A Series Solution for the Barotropic Vorticity Equation and its Application in the Study of Atmospheric Vortices

By JULIÁN ADEM, International Meteorological Institute in Stockholm[1]

(Manuscript received April 17, 1956)

Abstract

A method to obtain a series solution for the barotropic non-divergent, vorticity equation is given and is applied to the case of a vortex.

A formula that gives the development of a symmetrical vortex as a function of radius, maximum velocity and latitude is obtained. A numerical example is worked out.

The convergence of the series solution for the general case is discussed and shown to depend on the scale of motion to be forecast and conclusions are drawn regarding time-steps used in a forecast and the smoothing of the initial map.

The possibility of improving a forecast and reducing the number of time-steps used in it by the application of this series solution, follows as a general conclusion.

Finally, a theorem that gives the scaling of the variables in an experimental model of a barotropic flow is obtained.

I. Introduction

In order to describe the behaviour of the complex atmosphere in the form of equations which can be solved in a practical way, it is necessary to search for the *main* mechanisms responsible for the atmospheric changes and formulate a theoretical model, which takes into account these mechanisms.

The barotropic model of the atmosphere is the only model up till now, which has been extensively tested and subjected to a detailed study and the reports about it (BOLIN, 1955, 1956) are encouraging. In this model the primitive equations of motion can be combined into a single forecasting equation, the well known vorticity equation, which was apparently used for the first time in meteorological applications in 1939 by Rossby and his collaborators.

When considering the integration problem from the point of view of practical applications, one is forced to use the finite difference version of the forecasting differential equations. Consequently, although very much use has been made of the vorticity equation, very little, for example, is known about the nature and properties of its solution from the point of view of differential equations.

[1] The author is a member of the Instituto de Geofísica, Universidad Nacional de México and of the Instituto Nacional de la Investigación Científica, Mexico City. He undertook the investigation reported in this paper when he was on leave in Stockholm.

A technique has been developed in this paper by which a forecasting equation or system of equations can be integrated and it has been applied to the specific case of the non-divergent barotropic equation. The solution obtained by this method is a series in the time variable and should be particularly useful when one wishes to obtain approximate analytical solutions. The method should therefore lead to a better understanding of the influence of the different parameters that enter in a forecast.

The method is specially useful in the study of atmospheric vortices. The paper contains an example of its application in the case of a cyclone embedded in a resting atmosphere.

In the last part of this paper a theoretical study of the nature of the solution of the vorticity equation and its dependence on the scale of motion has been presented and some of the results arrived at have been applied to get information of practical value in numerical forecasting.

II. The solution of the Poisson equation

The solution of an equation of the type

$$\nabla^2 z = F(x, y) \tag{1}$$

where ∇^2 is the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ and $F(x, y)$ is a function of two variables x, y is of fundamental importance in numerical forecasting. We shall start by deriving a solution to such an equation in a form that can be useful to obtain some analytical solutions.

Writing (1) in polar coordinates (r, Θ) and considering that the function $F(x, y)$ can be expressed as a sum of terms of the type $F_1(r) \sin n\Theta$ and $F_2(r) \cos n\Theta$ where n is an integer, we shall deal with an equation of the type

$$\nabla^2 z = F_n(r)\begin{Bmatrix}\cos n\Theta \\ \sin n\Theta\end{Bmatrix} \tag{2}$$

A solution to this equation is

$$z = \left[\frac{r^n}{2n}\int r^{-n+1} F_n(r)dr - \frac{r^{-n}}{2n}\int r^{n+1} F_n(r)dr\right]\cdot\begin{Bmatrix}\cos n\Theta \\ \sin n\Theta\end{Bmatrix} \tag{3}$$

as can be easily checked by direct substitution in (2).

III. Series solution for the barotropic non-divergent vorticity equation

Let[1] $u = -\psi_y$ and $v = \psi_x$ where u and v are the components along the x- and y-axis, respectively, of the horizontal wind, the x-axis points to the east and the y-axis to the north. Substituting these values of u and v in the non-divergent barotropic vorticity equation we obtain

$$\nabla^2\psi_t = J(\nabla^2\psi, \psi) - \beta\psi_x \tag{4}$$

where $J(\nabla^2\psi, \psi) \equiv \psi_y(\nabla^2\psi)_x - \psi_x(\nabla^2\psi)_y$ and β is the variation of the Coriolis parameter.

Given the stream function ψ and boundary conditions for the tendency ψ_t, one can compute ψ_t from (4), by solving a Poisson equation.

