Solitary Waves in the One- and Two-Fluid Systems[1]

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Abstract

A theoretical discussion is given of the solitary wave. Part I is concerned with the wave on a water surface; Part II considers the solitary wave at the interface of two superimposed liquids of different density, bounded above and below by rigid surfaces.

In Part I higher approximations are obtained for the speed of propagation, and limits to the wave speed are derived from the momentum theorem. The results include a correction to the second approximation of Weinstein.

In Part II the first two approximations to the wave speed are derived. The equations of the second approximation yield limits on the wave amplitude.

PART I

THE SOLITARY WATER WAVE

I. Introduction

The solitary wave of elevation on a water surface, shown schematically in Fig. 1, was first studied systematically by RUSSELL (1844). His empirical formula for the speed of propagation, c, is

$$c^2 = g(h + a), \tag{1}$$

were g is the acceleration of gravity, h is the depth of the water at a great distance from the crest, and a is the height of the crest above the level h. This may be written

$$F^2 = 1 + \alpha, \tag{2}$$

where F^2 is the Froude number, c^2/gh, and α is a/h.

Equation (2) was obtained theoretically by BOUSSINESQ (1871) and RAYLEIGH (1876) as a first approximation. Later MCCOWAN (1891) in an attempt to improve the approximation, derived the following expressions defining the speed:

$$F^2 = \frac{\tan\beta}{\beta},$$

$$\alpha = \frac{2}{3\beta}\sin^2\beta\tan\frac{1}{2}\beta(1+\alpha). \tag{3}$$

It is easily verified that this agrees with the Boussinesq-Rayleigh approximation to the order α. In the same paper, however, McCowan chose as a "much closer approximation",

$$F^2 = \frac{\tan\beta}{\beta},$$

$$\alpha = \frac{2}{3\beta}\sin^2\beta\left(1+\frac{2}{3}\alpha\right)\tan\frac{1}{2}\beta(1+\alpha) \tag{4}$$

To the order α^2, this yields $F^2 = 1 + \alpha - 23/60\alpha^2$.

It is clear from McCowan's method that his coefficient of α^2 is too large in absolute value. However, WEINSTEIN (1926) obtained $F^2 = 1 + \alpha - 21/20\alpha^2$, using methods rigorous to the order α^2. In addition to the conflict with McCowan's result, the Weinstein formula does

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not agree with the results of the present paper. These two considerations led the author to the discovery of a numerical error in the last step of Weinstein's development. His corrected result is

$$F^2 = 1 + \alpha - \frac{1}{20}\alpha^2 + o(\alpha^3). \qquad (5)$$

Numerous other studies have been made of the solitary wave. McCowan (1894) investi-

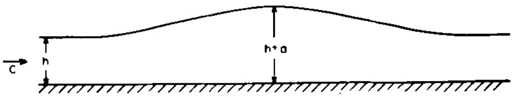

Fig. 1. Schematic drawing of solitary wave

gated the highest wave possible. Assuming a solution similar to that in his first paper, he concluded that the maximum wave would occur for β approximately equal to 1 in $F^2 = \tan\beta/\beta$. This yields $F^2 = 1.56$ and, from the energy equation, $\alpha = 0.78$.

More recently Packham (1952), using a technique developed by Davies (1951), obtained an approximate formula for the wave speed similar in form to that of McCowan. Finally Keller (1954) and Friedrichs and Hyers (1954) developed systematic approaches for first and higher approximations to the solution. In the latter paper, the existence of the wave for sufficiently small α was established using methods differing from those of Lavrentieff[1] who is credited with the earliest proof of existence.

The purpose of the present paper is to derive higher approximations to the solitary wave. By direct integration F^2 is obtained as a power series in α as far as the α^4 term. An application of the momentum theorem yields one additional term. The total result is

$$F^2 = 1 + \alpha - \frac{1}{20}\alpha^2 - \frac{3}{70}\alpha^3 - \frac{6}{175}\alpha^4 - \frac{2\,427}{77\,000}\alpha^5 + o(\alpha^6). \qquad (6)$$

The momentum theorem is further employed to obtain upper and lower limits on F^2 for values of α up to that of the maximum wave.

[1] See Friedrichs and Hyers (1954).

2. Higher approximations to the solitary wave

We assume the existence of a wave of permanent form with a single crest (Fig. 1). The height of the crest above the horizontal bottom $y = 0$ is $h + a$. The free surface lowers symmetrically on both sides of the crest to the height $y = h$ at $|x| = \infty$. The fluid motion is two-dimensional, irrotational and frictionless. The density is constant in time and uniform in space.

If we translate the axes of our coordinate system with the wave, the flow becomes steady, the wave motion superimposed on a basic current, c. The dynamic equation at the free surface is

$$q_s^2 = c^2 - 2g(y_s - h), \qquad (7)$$

where q is the fluid speed and "s" denotes quantities on the free surface. In view of the irrotationality and incompressibility, a velocity potential $\varphi(x, y)$ and a streamfunction $\psi(x, y)$ exist such that $\varphi + i\psi$ is an analytic function of the complex variable $x + iy$ and vice versa. Therefore

$$[c^2 - 2g(y_s - h)]\frac{\partial(x, y)}{\partial(\varphi, \psi)_s} = 1, \qquad (8)$$

the suffix "s" denoting derivatives at the free surface. Defining nondimensional quantities,

$$p \equiv \frac{1}{F^2} \equiv \frac{gh}{c^2}, \; \frac{x}{h} \equiv x', \; \frac{y}{h} \equiv y', \; \frac{\varphi}{ch} \equiv \varphi', \; \frac{\psi}{ch} \equiv \psi', \qquad (9)$$

we may write (8) as

$$[1 - 2p(y'_s - 1)]\frac{\partial(x', y')}{\partial(\varphi', \psi')_s} = 1. \qquad (10)$$

We now assume solutions of the form used originally by Rayleigh (1876):

$$x' = F - \frac{F^{(2)}\psi'^2}{2!} + \frac{F^{(4)}\psi'^4}{4!} - \ldots, \qquad (11)$$

$$y' = F^{(1)}\psi' - \frac{F^{(3)}\psi'^3}{3!} + \frac{F^{(5)}\psi'^5}{5!} - \ldots. \qquad (12)$$

