On Quasi-nondivergent Prognostic Equations and their Integration

By H.-L. KUO
Massachusetts Institute of Technology, Cambridge, Mass.

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Abstract

A general two-dimensional wind-pressure relationship is obtained as the first integral of the so-called "balance equation" which requires the horizontal divergence to remain small as compared with the vertical component of the relative vorticity for the large-scale motions. This wind-pressure relationship is used in determining the initial conditions and in deriving the governing equations. The use of this general system removes the error introduced by the quasi-geostrophic approximation.

An influence function is computed for a normal atmosphere. This function represents the reaction of the atmosphere to a point source perturbation and may be used to integrate the three-dimensional governing equation for arbitrary initial disturbances. A set of such functions may be computed, corresponding to different values of the parameter $\bar{f}^2 \cos^2 \varphi / \Gamma$, to meet different situations.

1. Introduction

The synoptically important large-scale motions of the atmosphere are characterized by the fact that at every moment the air is in a near-equilibrium state, with the forces acting on any parcel of air nearly balancing each other so that the wind is given approximately by the geostrophic-wind relation, though not exactly. The equations used in discussing the properties of these large-scale motions and in numerical forecasting are generally derived by making use of the geostrophic approximation, which eliminates the synoptically unimportant but mathematically troublesome sound waves and gravitational waves and lead to a nonhomogeneous partial differential equation in the first time-derivative of the pressure or geopotential, and can therefore be solved and extended into the future by iteration. The success of the application of simple equations derived under the geostrophic approximation in forecasting the motion of the upper atmosphere encourages meteorologists to try to improve the simplification in order to obtain still better results. (For references on quasi-geostrophic prognostic equations see KUO, 1953 a, CHARNEY and PHILLIPS, 1953.) Experiences show that the geostrophic approximation becomes poor in regions of strong wind, especially in the vicinity of intense cyclones (anticyclones) where it gives too high (low) wind and vorticity and therefore incurs fictitious development that does not exist. Since the problem of forecasting is usually more concerned with the development of cyclones and anticyclones, it would seem that the next nontrivial improvement of the quasi-geostrophic approach is to seek an approximation to the wind system better than the geostrophic which, when used as a first approximation in deriving the governing equations for the large scale motions, also eliminates the sound waves and the gravitational waves. As is usually stated, the synoptically important large-scale motions are characterized by the small value of the ratio between the horizontal divergence D and the relative vorticity ζ while the sound waves and gravitational waves are characterized by large values of the ratio D/ζ, and therefore the small-scale sound and gravitational waves will be excluded if we require the individual change of the horizontal divergence dD/dt to be small, so that it may be neglected in the divergence equation. This equation then reduces to the so called "balance equation", expressing a balance between the horizontal wind and the pressure distribution. Only when the nonlinear terms of this equation are neglected does it reduce to the geostrophic relation. Various attempts have been made to derive forecasting equations satisfying this balance equation: for example, CHARNEY (1955) has proposed to integrate the primitive equations with the balance equation used as an extra condition.

The purpose of this study is two-fold: first we shall obtain the integral of the balance equation in explicit form and then derive the governing equations by making use of this solution, and second: we shall discuss some general methods of integrating these equations.

In contrast to the quasi-geostrophic approximation, the dynamic equations we obtain under the quasi-nondivergent approximation become very complicated when expressed in terms of pressure or geopotential. However, in terms of the stream-function ψ, the equations retain their simple forms. We shall therefore take $\partial\psi/\partial t$ as the dependent variable. This also partly removes the difficulty of using the geostrophic relation at low latitudes, because the large-scale motions can still be represented by the ψ-field if it is nearly horizontal, which is usually the case even near the equator, since the normal atmosphere has a large stable stratification.

In order to derive a single forecasting equation with $\partial\psi/\partial t$ as the dependent variable alone we also express the thermodynamic equation in terms of ψ by making use of the integral of the balance equation. When the absolute vorticity is positive and the stratification is stable, the equation thus obtained is of elliptic type, similar in form with the quasi-geostrophic equation in $\partial\phi/\partial t$. However, since both the wind and the vorticity we use in the nonhomogeneous terms are nongeostrophic, the result we obtain represents the nongeostrophic flow.

When the region under consideration is very large, the integration of the three-dimensional elliptic partial differential equation by relaxation or operational methods usually become very difficult. It seems therefore more advantageous to use Green's method for such cases. In numerical applications the advantage of this method depends upon whether standard influence functions can be obtained. The variable coefficient of the elliptic equation depends upon the static stability factor Γ and the latitude. However, experience shows that the solution is not sensitive to variations of these factors. It is therefore permissible to replace the instantaneous distribution of Γ by a function representing a normal distribution. It is then possible to obtain the analytical form of the influence function by suitable transformations. Various sets of such influence functions can be computed corresponding to different values of Γ, which can be used under different situations.

In case the variation of Γ is irregular and therefore cannot be replaced by a normal distribution, the only way of solving the equation is by numerical integration, which amounts to replacing the differential equation by a system of finite difference equations for the various grid points.

2. Dynamic equations for the large-scale motions

Assuming hydrostatic approximation to be valid, and using the pressure p as the vertical coordinate, the equation of motion can be written in the following form

$$\frac{\partial V}{\partial t} + \omega \frac{\partial V}{\partial p} + (f+\zeta)\, k \times V = - \nabla_s\left(\phi + \frac{1}{2} V^2\right) + \vec{F} \qquad (1)$$

Here $V = ui + vj$ is the horizontal velocity vector, u and v are the two components, $\omega \equiv dp/dt$ is the individual change of pressure, i, j and k are unit vectors in the directions of increasing λ (longitude), φ (latitude) and z (elevation), f is the Coriolis parameter, ϕ is the geopotential of the isobaric surface. $\nabla_s = (a$

$\cos\varphi)^{-1}\, i\, \partial/\partial\lambda + a^{-1}\, j\, \partial/\partial\varphi$ is the spherical gradient operator, a being earth's radius, $\zeta = \nabla_s \times V$ is the vertical component of the relative vorticity and $\vec{F}$ is the frictional force.