Differentiating (4) with respect to t one obtains

$$\nabla^2\psi_{tt} = J(\nabla^2\psi_t, \psi) + J(\nabla^2\psi, \psi_t) - \beta\psi_{tx} \tag{5}$$

in which ψ_t can be computed from (4) and which is also a Poisson equation in ψ_{tt}.

Similarly, by successive differentiation with respect to t, we can obtain the equations giving the derivatives of higher order

$$\nabla^2\psi_{ttt} = J(\nabla^2\psi_{tt}, \psi) + 2(\nabla^2\psi_t, \psi_t) + J(\nabla^2\psi, \psi_{tt}) - \beta\psi_{ttx} \tag{6}$$

$$\nabla^2\psi_{tttt} = J(\nabla^2\psi_{ttt}, \psi) + 3J(\nabla^2\psi_{tt}, \psi_t) + 3J(\nabla^2\psi_t, \psi_{tt}) + J(\nabla^2\psi, \psi_{ttt}) - \beta\psi_{ttt} \tag{7}$$

We have therefore a possibility of obtaining a series solution for the vorticity equation by solving equations (4), (5), (6), (7), . . .which are all of type (1) in the highest derivative with respect to t.

The solution is obtained as a Taylor's series:

$$\Psi = \psi + \frac{\partial\psi}{\partial t}t + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2}t^2 + \frac{1}{3!}\frac{\partial^3\psi}{\partial t^3}t^3 + \dots \tag{8}$$

where ψ is the stream function at the initial time, and Ψ the stream function at time t. A general discussion of this solution will be given in section V of this paper.

[1] In this section the subscript notation for derivatives will be employed. For example, $\psi_t = \partial\psi/\partial t$, $\psi_{tx} = \partial^2\psi/\partial x\partial t$, etc.

IV. Series solution for a barotropic vortex

Let (r, Θ) be a polar coordinate system, and the stream function $\psi(r)$ initially be only a function of r, and such that at $r \geq r_0$ it is equal to zero. We assume that $\psi(r)$ and its three first derivatives are continuous differentiable functions.

Substituting $\psi(r)$ in (4) we obtain

$$\nabla^2\psi_t = -\beta \cos \Theta \frac{d\psi}{dr} \tag{9}$$

from which ψ_t can be obtained at once by applying formula (3) and specifying some boundary conditions. If we assume that ψ_t is zero at $r = R_0$ we get

$$\psi_t = -\frac{\beta \cos \Theta}{r}\int_0^r \psi(r) r dr + \frac{\beta r \cos \Theta}{R_0^2}\int_0^{r_0} \psi(r) r dr \tag{10}$$

We shall assume throughout the rest of this paper that $R_0 \gg r_0$ so that the last term of the second member of (10) can be neglected.

Substituting $\psi(r)$, (9) and (10) in (5) we obtain, after simple manipulation

$$\nabla^2\psi_{tt} = \beta \sin \Theta\left[-\frac{1}{r}\left(\frac{d\psi}{dr}\right)^2 + \frac{1}{r^2}\frac{d(\nabla^2\psi)}{dr}\int_0^r \psi(r) r dr\right] + \beta^2\left[\frac{\psi}{2} + \frac{\cos 2\Theta}{2r^2}\int_0^r \frac{d\psi}{dr} r^2 dr\right] \tag{11}$$

which, by applying again (3) and boundary conditions of the type used to determine ψ_t, yields ψ_{tt}.

In a similar manner, one can obtain the derivatives of higher order. For example, ψ_{ttt} contains $\beta \cos \Theta$, $\beta^2 \sin 2\Theta$ and $\beta^3 \cos 3\Theta$ terms.

If in the above computations we consider numerical values within the range of those corresponding to the actual atmospheric motions which we want to study, it can be seen that the terms in higher powers of β are small compared with those in the first power; for example, in (11) the β^2 term is, for practical purposes, negligible compared with the β term. For the purpose of this investigation, it is enough to consider only the β terms.

When solving (5) the terms $J(\nabla^2\psi_t, \psi)$ and $J(\nabla^2\psi, \psi_t)$ give rise to a β term, and $\beta\psi_{tx}$ to a β^2 term.

In (6), similarly, the only β terms are $J(\nabla^2\psi, \psi_{tt})$ and $J(\nabla^2\psi_{tt}, \psi)$; in (7) the β terms are $J(\nabla^2\psi, \psi_{ttt})$ and $J(\nabla^2\psi_{ttt}, \psi)$; and so forth.