F is a function of φ' alone, and $F^{(n)}$ is its n^{th} derivative. Choosing $\psi' = 0$ at the bottom of the channel $(y' = 0)$, we have $\psi' = -1$ at the free surface and we satisfy the kinematic conditions at both surfaces. Defining $F^{(1)} \equiv -1 - \eta(\varphi)$, at the free surface,

$$Dx'_s = -1 - \cos D \cdot \eta = -1 - \eta + \frac{\eta^{(2)}}{2!} - \frac{\eta^{(4)}}{4!} + \ldots, \qquad (13)$$

$$y'_s = 1 + \sin D \cdot Q\eta = 1 + \eta - \frac{\eta^{(2)}}{3!} + \frac{\eta^{(4)}}{5!} - \ldots, \qquad (14)$$

where D denotes the operator of differentiation with respect to φ', and $Q\eta$ is the indefinite integral of η. Setting $\varepsilon \equiv y'_s - 1$ we find from equation (10)

$$(1 - 2p\varepsilon)\left[(1 + \cos D \cdot \eta)^2 + (D\varepsilon)^2\right] = 1. \quad (15)$$

Using (14),

$$\cos D \cdot \eta = D \cot D \cdot \varepsilon = \varepsilon - \frac{\varepsilon^{(2)}}{3} - \frac{\varepsilon^{(4)}}{45} - \frac{2}{945}\varepsilon^{(6)} - \ldots - \frac{B_{2n-1}\, 2^{2n}\varepsilon^{(2n)}}{(2n)!} \ldots, \qquad (16)$$

where B_{2n-1} are the Bernoulli numbers. Equation (15) becomes

$$(1 - 2p\varepsilon)\left[(1 + D \cot D \cdot \varepsilon)^2 + (D\varepsilon)^2\right] = 1. \quad (17)$$

This equation may be solved by successive approximations by observing that ε is a quantity of the order α (it is equal to α at the crest where $x=0$), and assuming that the $2m^{th}$ derivative, $\varepsilon^{(2m)}$, is $o(\alpha^{m+1})$. It is evident *a posteriori* that such a solution exists provided the successive approximations converge. Assuming this we begin by retaining in (17) terms of the order α^2, i.e.,

$$2\varepsilon(1-p) + \varepsilon^2(1-4p) - \frac{2}{3}\varepsilon^{(2)} = o(\alpha^3). \quad (18)$$

Integrating

$$\frac{(D\varepsilon)^2}{3} = \varepsilon^2(1-p) + \frac{\varepsilon^3}{3}(1-4p) + o(\alpha^4), \quad (19)$$

where the constant of integration vanishes by the condition that $D\varepsilon = \varepsilon = 0$ at $|x| = \infty$. The condition that $D\varepsilon = 0$ at the crest where $\varepsilon = \alpha$ requires

$$\frac{1}{p} = F^2 = 1 + \alpha + o(\alpha^2). \qquad (20)$$

Equation (20) is the Rayleigh-Boussinesq approximation.

To continue the approximation we substitute (20) into (19) to obtain

$$(D\varepsilon)^2 = 3\varepsilon^2\alpha - 3\varepsilon^3 + o(\alpha^4). \qquad (21)$$

It is now useful to express equation (17) by solving for $D \cot D \cdot \varepsilon$ and expanding all terms in series. This yields

$$\varepsilon - \frac{\varepsilon^{(2)}}{3} - \frac{\varepsilon^{(4)}}{45} - \frac{2\varepsilon^{(6)}}{945} - \frac{\varepsilon^{(8)}}{4\,725} = p\varepsilon + \frac{3p^2\varepsilon^2}{2} + \frac{5p^3\varepsilon^3}{2} - \frac{(D\varepsilon)^2}{2} + p\varepsilon\frac{(D\varepsilon)^2}{2} + \frac{35p^4\varepsilon^4}{8} + \frac{(D\varepsilon)^2 p^2\varepsilon^2}{4} + \frac{63p^5\varepsilon^5}{8} + o(\alpha^6), \quad (22)$$

where terms of $o(\alpha^6)$ are omitted. For the next approximation we retain terms of $o(\alpha^3)$ in (22), substituting $p = 1 - \alpha + b_2\alpha^2$.

$$\frac{\varepsilon^{(2)}}{3} = \varepsilon - \frac{\varepsilon^{(4)}}{45} - \varepsilon(1 - \alpha + b_2\alpha^2) - \frac{3}{2}\varepsilon^2(1-2\alpha) - \frac{5}{2}\varepsilon^3 + \frac{(D\varepsilon)^2}{2} + o(\alpha^4), \quad (23)$$

We note that we can substitute for $(D\varepsilon)^2$ in (23) by using (21), and commit only the same error, $o(\alpha^4)$. Furthermore, differentiating (21) we obtain

$$\varepsilon^{(4)} = 9\alpha^2\varepsilon - \frac{135}{2}\alpha\varepsilon^2 + \frac{135}{2}\varepsilon^3 + o(\alpha^4), \quad (24)$$

which may be used to eliminate $\varepsilon^{(4)}$ in (23). We now have a polynomial on the right of (23). Integrating, the condition at the crest yields $b_2 = 21/20$ or

$$\left.\begin{aligned} p &= 1 - \alpha + \frac{21}{20}\alpha^2 + o(\alpha^3), \\ F^2 &= 1 + \alpha - \frac{1}{20}\alpha^2 + o(\alpha^3), \end{aligned}\right\} \qquad (25)$$

in agreement with Weinstein's corrected results.