For the sake of convenience we separate the horizontal velocity vector V into a nondivergent part V_1 and an irrotational part V_2 such that $V = V_1 + V_2$ and put

$$V_1 = k \times \nabla\psi, \quad V_2 = \nabla\chi \tag{2}$$

where ψ serves as the stream function for V_1 and χ is the potential function for V_2. The vorticity ζ is then given by $\zeta = \nabla_s^2\psi$ and the horizontal divergence D ($\equiv \nabla_s \cdot V$) is given by $D = \nabla_s^2\chi$, where $\nabla_s^2 = (a\cos\varphi)^{-2}\, \partial^2/\partial\lambda^2 + a^{-2}(\cos\varphi)^{-1}\, \partial/\partial\varphi \cos\varphi\, \partial/\partial\varphi$ is the surface spherical Laplacian operator. The continuity equation may now be written as

$$D \equiv \nabla^2\chi = -\frac{\partial\omega}{\partial p}. \tag{3}$$

Applying the Curl operator $\nabla_s\times$ to (1) we obtain the vorticity equation

$$\frac{\partial\zeta}{\partial t} + \frac{1}{a^2\cos\varphi}\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\varphi} - \frac{\partial\psi}{\partial\varphi}\frac{\partial\zeta}{\partial\lambda}\right) - (f+\zeta)^2\frac{\partial}{\partial p}\left(\frac{\omega}{f+\zeta}\right) \tag{4}$$
$$+ \frac{1}{a}\frac{\partial f}{\partial\varphi}v + kx\nabla_s\omega \cdot \frac{\partial V}{\partial p} + V_2 \cdot \nabla_s\zeta - \nabla_s \times F = 0$$

Instead of using the geostrophic approximation we assume V_2 to be always much smaller than V_1 and that the individual change of the horizontal divergence D may be neglected in the divergence equation (see PETTERSSEN, 1953) which is obtained by applying the operator $\nabla_s\cdot$ to (1). Thus our approximate divergence equation is[1]

$$(f+\zeta)\zeta - \left(\frac{\partial V}{\partial n}\right)^2 - \frac{1}{a}\frac{\partial f}{\partial\varphi}u - \nabla^2\phi - \nabla\omega \cdot \frac{\partial V}{\partial p}$$
$$- \frac{2\tan\varphi}{a^2\cos\varphi}u^2\frac{\partial}{\partial\lambda}\left(\frac{v}{u}\right) - \frac{\sec^2\varphi}{a^2}V^2 = 0 \tag{5}$$

[1] This approximation implies that every term of $\left(\frac{\partial}{\partial t} + V\cdot\nabla + \omega\cdot\frac{\partial}{\partial p}\right) D$ is assumed to be small.

where

$$\left(\frac{\partial V}{\partial n}\right)^2 = (a\cos\varphi)^{-2}\left\{\left(\frac{\partial u}{\partial\lambda}\right)^2 + \left(\frac{\partial v}{\partial\lambda}\right)^2 + \left(\cos\varphi\frac{\partial u}{\partial\varphi}\right)^2 + \left(\cos\varphi\frac{\partial v}{\partial\varphi}\right)^2\right\}.$$

Since $V_2 \ll V_1$, we may replace V by V_1 in this equation. The last three terms are also much smaller than the other terms and may be neglected. Using plane coordinates we then have

$$(f+\zeta)\zeta = \nabla^2\phi + \left(\frac{\partial V_1}{\partial n}\right)^2 + \frac{df}{dy}u_1 \tag{5 a}$$

or

$$f\nabla^2\psi + 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) + f_y\psi_y = \nabla^2\phi \tag{5 b}$$

where the subscripts denote differentiation. This equation is identical with the so called "balance equation" used by CHARNEY (1955), and BOLIN (1955). In this form the divergence equation has lost its prognostic property; instead it expresses a relation between the stream function ψ and the geopotential ϕ, or between V_1 and ϕ.

The first integrals of (5) are given by

$$\left.\begin{aligned}(f+\zeta)\frac{\partial\psi}{\partial x} &= \frac{\partial}{\partial x}\left(\phi + \frac{1}{2}V_1^2\right)\\ (f+\zeta)\frac{\partial\psi}{\partial y} &= \frac{\partial}{\partial y}\left(\phi + \frac{1}{2}V_1^2\right)\end{aligned}\right\} \tag{6}$$

where $V_1^2 = u_1^2 + v_1^2 = \psi_x^2 + \psi_y^2$. It is readily seen that this system represents a general steady state two-dimensional nondivergent flow with convective accelerations. It includes the gradient-wind relation as a particular case, which is obtained by assuming the motion is along circular isobars, such that $V_1 = V_\Theta = \partial\psi/\partial r$, $V_r = \partial\psi/r\partial\Theta = 0$, where r denotes the radius of curvature of the stream lines.

We note that $\frac{\partial u_1}{\partial t} + \omega\frac{\partial u_1}{\partial p} - (f+\zeta)v_2$ and $\frac{\partial v_1}{\partial t} + \omega\frac{\partial v_1}{\partial p} + (f+\zeta)u_2$ can be subtracted from the left sides of the two equations (6) respectively; the resulting equations are still solutions of the balance equation (5). Thus the balance condition (5), with V replaced by V_1, implies also $(\partial/\partial t + \omega\,\partial/\partial p)V_2 \ll \left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial p}\right)V_1$, and therefore we are in essence using the modified equation of motion

$$\left(\frac{\partial}{\partial t}+\omega\frac{\partial}{\partial p}\right)V_1+(f+\zeta)\,k\times V=-$$
$$=-\nabla_s\left(\phi+\frac{1}{2}V^2\right)+\vec{F} \qquad (1')$$

It can readily be shown that the equations (4) and (5) can be derived from this equation.