It is not difficult in this way to find that the solution can be written in the form

$$\Psi(r, \Theta, t) = \psi(r) + \beta \cos \Theta\left[F_1 t + \frac{F_3}{3!}t^3 + \frac{F_5}{5!}t^5 + \ldots\ldots\right] - \beta \sin \Theta\left[\frac{F_2}{2}t^2 + \frac{F_4}{4!}t^4 + \frac{F_6}{6!}t^6 + \ldots\right] + R \tag{12}$$

where

$$F_n = -\frac{1}{2r}\int_0^r F_n^\star r^2 dr + \frac{r}{2}\int_{r_0}^r F_n^\star dr$$

$$F_1 = -\frac{d\psi}{dr}$$

$$F_n^\star = \frac{1}{r}\left[\frac{d(\nabla^2\psi)}{dr}F_{n-1} - \frac{d\psi}{dr}F_{n-1}^\star\right]$$

and where R represents the sum of the terms containing β to a higher power than one (i.e. β^m, $m > 1$).

Let

$$\left.\begin{aligned} \psi(r) &= \psi_0\left[1 - \left(\frac{r}{r_0}\right)^2\right]^4 \quad \text{for } r \leq r_0 \\ \text{and } \psi(r) &= 0 \quad \text{for } r > r_0 \end{aligned}\right\} \tag{13}$$

This stream function represents a cyclone of radius r_0 when $\psi_0 < 0$ (and an anticyclone when $\psi_0 > 0$).

The maximum velocity $v_{\max} = (d\psi/dr)_{\max}$ is at $(r/r_0)^2 = 1/7$ and ψ_0 is given as a function of the radius and of the maximum velocity by

$$\psi_0 = -0.525\, r_0\left(\frac{d\psi}{dr}\right)_{\max}$$

Table 1

Terms in series (14) for $\beta = 1.7 \times 10^{-13}$ sec^{-1} cm^{-1}, $r_0 = 1000$ km, $v_{max} = 30$ m/sec, t in hours.

r/r_0 \ t	2	4	6	8	10	12	14
	βt term for $\sin \Theta = 1$						
.0	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}
.2	−12	−24	−36	−48	−60	−72	−84
.4	−18	−36	−54	−72	−90	−108	−126
.6	−19	−38	−56	−76	−95	−112	−133
.8	−15	−30	−45	−60	−75	−90	−105
1.0	−13	−26	−38	−52	−65	−76	−91
	βt^2 term for $\cos \Theta = 1$						
.0	0×10^{-4}	0×10^{-4}	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}
.2	15	60	14	24	37	54	73
.4	20	80	18	32	50	70	98
.6	12	48	11	19	30	43	59
.8	4	16	2	18	10	10	20
1.0	0	0	0	0	0	0	0
	βt^3 term for $\sin \Theta = 1$						
.0	0×10^{-5}	0×10^{-5}	0×10^{-4}	0×10^{-3}	0×10^{-3}	0×10^{-3}	0×10^{-3}
.2	5	40	13	3	6	10	17
.4	8	64	21	5	10	16	27
.6	10	80	26	6	12	21	34
.8	13	104	33	8	16	26	44
1.0	0	0	0	0	0	0	0
	Values of $\psi(r)/\psi_0$						
.0	1.00						
.2	.849						
.4	.498						
.6	.168						
.8	.017						
1.0	.0						

Substituting (13) in (12) we obtain

$$\frac{\Psi}{\psi_0} = \frac{\psi}{\psi_0} + r_0\,\beta \cos\Theta\left[G_1 t + G_3\left(\frac{v_{max}}{r_0}\right)^2 t^3 + G_5\left(\frac{v_{max}}{r_0}\right)^4 t^5 + \ldots\right] + r_0\,\beta \sin\Theta\left[G_2\left(\frac{v_{max}}{r_0}\right) t^2 + G_4\left(\frac{v_{max}}{r_0}\right)^3 t^4 + \ldots\right] + R \qquad (14)$$

where G_1, G_2, $G_3, \ldots$ are functions of the non-dimensional variable r/r_0.

This solution can be written also in the form

$$\left.\begin{aligned}\frac{\Psi}{\psi_0} = \frac{\psi}{\psi_0} &+ A\cos\Theta\left[G_1^\star \tau + G_3^\star \tau^3 + \ldots\right] + \\ &+ A\sin\Theta\left[G_2^\star \tau^2 + G_4^\star \tau^4 + \ldots\right] + R\end{aligned}\right\} \qquad (14')$$

where $G_1^\star$, $G_2^\star, \ldots$ are functions of the non-dimensional variable r/r_0 and $\tau = (v_{max}/r_0)t$ (or $\tau = (\psi_0/r_0^2)t$) is a non-dimensional time and $A = r_0^2\beta/v_{max}$ (or $A = r_0^3\beta/\psi_0$).

