Thermodynamical Study of the Entrainment of Air into a Cumulus

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Abstract

Formulas dealing with the entrainment of air into a cumulus are rationally established on the basis of the thermodynamics of polytherm systems. The hypothesis made in the course of the calculus are clearly put into evidence.

It is agreed (e.g. BYERS and BRAHAM, 1949) that when a cumulus forms, surrounding air mixes with the air composing the cloud. The mixture is made in such a manner that air in the interior of a cumulus remains saturated.

For a thermodynamical study of this phenomenon we have supposed

1) that the cloud is composed of saturated humid air containing liquid water and forms an open system,
2) that the surrounding air which is entrained into the cloud is composed of humid non-saturated air and also forms an open system,
3) that the composite formed by these two open systems constitutes a closed system in which are produced exchanges of dry air and water vapour,
4) that the transformations of this closed system are adiabatic.

According to these hypotheses, the thermodynamical study of the entrainment of air into a cumulus belongs to those adiabatic transformations of a polythermal closed system formed of a prime open system (cloud) determined by the temperature T', the pressure p and the masses m_a', m_v', m_e' of dry air, of water vapour and of liquid water and a second open system (air entrained into the cloud) determined by the temperature T'', the pressure p and the masses m_a'', m_v'' of dry air and of water vapour.

We write

$$m_a = m_a' + m_a'' , \quad m_v = m_v' + m_v'' . \tag{1}$$

The dry air being an inert constituent and the system formed by the ensemble of the two systems prime and second being closed, we have

$$dm_a = dm_a' + dm_a'' , \quad dm_v + dm_e' = 0 \tag{2}$$

from which it arises that of the mass variables determining the polythermal system, only three are independent.

Denoting by h_a', h_v', h_e' and h_a'', h_v'' the specific enthalpies of the constituents in the systems prime and second, we have (e.g. VAN MIEGHEM and DUFOUR, 1948, § 22)

$$\left.\begin{aligned} h_a' &= h_a^\circ + c_{pa}(T' - T_0) , \\ h_v' &= h_v^\circ + c_{pv}(T' - T_0) , \\ h_e' &= h_e^\circ + c_{pe}(T' - T_0) , \\ h_a'' &= h_a^\circ + c_{pa}(T'' - T_0) , \\ h_v'' &= h_v^\circ + c_{pv}(T'' - T_0) . \end{aligned}\right\} \tag{3}$$

where h_a°, h_v°, h_e° are the specific enthalpies at the temperature T_0 and c_{pa}, c_{pv}, c_{pe} the specific heats at constant pressure of dry air, water vapour and liquid water.

We recall that the enthalpies of the two systems prime and second are given by the relations

$$\left.\begin{aligned} H' &= m_a' h_a' + m_v' h_v' + m_e' h_e', \\ H'' &= m_a'' h_a'' + m_v'' h_v'' \end{aligned}\right\} \tag{4}$$

and that the heat of vaporisation at temperature T' is given by the relation

$$L_v' = h_v' - h_e'. \tag{5}$$

The differential equation of the adiabatic transformations of the closed system is thus written

$$d(H' + H'') - (V' + V'')\, dp = 0 \tag{6}$$

where V' and V'' are the volumes of the systems prime and second.

Using equations (1), (2), (3), (4) and (5), equation (6) can be written in the form

$$\left.\begin{aligned} &(m_a' c_{pa} + m_v' c_{pv} + m_e' c_{pe}) dT' \\ &+ (m_a'' c_{pa} + m_v'' c_{pv}) dT'' + L_v' dm_v \\ &+ (T'' - T')(c_{pa} dm_a'' + c_{pv} dm_v'') \\ &- (V' + V'') dp = 0. \end{aligned}\right\} \tag{7}$$

We neglect in the prime system the volume of the liquid phase compared with that of the gaseous phase. Thus one obtains, using Dalton's law

$$\left.\begin{aligned} V' &\simeq \frac{m_v' R_v T'}{p_v'} = \frac{m_a' R_a T'}{p - p_v'}, \\ V'' &= \frac{m_v'' R_v T''}{p_v''} = \frac{m_a'' R_a T''}{p - p_v''} \end{aligned}\right\} \tag{8}$$

where R_a, R_v are the gas constants (perfect gas) for dry air and water vapour and p_v', p_v'' the partial pressures in the systems prime and second.

Moreover, the mixing ratios in these two systems are given by the relations

$$\tau_v' = \frac{m_v'}{m_a'} = \frac{\varepsilon p_v'}{p - p_v'}, \quad \tau_v'' = \frac{m_v''}{m_a''} = \frac{\varepsilon p_v''}{p - p_v''} \tag{9}$$

where ε is the specific gravity of the water vapour. On differentiating the relation (9), we have

$$\left.\begin{aligned} dm_v' &= m_a' d\tau_v' + \tau_v' dm_a', \\ dm_v'' &= m_a'' d\tau_v'' + \tau_v'' dm_a''. \end{aligned}\right\} \tag{10}$$

Having regard to equations (2) and (10), equation (7) can be written

$$\begin{aligned} &(m_a' c_{pa} + m_v' c_{pv} + m_e' c_{pe}) dT' + (m_a'' c_{pa} + m_v'' c_{pv}) dT'' \\ &- [(c_{pa} + \tau_v'' c_{pv})(T' - T'') + L_v' (\tau_v' - \tau_v'')]\, dm_a'' \\ &+ m_a' L_v' d\tau_v' + m_a'' [L_v' - c_{pv}(T' - T'')] d\tau_v'' \\ &- R_a \left(\frac{m_a' T'}{p - p_v'} + \frac{m_a'' T''}{p - p_v''} \right) dp = 0. \end{aligned} \tag{11}$$

We note that this equation is a linear differential equation with six variables.