The equations (5) and (6) can be used to compute the wind and vorticity from the observed pressure distributions. These winds are needed in evaluating the initial transports in the prognostic equations. Such computations of ζ and V_1 can easily be performed by successive approximations, first using a rough estimate of V_1 (for example, the geostrophic wind or the gradient wind) in the right side of (5 a) and obtain $\zeta=\zeta_1$ as the first approximation of ζ, then using ζ_1 and the rough estimate of V_1 in (6) we obtain the first approximations of u_1 and v_1. The computation can be repeated with the result that it will converge rapidly.

It may be remarked that if the motion is exactly two-dimensional and nondivergent ($\omega=0$, $V_2=0$), the vorticity equation (4) alone is sufficient in determining the subsequent motions. The initial data needed are computed from (5 a) and (6) as discussed above, which are more general than the geostrophic approximation. The simple form of the vorticity equation is retained when expressed in terms of the stream function ψ, in analogy with the quasi-geostrophic vorticity equation.

3. The thermodynamic equation

However, we shall investigate the more general case of three-dimensional motions with horizontal divergence present. Since the vorticity equation (4) involves two dependent variables ψ and ω ($V_2=\nabla\chi$ is related to ω by (3)), we need another independent equation in these two variables in order to have a closed system. Such an equation is furnished by the first law of thermodynamics. Assuming that the hydrostatic relation

$$\frac{\partial\phi}{\partial p}=-\frac{1}{\varrho}=-\frac{RT}{p} \qquad (7)$$

holds, the thermodynamic equation may be written as

$$-\frac{p}{R}\left(\frac{\partial}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right)\frac{\partial\phi}{\partial p}+\Gamma\omega=\frac{Q}{c_p} \qquad (8)$$

where $\Gamma=T\partial\log\Theta/\partial p$, Θ being the potential temperature, R is the gas constant and Q is the heat added to unit mass per unit time. Our immediate problem now is to express also this equation in terms of ψ and ω.

To express $\partial^2\phi/\partial x\partial p$ and $\partial^2\phi/\partial y\partial p$ in terms of ψ we simply differentiate (6) with respect to p and obtain

$$\phi_{xp}=f\psi_{xp}+\frac{\partial}{\partial p}J(\psi,\psi_y) \qquad (9\text{ a})$$

$$\phi_{yp}=f\psi_{yp}-\frac{\partial}{\partial p}J(\psi,\psi_x) \qquad (9\text{ b})$$

where $J(a,b)=\partial a/\partial x\;\partial b/\partial y-\partial b/\partial x\;\partial a/\partial y$. On the other hand, to express $\partial^2\phi/\partial t\,\partial p$ in (8) in terms of ψ, we must integrate the relation (6) once again. It is readily seen that the integration of (6) yields,

$$\phi=\bar{f}\psi+\int(\psi_x\psi_{yy}-\psi_y\psi_{xy})\,\delta x+$$
$$+\int(\psi_y\psi_{xx}-\psi_x\psi_{xy})\,\delta y \qquad (10)$$

and therefore

$$\frac{\partial\phi}{\partial p}=\bar{f}\frac{\partial\psi}{\partial p}+\frac{\partial}{\partial p}\int(\psi_x\psi_{yy}-\psi_y\psi_{xy})\,\delta x+$$
$$+\frac{\partial}{\partial p}\int(\psi_y\psi_{xx}-\psi_x\psi_{xy})\,\delta y \qquad (10\text{ a})$$

Here $\bar{f}$ is the average value of f over a large area.

The last term of (10) do not yield to exact integration except under particular circumstances. However, it can be shown that their contributions in (10 a) are usually much smaller than $\partial\phi/\partial p$ and therefore may be neglected. To demostrate this, let us put

$$\psi_{xx}=\alpha\psi_{yy}$$

and assume α to be a constant. For the large scale motions we are considering in which the horizontal divergence D is much smaller than the vorticity ζ, α is usually positive and is nearly equal to 1. In the case of a circular vortex of uniform vorticity, we have exactly $\alpha=1$. On making use of this assumption in (10 a) we find

$$\frac{\partial\phi}{\partial p}=\bar{f}\frac{\partial\psi}{\partial p}+$$

$$+\frac{1}{2}\frac{\partial}{\partial p}\left\{\frac{1}{\alpha}\Delta_x(\psi_x^2-\psi_y^2)-\Delta_y(\psi_x^2-\alpha\psi_y^2)\right\} \tag{10 b}$$

where Δ_x and Δ_y are the increments in the x and y-directions, respectively. To estimate the order of magnitude of the last two terms, let us take a strong increase of the horizontal wind, from zero at sea level to 50 m sec^{-1} at 500 mb, so that $\frac{1}{2}\,\partial V_1^2/\partial p=2.5$ m^2 sec^{-2}/mb. On the other hand, taking $T=260°$ A, we find $\partial\phi/\partial p=-100$ m^2 sec^{-2}/mb. Since $(\psi_x^2-\alpha\psi_y^2)$ is much smaller than $V_1^2(=\psi_x^2+\psi_y^2)$, the last two terms on the right side of (10 b) must be very much smaller than $\partial\phi/\partial p$ and therefore may be neglected. The accuracy of this approximation is certainly above that of the other approximations. These arguments apply also to the time derivative of equation (10 b), since the change of the temperature itself at a point is ordinarily not related to the change of $\frac{\partial}{\partial p}(v_1^2-u_1^2)$ at that point. Thus we obtain

$$\frac{\partial^2\phi}{\partial t\partial p}\equiv-\frac{R}{p}\frac{\partial T}{\partial t}=\bar{f}\frac{\partial^2\psi}{\partial t\partial p} \tag{11}$$

Substituting (9 a), (9 b) and (11) in (8) we obtain

$$\frac{\partial^2\psi}{\partial t\partial p}+J(\psi,\psi_p)+V_2\cdot\nabla\psi_p+\frac{RQ}{\bar{f}c_p p}+\frac{Y}{\bar{f}}=\frac{R\Gamma}{\bar{f}p}\omega \tag{12}$$

where

$$Y=(u_2-\psi_y)\frac{\partial}{\partial p}J(\psi,\psi_y)-(v_2+\psi_x)\frac{\partial}{\partial p}J(\psi_1\,\psi_x) \tag{13}$$

Since Y is a small correction, we may neglect the terms containing u_2 and v_2 in it.