The above development illustrates completely the approximation procedure. At each stage all derivatives in (22) except $\varepsilon^{(2)}$ may be eliminated by using the results of the previous approximation. The resulting equation may then be integrated by quadratures. Each step is increasingly complicated but the difficulties in proceeding to higher approximations are merely numerical in nature. The method has been extended to obtain the α^3 and α^4 coefficients in the series for p or F^2. The results are contained in the following equations:

$$(D\varepsilon)^2 = \varepsilon^2\left(3\,\alpha - \frac{15}{4}\alpha^2 + \frac{9}{2}\alpha^3 - \frac{7\,629}{1\,400}\alpha^4\right) + \\ + \varepsilon^3\left(-3 + 12\alpha - \frac{126}{5}\alpha^2 + \frac{2\,997}{70}\alpha^3\right) + \\ + \varepsilon^4\left(-\frac{33}{4} + \frac{87}{2}\alpha - \frac{4\,821}{40}\alpha^2\right) + \\ + \varepsilon^5\left(-\frac{114}{5} + \frac{3\,588}{25}\alpha\right) + \\ + \varepsilon^6\left(-\frac{1\,509}{25}\right) + o\,(\alpha^7), \qquad (26)$$

$$p = 1 - \alpha + \frac{21}{20}\alpha^2 - \frac{37}{35}\alpha^3 + \frac{3\,083}{2\,800}\alpha^4 + o\,(\alpha^5), \qquad (27)$$

$$F^2 = 1 + \alpha - \frac{1}{20}\alpha^2 - \frac{3}{70}\alpha^3 - \frac{6}{175}\alpha^4 + o\,(\alpha^5) \qquad (28)$$

All quantities in (27) and (28) are obtained as by-products of the investigation in the next section. This provides a check on the accuracy of these two equations.

We note that the coefficient of ε^2 in equation (26) must be positive. This may be used to derive a maximum value of α. To the order of the Weinstein approximation, the first two terms yield $\alpha_{\max} = .800$. Equating the full coefficient to zero, $\alpha_{\max} = .813$. These values are slightly larger than that of McCowan ($\alpha_{\max} = .78$) and slightly smaller than that of APTÉ (1953) ($\alpha_{\max} = .844$).

3. Use of momentum theorem for additional approximation

The momentum equation is

$$\int_0^{y_s} (p' + \varrho u^2)\, dy = \text{constant}, \qquad (29)$$

where p' is the fluid pressure, u is the horizontal velocity and ϱ the density. The constant in (29) may be evaluated at a great distance from the crest where the pressure is hydrostatic, $y_s = h$ and $u = c$. Then, substituting for p' from the Bernoulli equation,

$$p' + \varrho\frac{(q^2 - c^2)}{2} + \varrho g(y - h) = 0, \qquad (30)$$

and integrating the known terms in (29),

$$\int_0^{y_s} (u^2 - v^2)\, dy = g\,(y_s - h)^2 - 2\,c^2\,(y_s - h) + c^2 y_s. \qquad (31)$$

In particular at the crest $y_s = h + a$, $v = 0$. In addition we may express u along the vertical at the crest as

$$u = \bar{u} + w, \quad \bar{u} = \frac{ch}{h + a} \qquad (32)$$

Hence

$$\int_0^{h+a} w dy = 0, \qquad (33)$$

from continuity considerations. Equation (31) becomes at the crest

$$\int_0^{1+\alpha} \frac{w^2}{c^2} dy' = p\alpha^2 - \frac{\alpha^2}{1 + \alpha}. \qquad (34)$$

We may express the perturbation velocity along the vertical under the crest as

$$\frac{w}{c} = a_0 + a_2\left(\frac{y'}{1 + \alpha}\right)^2 + a_4\left(\frac{y'}{1 + \alpha}\right)^4 + \ldots, \qquad (35)$$

in which a_2, $a_4, \ldots$ are proportional to the 2nd, 4th, ... derivatives of u with respect to y (or x) at the point ($x = 0$, $y = 0$). Consistent with our original assumption regarding derivatives of ε, $a_{2n} = 0\,(\alpha^{n+1})\,(n > 0)$. If we neglect a_4, $a_6 \ldots$ we may compute a_0 and a_2 from the conditions

$$\int_0^{1+\alpha} w dy' = 0 \qquad (36)$$

$$\frac{w_s}{c} = \frac{u_s}{c} - \frac{1}{1 + \alpha} = (1 - 2p\alpha)^{\frac{1}{2}} - \frac{1}{1 + \alpha}, \qquad (37)$$

where w_s and u_s in (37) are evaluated at the point ($x=0$, $y=h+a$). Obviously (37) equals $-\alpha^2/2+o(\alpha^3)$. This and (36) lead to

$$\left.\begin{aligned} a_0 &= \frac{\alpha^2}{4}+o(\alpha^3),\\ a_2 &= -\frac{3}{4}\alpha^2+o(\alpha^3). \end{aligned}\right\} \tag{38}$$

Substituting in (34) we find

$$p = 1-\alpha+\frac{21}{20}\alpha^2+o(\alpha^3). \tag{39}$$

This constitutes a simple independent derivation of the Weinstein approximation.

We will now show that (34) and (35) may be used to derive one higher approximation for p or F^2 than was obtained by the successive approximation method of the previous section. Thus we have from (11)

$$1+\eta_0 = -\frac{dx'}{d\varphi'}(y'=0), \tag{40}$$

or

$$\frac{u_0}{c} = \frac{1}{1+\eta_0} = a_0+\frac{1}{1+\alpha} \tag{41}$$

where u_0 is the velocity at ($x=0$, $y=0$) and η_0 is evaluated at $x=\varphi'=0$. Higher derivatives of (41) at ($x=0$, $y=0$) are

$$\frac{\partial^2}{\partial x'^2}\left(\frac{u}{c}\right) = -\frac{d^2}{dy'^2}\left(\frac{w}{c}\right) = -\frac{\eta_0^{(2)}}{(1+\eta_0)^4} = -\frac{2a_2}{(1+\alpha)^2}, \tag{42}$$

$$\frac{\partial^4}{\partial x'^4}\left(\frac{u}{c}\right) = \frac{d^4}{dy'^4}\left(\frac{w}{c}\right) = 10\frac{\eta_0^{(2)}\eta_0^{(2)}}{(1+\eta_0)^7}-\frac{\eta_0^{(4)}}{(1+\eta_0)^6} = \frac{4!\,a_4}{(1+\alpha)^4}, \tag{43}$$