Solution (14) expresses the behaviour of a vortex as a function of latitude, maximum velocity in the vortex and radius of the vortex, and lends itself to a comparative study of vortices by varying these three parameters.

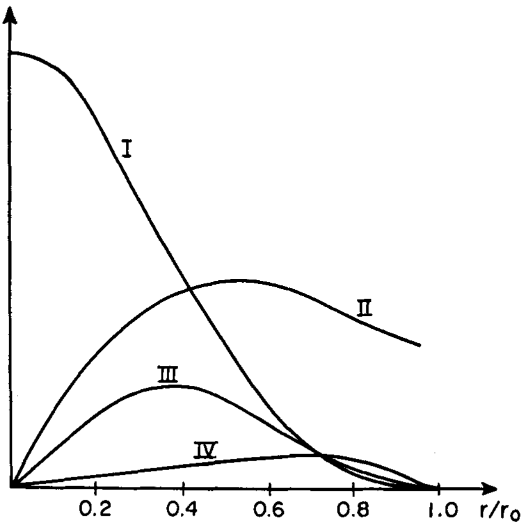


Figure 1. The line I represents the initial stream function used in the example of a vortex as a function of r/r_0; the lines II and IV represent the βt^3 terms of the series for $|\cos\Theta| = 1$ and $t = 10$ hours, and III the βt^2 term evaluated for $\sin\Theta = 1$ and the same time $t = 10$ hours. The initial stream function is represented with a vertical scale one fifth of that used for the other terms.

Let $\beta = 1.7 \times 10^{-13}$ sec^{-1} cm^{-1}, $r_0 = 1{,}000$ km and $v_{\max} = 30$ m/s

Some of the results of the computation using these data are summarized in Table 1, which gives the values of the βt, βt^2 and βt^3 terms for different values of r/r_0 and different times.

The results are also shown graphically in Fig. 1, in which the lines II and IV represent the βt and βt^3 terms of the series evaluated for for $|\cos\Theta| = 1$ and $t = 10$ hours. The line III, the βt^2 term for $\sin\Theta = 1$ and the same time $t = 10$ hours. Finally, the line I represents the initial stream function drawn with vertical scale one fifth of that used for the other terms.

These calculations show that a forecast using only the βt term can be obtained up to approximately 2 hours; afterwards the βt^2 term also becomes important, and up to 6 or 8 hours these two terms are the only important ones; afterwards the βt^3 term can no longer be neglected and up to $t = 10$ or 12 hours one can obtain a solution using only these three terms; afterwards other terms become increasingly important. If one wants to continue the forecast, one has to use the above solution up to, for example, 10 hours, and then start again a new series solution now with stream function at $t = 10$ hours as the initial field. This new initial ψ is a sum of terms of the type

$$F_n(r) \begin{Bmatrix} \sin n\Theta \\ \cos n\Theta \end{Bmatrix}$$

and therefore the integration can be continued using the same formula (3) obtained above.

The forecast for $t = 10$ hours is shown in Fig. 2. Fig. 2-a represents the initial stream function and Fig. 2-b the forecast for $t = 10$ hours.

The center of the vortex has moved 65 km in 10 hours, that it to say at the rate of 1.80 m/s.

We have used only the βt, βt^2 and βt^3 terms in this forecast. Fig. 3-a shows the βt term, Fig. 3-b the βt^2 term, Fig. 3-c shows the sum of the βt and βt^2 terms and, finally, Fig. 3-d the sum of the βt, βt^2 and βt^3 terms. Comparison of 3-d and 3-c shows that for $t = 10$ hours the βt^3 term is still not so important. Fig. 3 has been drawn with a scale different from that for Fig. 2.

The following general conclusions can be drawn from formula (14) and from the previous example:

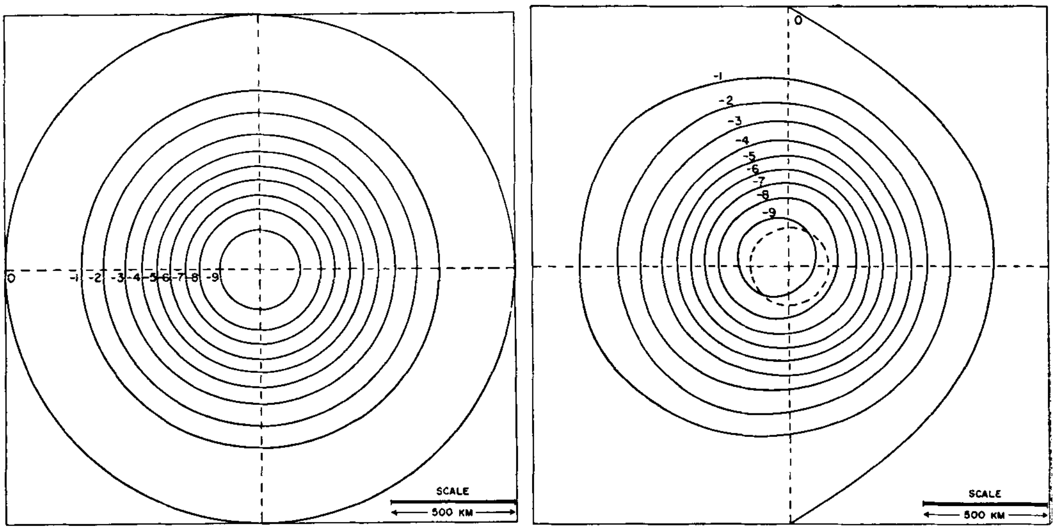
Due to the variation of the Coriolis parameter β, a barotropic non-divergent cyclone embedded in a resting atmosphere moves north-westwards.

Initially, before the symmetrical cyclone gets deformed, it moves westwards and its motion is given by the term $r_0 \beta G_1 t \cos\Theta$ in (14); as time increases, it gradually begins to move also northwards the component in this direction being given by $\beta G_2 t^2 v_{\max} \sin\Theta$. The initial westward component is, therefore, proportional to the radius of the vortex and the northward component is proportional to the maximum velocity in the cyclone.

It is interesting to note that in our solution, the βt^3 term has sign opposite to that of the βt term, and although we have computed only the terms up to the βt^3 term, it seems as if the terms in the series (14) have alternating signs.

The above results suggest the following physical interpretation:

The variation of the Coriolis parameter induces a resultant acceleration on the cyclone (Rossby, 1949) that tends to drive the material particles northwards; this northward translational motion combined with the rotational motion of the vortex produces the westward



2 a 2 b

Figure 2. Forecast for a cyclone with a maximum velocity of 30 m/s and a radius of 1 000 km. 2-a is initial stream function and 2-b the forecast for $t = 10$ hours.

movement of the vortex. This fact was pointed out by ROSSBY (1949) and can be seen by adding graphically the stream lines from Figs 2-a and 3-a. The initial westward displacement of the cyclone, apparently, induces the northward displacement represented by the βt^2 term in the series, that in turn induces the eastward component represented by the βt^3 term, and so forth.

The solution for an anticyclone can be obtained by substituting in the above solution a positive value for ψ_0 instead of the negative one that corresponds to a cyclone; and from (14) it can be shown that an anticyclone moves south-westwards and that the above results for a cyclone apply to an anticyclone if the word "north" is replaced by "south".

Solution (14′) shows that the time up to which the series can be used with the first n terms depends on the ratio $v_{\max}/r_0$ (or ψ_0/r_0^2); and because $\tau = (v_{\max}/r_0)\ t$, there exists a linear relation between the ratio $r_0/v_{\max}$ (or r_0^2/ψ_0) and the time up to which the solution using the first n terms is valid.

In Fig. 4 we have plotted lines $\tau =$ constant in a t, $r_0/v_{\max}$ (or r_0^2/ψ_0) diagram, t is the ordinate and $r_0/v_{\max}$ is the abscissa. By the side of each line we have written three numbers which are proportional to the magnitude of the maximum velocity that appears in the βt, βt^2 and βt^3 terms when evaluated for the corresponding constant value of τ.