The vorticity equation (4) and the first law of thermodynamics (12) are the two fundamental equations for the two dependent variables ψ and ω, while V_2 is related to ω by the continuity equation (3).

4. Governing equations in $\partial\psi/\partial t$ and ω

Although the equations (4) and (12) can be solved as two simultaneous equations, their integration will be greatly simplified if we eliminate ω. We then obtain the following forecasting equation in $\psi_t=\partial\psi/\partial t$

$$\nabla^2\psi_t-\frac{\bar{f}\eta^2}{R}\frac{\partial}{\partial p}\left(\frac{p}{\eta\Gamma}\frac{\partial\psi_t}{\partial p}\right)+\frac{p}{R}\nabla\psi_p\cdot\nabla\left(\frac{\bar{f}}{\Gamma}\psi_{pt}\right)$$
$$=-J(\psi,\nabla^2\psi+f)+\frac{\eta^2}{R}\frac{\partial}{\partial p}\left(\frac{Z}{\eta\Gamma}\right)-$$
$$-\frac{1}{R}\nabla\psi_p\cdot\nabla\frac{Z}{\Gamma}-V_2\cdot\nabla\zeta+\nabla\times\vec{F} \tag{14}$$

where

$$Z=\bar{f}J(\psi,p\psi_p)+fV_2\cdot\nabla(p\psi_p)+\frac{RQ}{c_p}+pY$$

and $\eta=f+\nabla^2\psi$ is the absolute vorticity. Equation (14) is of elliptic type when η and $-\Gamma$ are positive and when the horizontal temperature gradient is below the critical value for overturning or inertial instability.

We note that V_2 is involved in the right side of (14) in the two terms $V_2\cdot\nabla\zeta$ and $V_2\cdot\nabla\psi_p$ of Z. To include the effect of V_2, we may solve the equation for ω, which is obtained by eliminating ψ_t from (4) and (12):

$$\nabla^2\left(\frac{\Gamma}{f}\omega\right)-\frac{p}{R}\frac{\partial}{\partial p}\eta^2\frac{\partial}{\partial p}\left(\frac{\omega}{\eta}\right)+$$
$$+\frac{p}{R}\frac{\partial}{\partial p}(\nabla\psi_p\cdot\nabla\omega)$$
$$=-\frac{p}{R}\frac{\partial}{\partial p}\left\{J(\psi,\nabla^2\psi+f)+V_2\cdot\nabla\zeta-\nabla\times\vec{F}\right\}+$$
$$+\frac{1}{R}\nabla^2\left(\frac{Z}{f}\right). \tag{15}$$

Thus ω can be determined from the instantaneous velocity and temperature fields, after which we can solve for χ from the continuity equation (3) and obtain $V_2=\nabla\chi$.

However, although the effect of the transports by V_2 in (14) can be included by the procedure outlined above, too much extra computation is required to incorporate this small correction, since V_2 is assumed to be much smaller than V_1. It seems therefore both profitable and permissible to replace V_2 by a certain approximation in these terms in (14). Since V_1 is considered as the first approximation to the total horizontal velocity vector V and satisfies the equations (6) while V satisfies (1'), we may take

$$fV_2' = \left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial p}\right) V_1 \times k = -\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial p}\right) \nabla \psi \tag{16}$$

as an approximation for fV_2, which should be sufficient for these transport terms. We note that the first part of V_2' is equivalent to the isallobaric wind while the second part represents a motion in the direction of the horizontal temperature gradient, toward regions of higher temperature when the air is ascending and toward regions of lower temperature when it is descending. This second part is usually very small and can be neglected, its inclusion in $V_2 \cdot \nabla \psi_p$ of (12) merely changes the stability factor Γ into $\Gamma' = \Gamma\ (1 - 1/Ri)$, where $Ri = g\ \dfrac{\partial \ln \Theta}{\partial z} \left(\dfrac{\partial V_1}{\partial z}\right)^{-2}$ is the Richardson number, which is usually much larger than 1.

The approximation (16) brings out another interesting property of the large-scale motions. Taken as a part of the horizontal wind, the isallobaric wind represented by the first term is irrotational, which is a pertinent property of V_2. However, since this term is the local acceleration in the equations of motion, the motion produced by it at a later time will be nondivergent. This represents a continuous adjustment between the mass field and the velocity field.

Substituting (16) in (14) we obtain

$$\begin{aligned} \nabla^2 \psi_t - \frac{\bar{f}\eta^2}{R} \frac{\partial}{\partial p} \frac{p}{\eta \Gamma} \frac{\partial \psi_t}{\partial p} + \frac{2\bar{f}+\zeta}{R\Gamma} p \nabla \psi_p \cdot \nabla \psi_{pt} + \\ + \frac{\eta^2}{R} \nabla \left(\frac{\partial}{\partial p} \frac{p}{\eta \Gamma} \frac{\partial \psi}{\partial p}\right) \cdot \nabla \psi_t - \frac{1}{\bar{f}} \nabla \zeta \cdot \nabla \psi_t \quad (17) \\ = -J(\psi, \nabla^2 \psi + f) + \frac{\eta^2}{R} \frac{\partial}{\partial p} \frac{Z_1}{\eta \Gamma} - \\ - \frac{1}{R} \nabla \psi_p \cdot \nabla \frac{Z_1}{\Gamma} + \nabla x \vec{F} \end{aligned}$$

where

$$Z_1 = \bar{f} J(\psi, p \psi_p) + RQ/c_p + pY.$$

Since no other dependent variable is involved in (17), ψ can be extrapolated into the future by solving this equation alone.