The constants a_0, a_2, a_4 in (35) may be computed from (41)–(43) to order α^5 by using the final approximation of the previous section and substituting for η from (14):

$$\left.\begin{aligned} Q\eta &= \operatorname{cosec} D\cdot\varepsilon\\ \eta &= D\operatorname{cosec} D\cdot\varepsilon = \varepsilon+\frac{\varepsilon^{(2)}}{6}+\frac{7}{360}\varepsilon^{(4)}+\\ &+\frac{31}{15\,120}\varepsilon^{(6)}+\frac{127}{604\,000}\varepsilon^{(8)}+o(\alpha^6). \end{aligned}\right\} \tag{44}$$

Equations (36) and (37) then determine a_6 and a_8 to this order. The result is

$$\begin{aligned} \frac{w}{c} &= \frac{\alpha^2}{4}-\frac{3}{10}\alpha^3+\frac{121}{280}\alpha^4-\frac{87}{175}\alpha^5-\\ &-\left(\frac{y'}{1+\alpha}\right)^2\left(\frac{3}{4}\alpha^2-\frac{9}{8}\alpha^3+\frac{111}{64}\alpha^4-\frac{26\,301}{11\,200}\alpha^5\right)\\ &-\left(\frac{y'}{1+\alpha}\right)^4\left(\frac{3}{8}\alpha^3-\frac{27}{32}\alpha^4+\frac{1\,149}{640}\alpha^5\right)-\\ &-\left(\frac{y'}{1+\alpha}\right)^6\left(\frac{51}{320}\alpha^4-\frac{9}{16}\alpha^5\right)-\\ &-\left(\frac{y'}{1+\alpha}\right)^8\left(\frac{279}{4\,480}\alpha^5\right)+o(\alpha^6). \end{aligned} \tag{45}$$

Substituting in (34) we determine p to the order α^5. The method verifies the previous results and adds the α^5 term. The final result is

$$F^2 = 1+\alpha-\frac{1}{20}\alpha^2-\frac{3}{70}\alpha^3-\frac{6}{175}\alpha^4-\frac{2\,427}{77\,000}\alpha^5+o(\alpha^6). \tag{46}$$

4. Further applications of the momentum theorem

The momentum theorem, equation (34) is useful to obtain certain limits on the value of F^2 for any value of α. For example, the left-hand side of (34) is positive. The resulting inequality can be written

$$F^2 < 1+\alpha, \tag{47}$$

an upper limit to F^2 derived originally by Starr (1947). This result is in harmony with the negative coefficients of terms of greater order than in equation (46). (We may guess that the remaining coefficients are all negative.) From equation (7), evaluated at the crest, $F^2 \geq 2\alpha$. Combining this with (47),

$$F^2 < 2,\quad \alpha < 1, \tag{48}$$

also derived by Starr.

Obviously (48) are weak conditions for the speed and height of the maximum wave, since the condition that the integral over w^2/c^2 be positive is a weak condition for a wave of large amplitude. This observation is in accord with independent indications that the maximum α is about .80. Setting $F^2 = 2\alpha_{\max}$ in (46) we obtain $\alpha_{\max} = .891$.

It is of interest to note that (48) holds for the maximum elevation and Froude number of any steady wave disturbance that vanishes as $x \to -\infty$. In particular it may be applied to the so-called "undular jump" observed in open channel flow (ROUSE, 1938) in which a supercritical flow ($F^2 > 1$) with horizontal free surface changes downstream to a series of waves about an increased mean level. Such a system is observed without visible turbulence only when $F^2 < 2$. Above this, the waves break to form a turbulent hydraulic jump. This connection between experiment and theory may not be real, however, for BENJAMIN and LIGHTHILL (1954) believe that the undular type of jump must always involve some energy loss.

A lower limit on the wave speed is obtained in the following manner: we assume the existence of a unique solitary wave for a given value of a. The symmetry of the problem then requires that u be symmetrical about $x = 0$ (the vertical at the crest), v antisymmetrical about $x = 0$, and $v = 0$ at $x = 0$. Again denoting quantities at the free surface by the suffix "s", we have $v_s = q_s \sin\Theta$, where Θ is the angle of the surface to the horizontal. Assuming y_s decreases monotonically from the crest to $|x| = \infty$, $\Theta \geq 0$ left of the crest and $\Theta \leq 0$ right of the crest. Therefore $v_s \geq 0$ in the interval $(-\infty < x \leq 0)$, $v_s \leq 0$ in $(0 \leq x < \infty)$. Since $v = 0$ at the bottom of the channel and along $x = 0$, $v(x, y) \geq 0$ everywhere left of $x = 0$, $v(x, y) \leq 0$ everywhere right of $x = 0$; for $v(x, y)$ is a harmonic function and cannot be a maximum or minimum within the fluid. We have therefore shown that

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \leq 0, \quad \text{at } x = 0. \tag{49}$$

At the free surface

$$\left.\frac{\partial u}{\partial x}\right)_s = q_s^2 \frac{\partial(u, y)}{\partial(\varphi, \psi)_s} = q_s^2 \left[\frac{\partial u}{\partial \varphi}\frac{\partial y}{\partial \psi} - \frac{\partial u}{\partial \psi}\frac{\partial y}{\partial \varphi}\right]_s. \tag{50}$$

Since $u - iv$ and $x + iy$ are functions of $\varphi + i\psi$,

$$\left.\frac{\partial u}{\partial x}\right)_s = q_s^2 \left[\frac{du_s}{d\psi_s}\frac{dx}{d\varphi_s} - \frac{dv_s}{d\varphi_s}\frac{dy_s}{d\varphi_s}\right]. \tag{51}$$