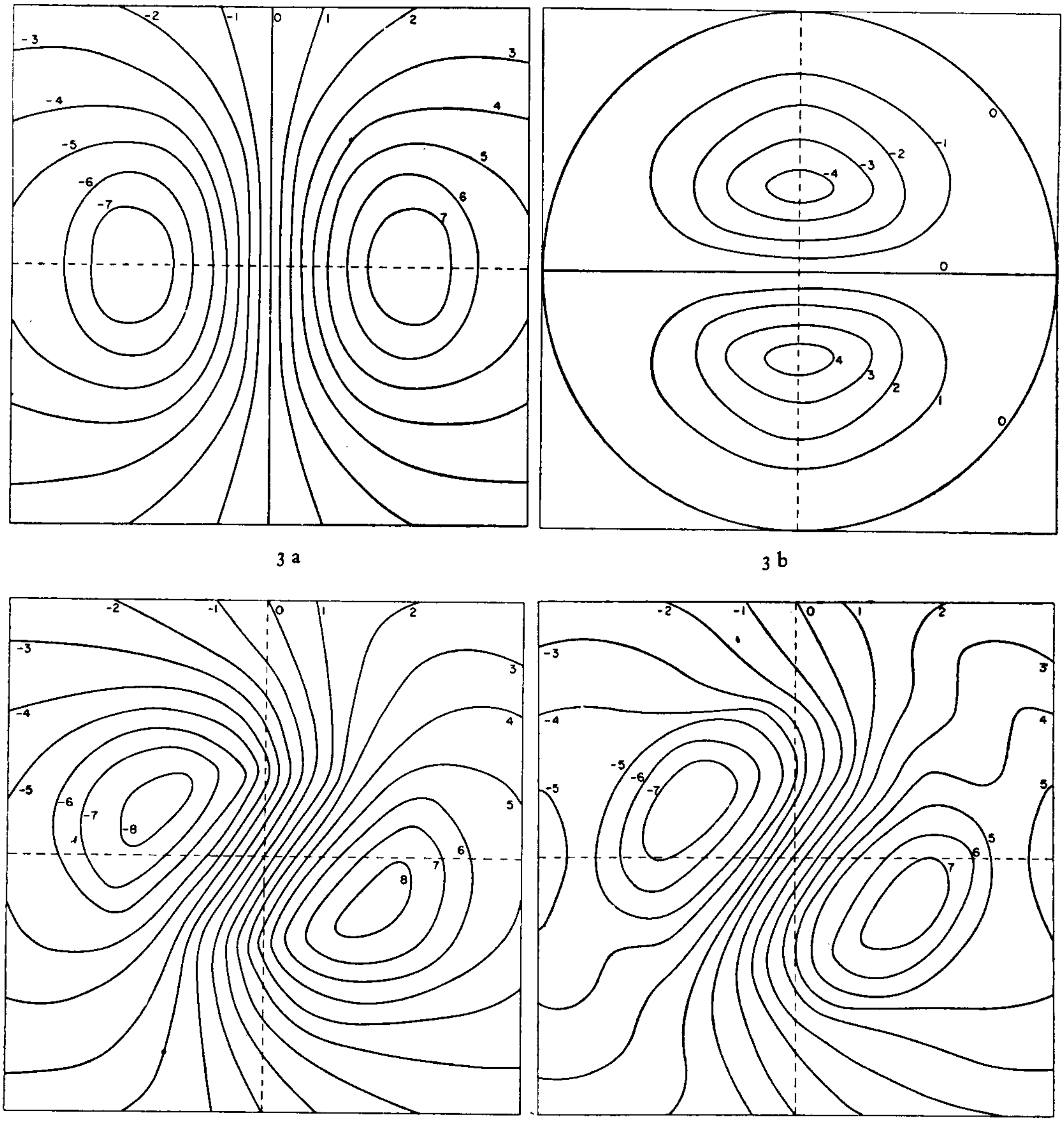
In this diagram, times 2, 8, 12 hours of our previous example, correspond to 8, 32 and 48 hours respectively for a vortex with $r_0 = 2{,}000$ km and $v_{\max} = 15$ m/s. In other words, in this case, a forecast with only the βt term can be obtained up to 8 hours, instead of 2 hours; up to 32 hours instead of 8 hours using only two terms and up to 48 hours instead of 12 hours, using three terms. On the other hand for a vortex of radius $r_0 = 500$ km and $v_{\max} =$ $= 40$ m/s. one gets 1 hour, 3 hour and 5 hour respectively.

V. Solution for a general non-divergent barotropic flow

A stream function can always be written in the form

$$\psi = \psi_0\ P_1\ (x/r_0,\ y/r_0) \qquad (15)$$

where P_1 is a function of the non-dimensional variables x/r_0 and y/r_0; ψ_0 and r_0 are scaling factors from a normalized height of the stream function and a normalized x—y plane respectively. ψ_0 and r_0 will be considered as variable parameters. For example in the case of a vortex, we have chosen r_0 as the radius of the vortex and ψ_0 as the value of the stream function at the center of the vortex.



3 a 3 b 3 c 3 d

Figure 3. Graphical representation of the first terms of the series solution for a vortex of radius $r = 1\,000$ km. and maximum velocity equal to 30 m/s., for $t = 10$ hours. 3-a is the βt term, 3-b the βt^2 term; 3-c is the sum of the βt and βt^2 terms; 3-d the sum of the βt, βt^2 and βt^3 terms. The scale is different from that used in Fig. 2-a.