It may be remarked that this equation is capable of representing the slowly varying convective mean meridional circulation in the atmosphere, as can be demonstrated by assuming ψ to be independent of λ and replacing ψ_{yt} by $-fv_2$. The left side of this equation then becomes identical with the equation governing the slowly varying mean meridional circulation (see Kuo, 1956).

We note that the second and third terms on the left, which originate partly from the divergence term and partly from the twisting term $\nabla \omega \cdot \nabla \psi_p$ of the vorticity equation, represent an overturning or convective process in a vertical plane perpendicular to the horizontal isotherms. These terms are of some importance in regions of strong horizontal temperature gradient, but under normal conditions are usually small and can therefore be neglected or included through some approximation. They can also be incorporated in the first term by using Θ as the vertical coordinate so as to measure the horizontal differentials along the isentropic surfaces. However, because of the rapid variation of Θ such a coordinate system is not very desirable. Since these terms are usually small, their effect can be included by successive approximations. Denoting the sum of these terms by M and transforming it to the right side of the equation we may write equation (17) in the following form

$$\nabla^2 \psi_t - \frac{\bar{f}^2}{R} \frac{\partial}{\partial p} \frac{p}{\Gamma} \frac{\partial \psi_t}{\partial p} = -H(\lambda, \varphi, p) = -\sum_{i=1}^{5} H_i \tag{17 a}$$

with

$$\begin{aligned} H_1 &= J(\psi, \nabla^2 \psi + f), \\ H_2 &= -\frac{\bar{f}\eta^2}{R} \frac{\partial}{\partial p} \frac{p}{\eta \Gamma} J(\psi, \varphi_p) \\ H_3 &= -\frac{\eta^2}{R} \frac{\partial}{\partial p} \frac{1}{\eta \Gamma} \left(\frac{RQ}{c_p} + pY\right) + \frac{1}{R} \nabla \psi_p \cdot \nabla \frac{Z_1}{\Gamma} \\ H_4 &= -\nabla \times \vec{F} \\ H_5 &= -M = \frac{2\bar{f}p}{R\Gamma} \nabla \psi_p \cdot \nabla \frac{\partial \psi_t}{\partial p} + \\ &\quad + \nabla \left(\frac{\bar{f}}{R} \frac{\partial}{\partial p} \frac{p}{\Gamma} \frac{\partial \psi}{\partial p} + \frac{1}{\bar{f}} \zeta\right) \cdot \nabla \psi_t - \\ &\quad - \frac{\bar{f}\zeta^2}{R} \frac{\partial}{\partial p} \frac{p}{\zeta \Gamma} \frac{\partial \psi_t}{\partial p} \end{aligned} \tag{18}$$

The correction M can be included by first solving equation (17 a) with M omitted, then evaluating M from the solution and then adding it to the right hand side and solving again. Another way of including M is to replace the time derivative in M by that of the immediate past, i.e., $\partial\psi/\partial t)_{t=e\Delta t} = \{(\psi)_{t=e\Delta^t} - -\psi)_{t=(e-1)\Delta t}\}/\Delta t$; in this way M also becomes a known function of space and can therefore be considered as a nonhomogeneous term.

Equation (14) or (17) is linear in ψ_t, with nonhomogeneous terms which also change with ψ, therefore these terms must be determined at frequent time intervals. The stability factor Γ which occurs both in the coefficient of ψ_t and in the nonhomogeneous terms is also related to ψ through the following relation

$$\Gamma = -\frac{\bar{f}}{R}\left(p\frac{\partial^2\psi}{\partial p^2} + \frac{c_v}{c_p}\frac{\partial\psi}{\partial p}\right).$$

In solving these equations over a large area, it is more convenient to employ a nondimensional horizontal coordinate-system defined by

$$x = \lambda, \quad y = \int\frac{d\varphi}{\cos\varphi} = \log(\tan\varphi + \sec\varphi) \quad (19\text{ a})$$

which represents a Mercator projection. In this projection, a point on the surface of the earth is projected from the center on a cylindrical surface which is tangent to the earth along the equator, with its axis parallel to that of the earth. Denoting the differential operators in the (λ, φ) system by the subscript s and those in the new (x, y) system by the subscript e, we then have

$$\nabla_e = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} = a\cos\varphi\,\nabla_s,$$

$$J_e(\alpha_1, \alpha_2) = \frac{\partial\alpha_1}{\partial x}\frac{\partial\alpha_2}{\partial y} - \frac{\partial\alpha_2}{\partial x}\frac{\partial\alpha_1}{\partial y} =$$

$$= a^2\cos^2\varphi\, J_s(\alpha_1, \alpha_2)$$

$$\nabla_e^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = a^2\cos^2\varphi\,\nabla_s^2. \quad (19\text{ b})$$

Thus it is seen that these new operators have the same forms as in a cartesian coordinate-system and that the differentials $\partial/\partial x$ and $\partial/\partial y$ are commutative. When the equations (4), (12), (14) and (17) are multiplied by $a^2\cos^2\varphi$, then all the horizontal differentiations are expressed in terms of the new operators. In particular, eq. (17 a) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{a^2\bar{f}^2\cos^2\varphi}{R}\frac{\partial}{\partial p}\frac{p}{\Gamma}\frac{\partial}{\partial p}\right)\psi_t =$$

$$= -H_e(x, y, p),$$

$$H_e(x,y,p) = J_e(\psi, \nabla^2\psi + f) - \frac{a^2\cos^2\varphi\,\eta^2}{R}\frac{\partial}{\partial p}\frac{Z_1}{\eta\Gamma}$$

$$+\frac{1}{R}\nabla_e\psi_p\cdot\nabla_e\frac{Z_1}{\Gamma} + a\cos\varphi\,\nabla_e\times\vec{F} - M_e \quad (20)$$

where all horizontal differentiations are with respect to the new (x, y) system defined by (19 a).