Using $u_s = q_s \cos\Theta$, $v_s = q_s \sin\Theta$ and the pressure equation at the free surface,

$$\left.\frac{\partial u}{\partial x}\right)_s = -\frac{2}{q_s} \sin\Theta \cos\Theta \left(\frac{q_s^2}{r} + \frac{g}{2}\frac{\cos 2\Theta}{\cos\Theta}\right), \tag{52}$$

where r is the radius of curvature of the curve of the free surface, given by

$$\frac{1}{r} = \frac{d\Theta}{ds} \tag{53}$$

In the interval $(-\infty < x < 0)$ the derivative of u with respect to x (52) will be negative if r is positive. If r is negative (in the vicinity of the crest) $(\partial u/\partial x)_s$ will remain negative left of the crest unless $q_s^2/|r| > g \cos 2\Theta / 2 \cos\Theta$. Obviously this cannot occur unless the wave has a large amplitude. However, for the maximum wave, in the vicinity of the crest and along the vertical below the crest, $q^2 = g(h + a - y)$, according to Stokes (LAMB 1932). The radius of curvature of a streamline is given by the vorticity equation

$$\frac{\partial q}{\partial y} = \frac{q}{r} \tag{54}$$

This leads to

$$\frac{q_s^2}{|r|} = \frac{g}{2} \tag{55}$$

Thus the quantity in parenthesis in (52) falls to zero at the crest in the maximum wave. Elsewhere along the surface of the maximum wave $r \geq 0$ and $(\partial u/\partial x)_s$ is negative left of the crest and positive right of the crest. It seems probable that for smaller waves $q_s^2/r + g \cos 2\Theta/2 \cos\Theta$ is everywhere positive. Assuming this

$$\left.\begin{aligned} \left.\frac{\partial u}{\partial x}\right)_s &\leq 0 \text{ in } (-\infty < x \leq 0), \\ \left.\frac{\partial u}{\partial x}\right)_s &\geq 0 \text{ in } (0 \leq x < \infty). \end{aligned}\right\} \tag{56}$$

In addition $\partial u/\partial x = 0$ at $x = 0$ from symmetry considerations, and $\partial u/\partial x = -\partial v/\partial y \leq 0$ at $y = 0$ left of the crest, $\partial u/\partial x \geq 0$ at $y = 0$ right of the crest. The harmonic nature of $\partial u/\partial x$ then insures that $\partial u/\partial x \leq 0$ everywhere left of $x = 0$, $\partial u/\partial x \geq 0$ everywhere right of $x = 0$. Therefore

$$\frac{\partial^2 u}{\partial x^2} \geq 0 \text{ or } \frac{\partial^2 u}{\partial y^2} \leq 0 \text{ at } x = 0. \tag{57}$$

Finally we can also show that the derivative of u with respect to y is actually negative for $(0 < y \leq h+a)$ at $x=0$: Suppose $\partial u/\partial y = 0$ at a point y_1 in this interval. Then, either $\partial u/\partial y \equiv 0$ in the interval $0 \leq y \leq y_1$ or $\partial u/\partial y$ is negative at a point y_0, $(0 < y_0 < y_1)$. If $\partial u/\partial y \equiv 0$ in $0 \leq y \leq y_1$ the derivatives of u with respect to y all vanish at the point (0,0). It is easily shown that this would require $u \equiv$ const, $v \equiv 0$ in the whole field of flow. This is impossible for a finite wave. On the other hand if $\partial u/\partial y < 0$ at y_0, the mean value theorem shows that $\partial^2 u/\partial y^2 > 0$ somewhere in $0 \leq y_0 \leq y_1$, contrary to (57).

These results require a velocity profile of the form shown in Fig. 2. To the order of the Weinstein approximation, equations (35) and (38) define a parabola.

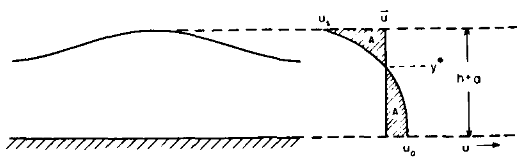

Fig. 2. Form of velocity profile $u(y)$ along vertical at crest.

Referring to Fig. 2 it is evident that

$$\int_{y^*}^{h+a} w^2 dy \leq w_s^2 \int_{y^*}^{h+a} \left(1 - \frac{h+a-y}{h+a-y^*}\right)^2 dy = \frac{w_s^2}{3}(h+a-y^*). \tag{58}$$

Since $y^* \geq (h+a)/2$,

$$\int_{y^*}^{h+a} w^2 dy \leq \frac{w_s^2}{6}(h+a). \tag{59}$$

To compute the remaining contribution to the integral over w^2, we define first

$$\chi \equiv \int_0^y w dy, \quad A \equiv \int_0^{y^*} w dy = \int_0^A d\chi. \tag{60}$$

If w_0 is the perturbation velocity at the point (0,0),

$$\int_0^{y^*} w^2 dy = \int_0^A w d\chi \leq w_0 A, \tag{61}$$

since w_0 is the maximum value of w in the integrand. Furthermore

$$\frac{h+a-y^*}{2}|w_s| > A > \frac{w_0 y^*}{2}, \tag{62}$$

so that

$$\int_0^{y^*} w^2 dy \leq w_0 A \leq \frac{2A^2}{y^*} \leq w_s^2 \frac{(h+a-y^*)}{2y^*} \leq \leq w_s^2 \frac{(h+a)}{4} \tag{63}$$

Combining (59) and (63) we obtain

$$\int_0^{h+a} w^2 dy \leq 5/12\, w_s^2 (h+a). \tag{64}$$

Using equation (34), inequality (64) leads to

$$F^2 \geq \frac{B+C}{D}, \tag{65}$$

where

$$\left.\begin{aligned} B &= 220 + 610\alpha + 764\alpha^2 + 374\alpha^3 \\ C &= 120(1+\alpha)^{3/2}\left(1 - \frac{11}{6}\alpha\right)^{1/2} \\ D &= 340 + 340\alpha + 289\alpha^2 \end{aligned}\right\} \tag{66}$$

For α greater than 6/11 we have $F^2 \geq 2\alpha$. We conclude that for the maximum wave

$$F_m^2 \geq 1.0909, \quad \alpha_m \geq .5454. \tag{67}$$

This is slightly stronger than a result by STARR (1947), $\alpha_m \geq .5000$. Computations based on (65) and (66) are shown in Table I, and the corresponding curve in Fig. 3.

The numerical results of this paper as shown in Table I specify the speed of a solitary wave with extremely little uncertainty for values of α less than .30 and with reasonable accuracy up to .45. Above this the limits leave much to be desired, although further considerations of the general properties of the solitary wave would probably permit considerable improvement. For example, it seems quite likely that an upper limit to the value of the integral in (64) is given by a parabolic velocity profile for $w(y)$. If so, the coefficient in (64) becomes 1/5 instead of 5/12. If this could be proved, a lower limit is obtained up to $\alpha = 5/7$ instead

Table 1.