We shall use also as a variable parameter the variation of the Coriolis parameter β, which for the sake of simplicity is considered as a constant.

Substituting (15) in (4) one obtains

$$\nabla^2 \frac{\partial \psi}{\partial t} = \frac{\psi_0^2}{r_0^4} P_3 - \beta \frac{\psi_0}{r_0} P_4 \qquad (16)$$

where P_3 and P_4 are functions in the non-dimensional variables x/r_0 and y/r_0.

If we assume that $(\psi_0/r_0^3\beta) =$ fixed constant, when integrating (16) we obtain

$$\frac{\partial \psi}{\partial t} = \psi_0 s P_5$$

where $s = \psi_0/r_0^2$ and P_5 is a function of x/r_0 and y/r_0.

In a similar way, one can obtain, under the same assumption $(\psi_0/r_0^3\,\beta) =$ constant,

$$\frac{\partial^2\psi}{\partial t^2} = \psi_0 s^2 P_6$$

$$\frac{\partial^3\psi}{\partial t^3} = \psi_0 s^3 P_7$$

. . . .

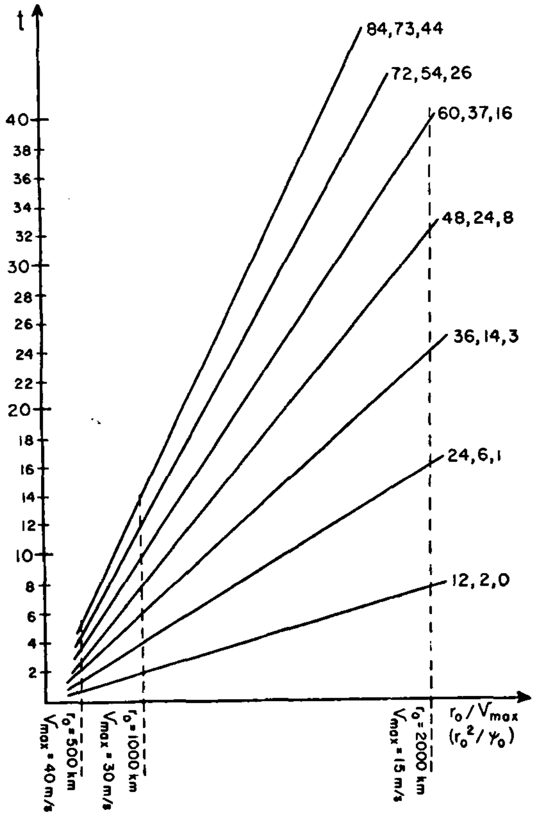


Figure 4 Time-scale of motion, diagram. The ordinate represents the time in hours and the abscisa the ratio r_0/v_{max} where v_{max} is the maximum velocity in the vortex and r_0 is the radius; in general, the abscisa can be taken as r_0^2/ψ_0 where ψ_0 and r_0 are scaling factors for the stream function and the x — y plane respectively. The straight lines represent lines of constant non-dimensional time $\tau = (\psi_0/r_0^2)\,t$ of the series solution. Note that when r_0^2/ψ_0 increases, the permissible time step in a forecast increases. The three numbers on the right hand side of each of these lines are proportional to the magnitude of the maximum velocity that appears in the βt, βt^2 and βt^3 terms of the series evaluated for the corresponding constant value of τ, for the case of a vortex.

where P_6, P_7, . . . are functions in the non-dimensional variables x/r_0 and y/r_0.

Substituting the above values of $\partial\psi/\partial t$, $\partial^2\psi/\partial t^2$, $\partial^3\psi/\partial t^3$, . . . in the series solution (8) we obtain the following result:

In an incompressible, non-divergent barotropic flow let the initial stream function ψ be given by (15) and let β be a constant. If we choose the variable parameters ψ_0, r_0 and β such that $(\psi_0/r_0^3\beta)$ = constant, then the stream function Ψ at a later time can be written as series in the non-dimensional time $\tau = (\psi_0/r_0^2\,t)$ in the form

$$\frac{\Psi}{\psi_0} = P_1 + P_2\tau + P_3\tau^2 + \ldots .$$

where P_1, P_2, P_3, . . . are functions of the non-dimensional variables x/r_0 and y/r_0.

Since the solution is independent of the arbitrary choice of ψ_0, r_0 and β, we can, without losing generality, always choose such values that $(\psi_0/r_0^3\beta)$ = constant, but still vary ψ_0, r_0 and β for a comparative purpose. We see, therefore, that a figure similar to Fig. 4 can be drawn showing that the time to which the n first terms of the series can approximate the stream function is proportional to the ratio r_0^2/ψ_0.