Certain conditions relating the distribution of ψ_t or its normal derivative on the boundaries must be specified in order that the solutions of the elliptic equation (14) or (20) should be unique. On the earth's surface $z = z^\star(x, y)$ or $p = p^\star(x, y)$ we must have

$$w = V\cdot\nabla z^\star \quad (21\text{ a})$$

Substituting this w and $V \doteq V_1 = k\times\nabla\psi$ in

$$\omega \equiv \frac{dp}{dt} = \varrho\left(\frac{\partial\phi}{\partial t} + V\cdot\nabla\phi - gw\right)$$

we obtain

$$\bar{f}\frac{\partial\psi}{\partial t} - J(\psi_1\phi^\star) = \frac{1}{\varrho}\,\omega \quad \text{for } p = p^\star, \quad (21\text{ b})$$

where $\nabla\phi^\star = g\nabla(z^\star - h)$, $\nabla(z^\star - h)$ being the slope of the earth's surface relative to the isobaric surface. Substituting (21 b) in the thermodynamic equation (12), with the term $V_2\cdot\nabla\,\partial\psi/\partial p$ omitted, we obtain for $p = p^\star$,

$$\frac{\partial\psi_t}{\partial p} + s\psi_t + J(\psi, \psi_p) - \frac{s}{\bar{f}}J(\psi, \phi^\star) + \frac{RQ}{c_0 p_0\bar{f}} = 0 \quad (21\text{ c})$$

where

$$s = -\partial\ln\Theta/\partial p = -\Gamma/T.$$

On the top of the atmosphere $p = 0$ we have

$$\omega|_{p=0} = 0. \quad (22\text{ a})$$

If we assume the kinetic energy per unit volume vanishes there ($\varrho V^2 \to 0$), we must also have

$$\lim_{p\to 0}\varrho^{1/2}\psi_t = 0. \quad (22\text{ b})$$

Since there is no natural side-boundary in the atmosphere, any physical condition specified on such imaginary boundaries will represent certain restriction on the motion. However, for the investigation of some general physical processes in the atmosphere we may assume symmetry about the equator.

5. Integration by Green's method

When the region under consideration is very large, it is more convenient to integrate the equation (14) or (17) by Green's method. For simplicity, we shall discuss equation (17 a). The form of the Green's function depends upon the stability factor Γ, which usually remains fairly constant in the troposphere but becomes inversely proportional to p in the stratosphere. We shall therefore represent Γ by the formula

$$\Gamma_0 = -\frac{Cp}{(p+\alpha p_0)^2}, \tag{23}$$

and choose $\alpha = 0$ in the stratosphere and $\alpha \simeq 1/2$ in the troposphere. The effect of the departure from this mean Γ can be taken into consideration by transferring the correction to the right side of the equation.

For this stratification, the substitutions

$$\begin{aligned} \psi_t &= (p/p_0+\alpha)^{-1/2}\,\tau \\ q &= \ln\frac{1+\alpha}{p/p_0+\alpha} \end{aligned} \tag{24}$$

transfor (17 a), (21 c) and (22 b) into the following forms

$$\frac{\partial^2\tau}{\partial x^2}+\frac{\partial^2\tau}{\partial y^2}+l^2\left(\frac{\partial^2\tau}{\partial q^2}-\frac{1}{4}\tau\right)= = -(p/p_0+\alpha)^{1/2}H_e = -H'(x,y,q), \tag{25}$$

$$\frac{\partial\tau}{\partial q}+h\tau = B \quad \text{for } q=0, \tag{25 a}$$

$$\lim_{q\to\infty}\tau = 0. \tag{25 b}$$

where H' is the transformation of H,

$$l^2 = a^2\bar{f^2}\cos^2\varphi/RC,\ h = 1/2+(1+\alpha)\,p_0\frac{\partial\ln\Theta}{\partial p}$$

and $B = -(1+\alpha)^{1/2}J_s(\psi,\psi_q)+\dfrac{R}{fc_p}(1+\alpha)^{3/2}Q.$

To solve this set of equations let us split τ into two parts, τ_1 and τ_2, such that τ_1 satisfies the nonhomogeneous equation (25) and the homogeneous parts of the boundary conditions (25 a) and (25 b), while τ_2 is a particular solution of the homogeneous part of (25) but satisfies the complete nonhomogeneous boundary conditions (25 a) and (25 b). It is easy to see that the sum $\tau_1+\tau_2$ satisfies both the nonhomogeneous equation (25) and the complete conditions (25 a, b). We shall assume that the region of integration covers the entire atmosphere.

The solution τ_1 is readily obtained by Green's method, and is given by $\tau_1 = \int_V GH'dV$, where the integration is to cover the whole atmosphere. The function G has to satisfy the following conditions:

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}+l^2\left(\frac{\partial^2 G}{\partial q^2}-\frac{1}{4}G\right)=0 \text{ for all interior points,} \tag{26 a}$$

$$\frac{\partial G}{\partial q}+h\,G=0 \quad \text{for } q=0, \tag{26 b}$$

$$\lim_{q\to\infty} G = 0, \tag{26 c}$$

$$\lim_{P\to Q} G(P,Q)\to\frac{1}{4\pi r} \text{ (condition of unit source)}$$

$$r^2 = (x-\xi)^2+(y-\eta)^2+\frac{1}{\bar{l}^2}(q-q')^2. \tag{26 d}$$

$\bar{l}$ being the mean value of l between the two points P and Q. As has been pointed out before, equation (20) is self-adjoint for all forms of Γ, therefore the Green's function G also satisfies the reciprocity relation $G(P,G) = G(Q,P)$. Because of this reciprocity and condition (26 d), let us assume the principal part of G to be a function of r alone. We then find that the one fundamental solution which satisfies both (26 c) and (26 d) is $e^{-\frac{\bar{l}r}{2}}/r$, therefore we shall construct our Green's function by a manipulation of this "principal solution", requiring that G also satisfies the boundary condition (26 b).