α	.0000	.0500	.1000	.1500	.2000	.2500	.3000	.3500	.4000
F^2 (Eq. 6)	1.0000	1.0499	1.0985	1.1488	1.1976	1.2461	1.2940	1.3415	1.3880
$F^2 \geq$ (Eq. 66)	1.0000	1.0497	1.0988	1.1471	1.1943	1.2403	1.2845	1.3291	1.3649
$F^2 \leq$ (Eq. 47)	1.0000	1.0500	1.1000	1.1500	1.2000	1.2500	1.3000	1.3500	1.4000
α	.4500	.5000	.5500	.6000	.6500	.7000	.7500	.8000	.8500
F^2 (Eq. 6)	1.4340	1.4790	1.5258	1.5657	1.6073	1.6473	1.6854	1.7217	1.7557
$F^2 \geq$ (Eq. 66.) . . .	1.3984	1.4193	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000
$F^2 \leq$ (Eq. 47) . . .	1.4500	1.5000	1.5500	1.6000	1.6500	1.7000	1.7500	1.8000	1.8500

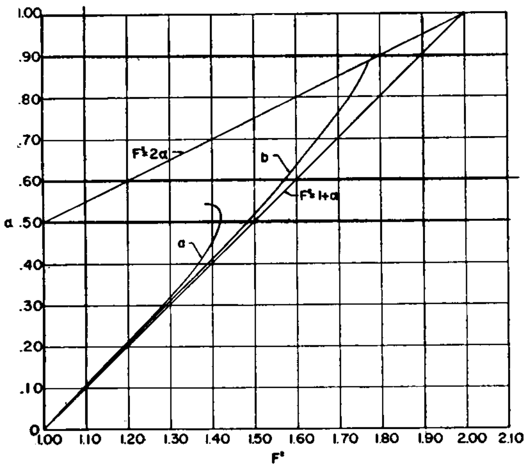

Fig. 3. Curve (a) is the lower limit of the wave speed, (b) is the curve of equation (5).

of 6/11 and the uncertainty in the speed is greatly reduced. Part of the proof may involve showing that all derivatives of w with respect to y are negative at the point (0,0).

Part II

THE SOLITARY WAVE IN A TWO-FLUID SYSTEM

1. Equations of the two-layer model

The mathematical techniques of Part I may be applied to a study of a solitary wave at the interface of two superimposed layers of immiscible liquids of different density. Such waves have been observed in experiments by Keulegan (1953) and Long (1955). Recently Abdullah (1955) has offered some evidence that phenomena of this type may occur in the atmosphere. Keulegan[1] derived the first approximation to the speed and form of the wave, using the approach of Boussinesq (1871). In the present analysis a second approximation is obtained. This permits a discussion of the maximum amplitude of the wave.

The model adopted for this investigation is shown in Fig. 4. The two fluids are in a channel of infinite length, bounded by two rigid planes at $y=0$ and $y=H$. We assume the existence of a wave of permanent form with a single crest (or trough) at the interface of two immiscible superimposed liquids of density ϱ_1 and ϱ_2. The height of the crest above the horizontal bottom is $h+a$. The fluids are incompressible and frictionless.

The dynamic equation at the interface is

$$\varrho_1(q_1^2 - u_1^2) - \varrho_2(q_2^2 - u_2^2) + + 2(\varrho_1 - \varrho_2) g (y_s - h) = 0, \tag{1}$$

where q is the fluid speed, g is gravity, and y_s is the height of the interface above $y=0$. We choose solutions for the velocity potential and streamfunctions similar in form to those of Part I.

$$\varphi_1 = M - M^{(2)}\frac{y^2}{2!} + M^{(4)}\frac{y^4}{4!} - \ldots, \tag{2}$$

$$\psi_1 = M^{(1)}y - M^{(3)}\frac{y^3}{3!} + M^{(5)}\frac{y^5}{5!} - \ldots, \tag{3}$$

$$\varphi_2 = N - N^{(2)}\frac{(y-H)^2}{2!} + N^{(4)}\frac{(y-H)^4}{4!} - \ldots, \tag{4}$$

[1] Keulegan's derivation was for the special case of $\Delta\varrho/\varrho$ small and $u_1 = u_2$.

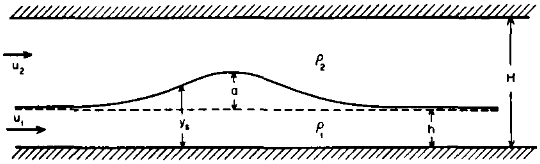

Fig. 4. Theoretical model of a two-fluid system.

$$\psi_2 = N^{(1)}(y-H) - N^{(3)}\frac{(y-H)^3}{3!} + N^{(5)}\frac{(y-H)^5}{5!} - \ldots - u_1 h - u_2(H-h), \quad (5)$$

where M, N are functions of x. These solutions satisfy the kinematic conditions at $y=0$ and $y=H$. The kinematic condition at the interface yields

$$-u_1 h = M^{(1)}y_s - M^{(3)}\frac{y_s^3}{3!} + \ldots, \quad (6)$$

$$u_2(H-h) = N^{(1)}(y_s - H) - N^{(3)}\frac{(y_s-H)^3}{3!} + \ldots. \quad (7)$$

