the initial profile of the mean zonal wind tends to "split" into two jets of lesser intensity, which then rapidly move apart to form the "double-jet" structure typical of widespread "blocking". A similar phenomenon in the atmosphere has been described in highly evocative terms by CRESSMAN (1950). Since the sole physical content of the present theory is expressed in the equations for barotropic flow, we tentatively conclude that *a mechanism for the initiation and development of blocking is contained in the theory of barotropic flow*.

A rather crude theoretical estimate of the "migration speed" of a pronounced mean zonal wind maximum was made from hemispheric data for 14 October 1945, previously published by RIEHL (1950). The theoretical estimate of 220 miles per day is to be compared with an observed speed of 197 miles per day, taken from independent data published by CRESSMAN (1950). One may conclude, at the least, that the theoretical speed of migration is of the correct order of magnitude.

In summary, the theory of mean motion outlined in the foregoing sections of this paper appears to describe—qualitatively and semiquantitatively—the most striking and regular features of long-period velocity variations in the atmosphere that have yet come to light.

The type of mean velocity variations discussed above has a characteristic time-scale of the order of a week or two, rather than a day or two. Thus, one is tempted to speculate that, if the theory of barotropic flow is adequate to describe *those* variations, then it is probably capable of predicting the general features of the large-scale circulation patterns over like periods of time, when taken *in the mean*. If this is indeed the case, the role of purely mechanical processes of the atmosphere (as distinguished from its thermodynamical processes) would be much more important from the standpoint of extended-range forecasting than previous studies of the *unaveraged* equations have indicated.

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