An intuitive and direct method of obtaining G is the image method. From the form of the condition (26 b) we notice at once that we need not only an isolated reflected point source at $q=-q'$, but also a continuous sequence of sources at all points $q<-q'$. The effects of these sources are given by $Ae^{-\frac{\bar{l}r'}{2}}/r'$ and $\int_{q_1}^{\infty} b(\beta)\frac{e^{-\frac{\bar{l}r'}{2}}}{r'(\beta)}\,d\beta$, respectively, where A and $b(\beta)\,d\beta$ are the yields of these sources and

$$r'^2=(x-\xi)^2+(y-\eta)^2+\frac{1}{\bar{l}^2}(q+q')^2,$$

$$r'^2(\beta)=(x-\xi)^2+(y-\eta)^2+\frac{1}{\bar{l}^2}(q+\beta)^2.$$

It is easy to see that these functions satisfy (26 a, c) and the reciprocity requirement. Since they are regular functions for all points occupied by the atmosphere, the function

$$4\pi G=\frac{e^{-\frac{\bar{l}r}{2}}}{r}+A\frac{e^{-\frac{\bar{l}r'}{2}}}{r'}+\int_{q'}^{\infty} b(\beta)\frac{e^{-\frac{\bar{l}r'}{2}(\beta)}}{r'(\beta)}\,d\beta$$

satisfies all the conditions (26 a, c, d, e). We have only to determine the constant A and the function $b(\beta)$ to make it also satisfy (26 b), which requires $A=1$, $(1+A)\,h=b(q')$ and $db/d\beta=hb$. The solution of the last two conditions is $b=2\,h\,e^{-h(\beta+q')}$, therefore the result is

$$4\pi G=\frac{e^{-\frac{\bar{l}r}{2}}}{r}+\frac{e^{-\frac{\bar{l}r'}{2}}}{r'}+2\,h\,e^{-h(q+q')}\,I, \qquad (27)$$

where $I=\int_{q'}^{\infty} Exp\left\{h\,(q+\beta)-\frac{\bar{l}r'(\beta)}{2}\right\}\frac{d\beta}{r'(\beta)}$. This influence function is similar to the one obtained by the writer in the previous study (KUO, 1953 a) under the assumption of a more restricted form of stratification.

For numerical application, the influence function G given by (27) can be computed for different values of the stability factor C of (23) and different latitudes. The first two terms of G are easy to compute. To compute the integral I, it is convenient to put it into another form. Writing Δa for $\{(x-\xi)^2+(y-\eta)^2\}^{1/2}$, we distinguish two cases:

(a) $\Delta a=0$ and (b) $\Delta a>0$, where $\Delta\,a$ is always taken as positive.

(a) $\Delta a=0$. We have $r'(\beta)=q+\beta=\varepsilon$ and

$$I=\int_{q+q'}^{\infty}\frac{e^{-(1/2-h)\varepsilon}}{\varepsilon}\,d\varepsilon=-Ei\,(-\lambda_1) \qquad (28\text{ a})$$

where $\lambda_1=\frac{C}{T_0}\left(\frac{q+q'}{1+\alpha}\right)$. For the tabulation of the exponential integral $-Ei\,(-x)$ see JAKNKE-EMDE (1943).

(b) $\Delta\alpha>0$. For this case we put $q+\beta=\bar{l}\Delta a\tan\alpha$, $r'(\beta)=\Delta a$ sec α. The integral I then becomes

$$I=\int_{\alpha_1}^{\frac{\pi}{2}}\sec\alpha\ exp\left\{l\Delta a\,(h\tan\alpha-\frac{1}{2}\sec\alpha)\right\}d\alpha, \qquad (28\text{ b})$$

where $\alpha_1=arc\ \tan\ \frac{q+q'}{\bar{l}\Delta a}$. This integral can easily be computed for given values of $\bar{l}$ and h.

It may be remarked that the first term of G, i.e., the principal solution $e^{-\frac{\bar{l}r}{2}}/r$, decreases more rapidly with increasing horizontal distance than the corresponding function in two-dimensional motions, so does also the image $e^{-\frac{\bar{l}r'}{2}}/r'$. However, the third term of G decreases more slowly with horizontal distance; therefore it increases the importance of far away points. The contribution from this term depends largely upon the factor $h=\frac{1}{2}-\frac{C}{(1+\alpha)\,T_0}$, and is smaller for an atmosphere with very stable stratification near the ground. We also note that since G is a function of the distance between the two points P and Q in the (x, y, q)-system, it is not symmetric in the linear system (λ, φ, p). Because the north-south distance $\Delta\varphi$ is magnified according to $\Delta y=\Delta\varphi/\cos\varphi$, G decreases more rapidly toward higher latitudes. However, this effect becomes appreciable only for the region beyond latitude 50°.

Some such influence functions have been computed for an atmosphere with $\alpha=0$, $C=C_1=31.5°$ A for $p>250$ mb, and $c=c_2=63°$ A for $p<250$ mb, $\bar{l}h=1.26$ and $\varphi_0=45°$. The values of this function are illustrated in

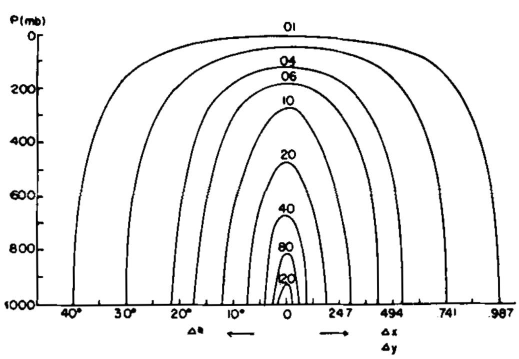

Fig. 1.