Defining the quantities

$$\frac{y_s}{h} - 1 = \varepsilon, \quad -1 - \frac{M^{(1)}}{u_1} = \eta,$$
$$-1 - \frac{N^{(1)}}{u_2} = \zeta, \quad \frac{H}{h} = r, \quad (8)$$

equations (6) and (7) may be written

$$\eta = -\varepsilon - \eta\varepsilon + \frac{\eta^{(2)}}{3!}(1+\varepsilon)^3 - \frac{\eta^{(4)}}{5!}(1+\varepsilon)^5 + \ldots, \quad (9)$$

$$\zeta(1-r) = -\varepsilon - \zeta\varepsilon + \frac{\zeta^{(2)}}{3!}(1-r+\varepsilon)^3 - \frac{\zeta^{(4)}}{5!}(1-r+\varepsilon)^5 + \ldots, \quad (10)$$

where superscripts now denote differentiation with respect to x/h. We observe that ε is a quantity of order $\alpha \equiv a/h$ (it is equal to α at the crest). We assume that the n^{th} derivative of ε, η and ζ are of order $\alpha^{1+\frac{n}{2}}$. With this assumption we may solve for η and ζ from (9) and (10) by successive approximations:

$$\eta = -\varepsilon + \varepsilon^2 - \varepsilon^3 - \frac{\varepsilon^{(2)}}{6} + \frac{\varepsilon^{(1)2}}{3} - \frac{7}{360}\varepsilon^{(4)} + o(\alpha^4), \quad (11)$$

$$\zeta = -\frac{\varepsilon}{(1-r)} + \frac{\varepsilon^2}{(1-r)^2} - \frac{\varepsilon^3}{(1-r)^3} - \frac{\varepsilon^{(2)}}{6}(1-r) + \frac{\varepsilon^{(1)2}}{3} - \frac{7}{360}\varepsilon^{(4)}(1-r)^3 + o(\alpha^4), \quad (12)$$

where terms of order α^4 are neglected. We now compute q_1^2 and q_2^2 in equation (1) by differentiating equations (2) and (4) and substituting from (8). We obtain for example

$$\frac{q_1^2}{u_1^2} - 1 = 2\eta + \eta^2 - \eta^{(2)}(1+2\varepsilon) + \frac{\eta^{(4)}}{12} - \eta\eta^{(2)} + \eta^{(1)2} + o(\alpha^4) \quad (13)$$

Using (11) we may eliminate η in (13). This procedure yields

$$\frac{q_1^2}{u_1^2} - 1 = -2\varepsilon + 3\varepsilon^2 - 4\varepsilon^3 + \frac{2}{3}\varepsilon^{(2)} - \frac{1}{3}\varepsilon^{(1)2} + \frac{2}{45}\varepsilon^{(4)} - \frac{2}{3}\varepsilon\varepsilon^{(2)} + o(\alpha^4), \quad (14)$$

$$\frac{q_2^2}{u_2^2} - 1 = -\frac{2\varepsilon}{1-r} + \frac{3\varepsilon^2}{(1-r)^2} - \frac{4\varepsilon^3}{(1-r)^3} + \frac{2}{3}\varepsilon^{(2)}(1-r) - \frac{\varepsilon^{(1)2}}{3} + \frac{2}{45}\varepsilon^{(4)}(1-r)^3 - \frac{2}{3}\varepsilon\varepsilon^{(2)} + o(\alpha^4). \quad (15)$$

Substituting in equation (1)

$$\varrho_1 u_1^2\left[-2\varepsilon + 3\varepsilon^2 - 4\varepsilon^3 + \frac{2}{3}\varepsilon^{(2)} - \frac{1}{3}\varepsilon^{(1)2} + \frac{2}{45}\varepsilon^{(4)} - \frac{2}{3}\varepsilon\varepsilon^{(2)}\right] - \varrho_2 u_2^2\left[-\frac{2\varepsilon}{1-r} + \frac{3\varepsilon^2}{(1-r)^2} - \frac{4\varepsilon^3}{(1-r)^3} + \frac{2}{3}\varepsilon^{(2)}(1-r) - \frac{1}{3}\varepsilon^{(1)2} + \frac{2}{45}\varepsilon^{(4)}(1-r)^3 - \frac{2}{3}\varepsilon\varepsilon^{(2)}\right] + 2(\varrho_1 - \varrho_2)gh\varepsilon = o(\alpha^4). \quad (16)$$

2. First approximation

We solve (16) by retaining first, terms of order α^2. The simplified differential equation may be integrated by quadratures by using ε instead of x/h as the independent variable. If we define

$$a_n = \varrho_1 u_1^2 - \varrho_2 u_2^2 (1 - r)^n, \qquad (17)$$

this integral may be written

$$\frac{\varepsilon^{(1)^2}}{3} = \frac{a_{-1}}{a_1}\varepsilon^2 - \frac{a_{-2}}{a_1}\varepsilon^3 - (\varrho_1 - \varrho_2) g \frac{h}{a_1}\varepsilon^2 + o(\alpha^4). \qquad (18)$$

In (18) the constant of integration vanishes by the condition that $\varepsilon^{(1)} = \varepsilon = 0$ at $|x| = \infty$. Defining

$$F_1^2 = \frac{u_1^2}{g\dfrac{(\varrho_1 - \varrho_2)}{\varrho_1}h}, \quad F_2^2 = \frac{u_2^2}{g\dfrac{(\varrho_1 - \varrho_2)}{\varrho_1}h},$$

$$R = \frac{h}{H} = \frac{1}{r}, \quad \frac{\varrho_2}{\varrho_1} = s, \qquad (19)$$

the condition at the crest, $\varepsilon^{(1)} = 0$, $\varepsilon = \alpha$, yields

$$F_1^2(1 - \alpha) + \frac{sR}{1 - R}F_2^2\left(1 + \frac{\alpha R}{1 - R}\right) = 1 + o(\alpha^2). \qquad (20)$$

Since the right hand side of (18) must be positive

$$F_1^2(1 - \varepsilon) + \frac{sR}{1 - R}F_2^2\left(1 + \frac{\varepsilon R}{1 - R}\right) \geq 1. \qquad (21)$$

Subtracting (20) from (21) we find

$$\left.\begin{aligned} \varepsilon > 0 \text{ if } F_1^2 - \frac{sR^2F_2^2}{(1 - R)^2} > 0, \\ \varepsilon < 0 \text{ if } F_1^2 - \frac{sR^2F_2^2}{(1 - R)^2} < 0. \end{aligned}\right\} \qquad (22)$$

These are the criteria for the wave to be one of elevation or of depression. If the density difference is small $s \cong 1$. If, in addition, the wave advances at a speed c on the interface of two fluids at rest at infinity, $u_1 = u_2 = c$ and $F_1 = F_2 = F$. Then, from (20)

$$F^2 = (1 - R)\left[1 + \alpha\frac{(1 - 2R)}{1 - R}\right] + o(\alpha^2). \qquad (23)$$

It this case (22) becomes:

$$\left.\begin{aligned} \varepsilon > 0 \text{ if } R < \frac{1}{2} \\ \varepsilon < 0 \text{ if } R > \frac{1}{2} \end{aligned}\right\} \qquad (24)$$

Relations (23) and (24) were derived by KEULEGAN (1953).