As a corollary of the above non-dimensional time series solution the following result can be obtained:

Let $\psi = \psi_0 F(x/r_0,\ y/r_0,\ \beta_0)$ represent an initial stream function, where ψ_0, r_0 represent the scaling of normalized quantities and β_0 the variation of Coriolis parameter; then the normalized stream function Ψ/ψ_0 at $t = t_0$ is identical with the one corresponding to an initial stream function $\psi_1 F(x/r_1, y/r_1, \beta_1)$ at time $t = t_1$, where $t_1 = (\beta_0 r_0/\beta_1 r_1)$ and $(r_1^3\beta_1/\psi_1) = (r_0^3\beta_0/\psi_0)$.

This result can be of importance in the construction of experimental models.

VI. Applications to numerical forecasting

Since the solution is a series in the non-dimensional time $\tau = (\psi_0/r_0^2)t$, it is evident that the smoothing of a map from which a forecast of large scale motions is to be made, is a necessity from the point of view of the nature of the solution of the vorticity equation. In fact the smoothing will eliminate small scale disturbances and motions for which r_0^2/ψ_0 is smaller than a certain value of C and we then forecast large scale motions and disturbances such that $r_0{}^2/\psi_0 \geq C$.

An important conclusion that can be drawn from our discussion is the dependence of the time step used in a forecast on the scale of motions or disturbances that one wants to forecast. To forecast large scale systems, one needs to forecast from a smoothed initial stream function such that $r_0{}^2/\psi_0 \geq C$ and use a certain time step which depends on the value of C; while to forecast small scale systems, one needs shorter time steps, since in this case, C is smaller.

There are two meteorologically important cases in which r_0^2/ψ_0 is small; the case of a tropical cyclone, in which one has large velocities concentrated in a small radius; and the case of a jet in which also large velocities are concentrated in a small cross section.

In the case of a cyclone as has already been pointed out when $r_0 = 500$ km and $v_{\max} = 40$ m/s the forecast with one term is valid up to one hour, while when $r_0 = 1{,}000$ km and $v_{\max} = 30$ m/s it is valid up to 2 hours; and when $r_0 = 2{,}000$ km and $v_{\max} = 15$ m/s it is valid up to 8 hours.

An important conclusion that can be drawn from our analysis is the possibility of using longer time steps in a forecast by including in the computational scheme of the forecast the $\partial^2\psi/\partial t^2$ term (and also the higher derivative terms) of the series expansion. In fact, in our example of a vortex of $r_0 = 1{,}000$ km and $v_{\max} = 30$ m/s a forecast using only the $\partial\psi/\partial t$ term is valid up to 2 hours, while by considering also the $\partial^2\psi/\partial t^2$ term the forecast can be extended up to 6 or 8 hours, in a single time step.

It may, therefore, be worth suggesting that, instead of using

$$\Psi = \psi + \frac{\partial\psi}{\partial t} \triangle t$$

with time steps of 2 hours (for example), we may solve for $\partial\psi/\partial t$ and $\partial^2\psi/\partial t^2$ from the Poisson equations (4) and (5) and then use

$$\Psi = \psi + \frac{\partial\psi}{\partial t} \triangle t + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} (\triangle t)^2$$

with larger time steps (for example 6 or 8 hours).

Another alternative is to use the series solution with as many terms as possible and reduce the number of time steps to a minimum.

Of course the above considerations have not taken into account the limitations set up by the actual practical procedure of using finite difference equations. For example, we know that the distance between grid-points in an actual case sets up a limit to the time steps used in a forecast, and requires special study (Bolin, 1955, p. 28).

In conclusion, it may be stated that this technique for obtaining a series solution for the barotropic non-divergent vorticity equation can also be applied to other forecasting equations or systems of equations. It should thus find application in a wide variety of problems in numerical forecasting.

Acknowledgement

The author wishes to express his indebtedness to professor C.-G. Rossby for giving him the benefit of several stimulating discussions and to Dr B. Bolin for reading most of the manuscript and making some valuable comments.

REFERENCES

Bolin, B., 1955: Numerical forecasting with the barotropic model. *Tellus*, **7**, pp. 27—49 m.

Bolin, B., 1956: An improved barotropic model and some aspects of using the balance equation for three-dimensional flow. *Tellus*, **8**, pp. 1—15.

Rossby, C.-G., 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. Marine Res.*, **2**, pp. 38—55.

Rossby, C.-G., 1949: On a mechanism for the release of potential energy in the atmosphere. *J. Meteor.*, **4**, pp. 135—162.