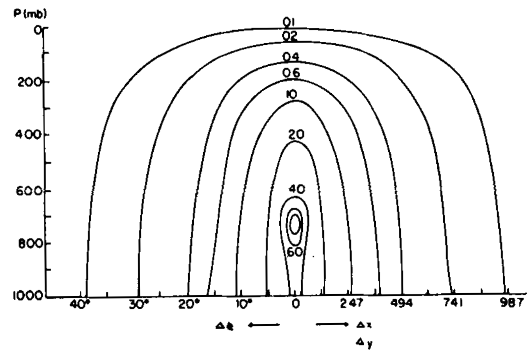

Fig. 2.

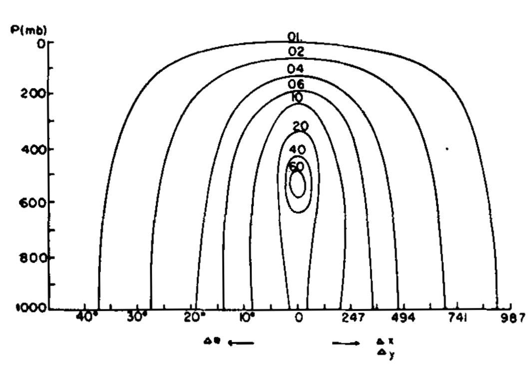

Fig. 3.

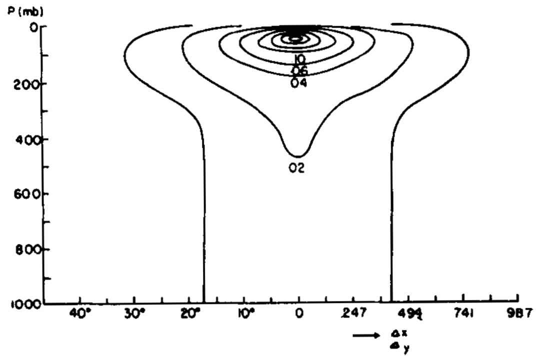

Fig. 4

Fig 1—4. The influence function $K(P, Q) = G(P, Q)/l$ for $c_1 = 31.5°$ C and $c_2 = 63°$ C with source points at $p' = 1\,000$ mb, 700 mb, 500 mb and 50 mb, respectively, and at $\varphi_0 = 45°$.

the figures 1 - 4, in the form of $K(P, Q) = G(P, Q)/l$.

The second part of the solution τ_2 can easily be found by expanding τ_2 and the function B in (25 a) into double Fourier series,

$$\tau_2 = \sum_{m,n=-\infty}^{\infty}\sum C_{mn} N_{mn}(q)\, e^{i(mx+ny)} \qquad (29)$$

Equations (25), (25 a) and (25 b) then require

$$\left.\begin{aligned} C_{mn} &= \frac{A_{mn}}{h-\lambda_{mn}}, \\ N_{mn}(q) &= e^{-\lambda_{mn} q}; \\ \lambda_{mn} &= \frac{1}{4} + \frac{m^2+n^2}{l^2}, \\ A_{mn} &= \frac{1}{4\pi^2} \iint B(\xi, \eta)\, e^{-i(m\xi+n\eta)}\, d\xi\, d\eta \end{aligned}\right\} \qquad (30)$$

Therefore τ_2 decreases rapidly with elevation, especially for the high wave-number components.

We may now write down the complete solution of our problem. Returning to the variable ψ_t we have

$$\frac{\partial\psi}{\partial t} = $$

$$= \frac{1}{(p+\alpha p_0)^{\frac{1}{2}}} \iiint K(P,Q) H_s(Q) \frac{dp'}{(p+\alpha' p_0)^{\frac{1}{2}}} dA_s + \sum_{m,n=-\infty}^{\infty}\sum \frac{A_{mn}}{h-\lambda_{mn}} \left(\frac{p}{p_0}+\alpha\right)^{\lambda_{mn}-1/2} e^{-i(mx+ny)} \qquad (31)$$

where $K(p, Q) = G(P, Q)/l$ and H_s is the sum of the nonhomogeneous terms given in (18), computed in the surface spherical system, and $dA_s = a^2 \cos\varphi^1\, d\lambda^1\, d\varphi^1$ is the actual area over the spherical earth. The triple integral can be obtained numerically when H_s has been computed. Since the influence function K decreases more rapidly with actual horizontal distance in higher latitudes, the last form of this equation shows that the polar region has a smaller influence area-wise.

In the treatment above no lateral boundary condition has been assumed, therefore the integral of (31) is to be taken over the entire atmosphere. For some theoretical problems certain lateral boundary conditions can also be specified, even though no such actual boundary exists. Thus if the difference between the two hemispheres is disregarded, we may assume symmetry about the equator, which requires ψ or $\partial\psi/\partial\varphi$ to be zero along the equator. These requirements are satisfied by using $G^1 = G \mp G_1$ as the influence function, where G_1 is the image of G about the equator, a function obtained from G by changing $y-\eta$ into $y+\eta$. When similar conditions are supposed for another latitude circle near the pole, the introduction of one more reflection will be sufficient. On the other hand, if values of ψ_t or its normal derivative are specified along a lateral boundary, then a surface integral has to be added to the solution (32).

In actual computations, the nonhomogeneous term H can be calculated by finite differences. The equations themselves may also be written in finite difference forms and solved by relaxation method (see HINKELMANN 1953, KUO 1955).

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