In the general case (22) shows that the existence of a wave of elevation is favored by a large density difference, high velocity in the upper layer in the direction of wave propagation, and by a shallow lower layer. If the density of the upper fluid is zero, only a wave of elevation is possible and the wave speed is given by

$$\frac{c^2}{gh} = 1 + \alpha + o(\alpha^2), \qquad (25)$$

in agreement with the Rayleigh-Boussinesq approximation. Equation (25) also holds if the upper fluid is at rest.

If h is finite and the total depth H of the fluids is infinite, (19) becomes

$$F_1^2 = 1 + \alpha + o(\alpha^2) \qquad (26)$$

Thus a wave on a very shallow lower layer behaves like a water wave in a gravity field $(\varrho_1 - \varrho_2)/\varrho_1$.

Equation (18) may be integrated again to yield the profile of the wave:

$$\varepsilon = \alpha\, sech^2 \frac{x}{2\gamma h}, \qquad (27)$$

$$\gamma^2 = \frac{F_1^2 + sF_2^2\dfrac{(1 - R)}{R}}{3\alpha\left[F_1^2 - \dfrac{sF_2^2R^2}{(1 - R)^2}\right]} \qquad (28)$$

3. Second approximation

To continue the approximation we write (18) as

$$\varepsilon^{(1)^2} = 3\frac{a_{-2}}{a_1}(\varepsilon^2\alpha - \varepsilon^3) + o(\alpha^4). \qquad (29)$$

Differentiating

$$\left.\begin{aligned} \varepsilon^{(2)} &= \frac{3}{2}\frac{a_{-2}}{a_1}\left(2\varepsilon\alpha - 3\varepsilon^2\right) + o\left(\alpha^3\right), \\ \varepsilon^{(4)} &= \\ &= \frac{a_{-2}^2}{a_1^2}\left(9\varepsilon\alpha^2 - \frac{135}{2}\varepsilon^2\alpha + \frac{135}{2}\varepsilon^3\right) + o\left(\alpha^4\right) \end{aligned}\right\} \quad (30)$$

Using (29) and (30) we may eliminate all derivatives in (16) (except where ε^2 occurs alone) and commit only the same error $o(\alpha^4)$. The resulting equation may be integrated by quadratures. This yields

$$a_1 \frac{\varepsilon^{(1)^2}}{3} = a_{-1}\varepsilon^2 - a_{-2}\varepsilon^3 + a_{-3}\varepsilon^4 - \frac{2}{45}a_3\frac{a_{-2}^2}{a_1^2}\left(\frac{9}{2}\varepsilon^2\alpha^2 - \frac{45}{2}\varepsilon^3\alpha + \frac{135}{8}\varepsilon^4\right) + \frac{a_0 a_{-2}}{a_1}\left(\varepsilon^3\alpha - \varepsilon^4\right) - \\ - \left(\varrho_1 - \varrho_2\right) gh\varepsilon^2 + o\left(\alpha^5\right). \quad (31)$$

The condition at the crest yields

$$F_1^2 + \frac{sF_2^2 R}{1-R} - \alpha\left\{F_1^2 - \frac{sF_2^2R^2}{(1-R)^2}\right\} + \\ + \alpha^2\left\{F_1^2 + \frac{sF_2^2R^3}{(1-R)^3} + \frac{1}{20}\left[F_1^2 + \frac{sF_2^2(1-R)^3}{R^3}\right] \frac{\left[F_1^2 - \dfrac{sF_2^2R^2}{(1-R)^2}\right]^2}{\left[F_1^2 + \dfrac{sF_2^2(1-R)}{R}\right]^2}\right\} = 1 + o\left(\alpha^3\right). \quad (32)$$

If $s=0$, and $u_1 = u_2 = c$, we find

$$\frac{c^2}{gh} = 1 + \alpha - \frac{\alpha^2}{20} + o\left(\alpha^3\right), \quad (33)$$

which agrees with the results of Part I.

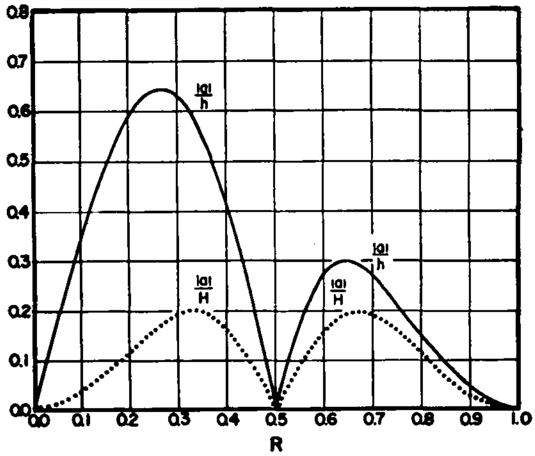

Fig. 5. Limits on wave amplitude.

Limits to the wave amplitude may be obtained by noting that to the present order of approximation the right-hand side of (31) must be positive. This may be written

$$(\alpha - \varepsilon)\left[a_{-2} - a_{-3}(\varepsilon + \alpha) + \frac{1}{4}a_3\frac{a_{-2}^2}{a_1^2}(3\varepsilon - \alpha) + \right. \\ \left. + a_0\frac{a_{-2}}{a_1}\varepsilon\right] \geq 0. \quad (34)$$

In the case $s \cong 1$, $F_1^2 \cong F_2^2$ this leads to

$$|\alpha|_{\max} \leq |\alpha| \leq \frac{4R(1-R)^2|1-2R|}{1-3R+3R^2}. \quad (35)$$

The curves of this inequality are shown in Fig. 5.

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