We now introduce the hypothesis that the prime system remains saturated during the course of the adiabatic transformation. To do this we suppose that the law of displacement of equilibrium is satisfied, that is to say

$$\frac{\varepsilon L_v'}{(T')^2} dT' = R_a \frac{dp_v'}{p_v'}. \tag{12}$$

On the other hand, on differentiating the first relation (9), we have

$$d\tau_v' = \frac{(\varepsilon + \tau_v') dp_v'}{p - p_v'} - \frac{\tau_v' dp}{p - p_v'}. \tag{13}$$

On eliminating dp_v' and $d\tau_v'$ from the equations (11), (12) and (13), we obtain, having regard to equation (9) which allows us to replace $\varepsilon p_v' / p - p_v'$ by τ_v'

$$\frac{dT'}{dp} = \frac{x}{y} \tag{14}$$

where

$$\begin{aligned} x = {} & \frac{m_a' R_a T'}{p - p_v'} \left[\left(1 + \frac{\tau_v' L_v'}{R_a T'} \right) + \frac{m_a''}{m_a'} \frac{T''}{T'} \frac{p - p_v'}{p - p_v''} \right] \\ & - (m_a'' c_{pa} + m_v'' c_{pv}) \frac{dT''}{dp} + \\ & + [(c_{pa} + \tau_v'' c_{pv})(T' - T'') + L_v' (\tau_v' - \tau_v'')] \frac{dm_a''}{dp} \\ & - m_a'' [L_v' - c_{pv}(T' - T'')] \frac{d\tau_v''}{dp}, \end{aligned} \tag{15}$$

$$y = (m_a' c_{pa} + m_v' c_{pv} + m_e' c_{pe}) + \frac{m_a' \tau_v' (\varepsilon + \tau_v')(L_v')^2}{R_a (T')^2}. \tag{16}$$

We note that this differential equation has no more than five variables, the sixth variable of equation (11) having been eliminated by means of equation (12).

The expression (15) is considerably simplified if the following hypotheses are made.

The first, which can hardly be avoided, is that the transport of dry air and that of the water vapour from the exterior into the cumulus are not accomplished independently. That is to say that the mixing ratio remains constant in the second system, or indeed that

$$d\tau_v'' = 0. \tag{17}$$

The second, which is more disputable, is that the transformations of the open second system are adiabatic,[1] that is to say that (e.g. VAN MIEGHEM and DUFOUR, 1948, § 41)

$$dH'' - h''(dm_a'' + dm_v'') - V''dp = 0 \tag{18}$$

where $h'' = H''/m_a'' + m_v''$ is the specific enthalpy of the second system.

Having regard to (3), (4) and (10), the equation (18) takes the following form

$$(m_a''c_{pa} + m_v''c_{pv})dT'' + \frac{(m_a'')^2(h_a'' - h_v'')d\tau_v''}{m_a'' + m_v''} - V''dp = 0 \tag{19}$$

or, considering (17),

$$(m_a''c_{pa} + m_v''c_{pv})dT'' - V''dp = 0 \tag{20}$$

which is nothing other than the equation of adiabatic transformations of humid air in the second system when it is supposed closed.

It is interesting to note that hypothesis (18), considering hypothesis (17), is identical to hypothesis (20).

On introducing the hypothesis (17) and (20) into the relation (15), the equation (14) becomes, having regard to (8)

$$\frac{dT'}{dp} = \frac{\dfrac{m_a'R_aT'}{p - p_v'}\left(1 + \dfrac{\tau_v'L_v'}{R_aT'}\right) + [(c_{pa} + \tau_v''c_{pv})(T' - T'') + L_v'(\tau_v' - \tau_v'')]\dfrac{dm_a''}{dp}}{(m_a'c_{pa} + m_v'c_{pv} + m_e'c_{pe}) + \dfrac{m_a'\tau_v'(\varepsilon + \tau_v')(L_v')^2}{R_a(T')^2}}. \tag{21}$$

It is generally agreed among meteorologists that p_v' is negligible compared with p and that it is the same for $\tau_v''c_{pv}$ compared with c_{pa}, for τ_v' compared to ε and for $m_v'c_{pv} + m_e'c_{pe}$ compared with $m_a'c_{pa}$. Employing these approximations, the equation (21) takes the following form

$$\frac{dT'}{dp} = \frac{\dfrac{R_aT'}{c_{pa}p}\left(1 + \dfrac{\tau_v'L_v'}{R_aT'}\right) + \left[(T' - T'') + \dfrac{L_v'}{c_{pa}}(\tau_v' - \tau_v'')\right]\dfrac{1}{m_a'}\dfrac{dm_a''}{dp}}{1 + \dfrac{\varepsilon\tau_v'(L_v')^2}{c_{pa}R_a(T')^2}} \tag{22}$$

Stommel seems to be the first who studied thermodynamically the problem of entrainment of air into a cumulus. He supposes that the complicated transformations which occur in the cloud can be replaced by the three following transformations. First, the temperature decreases adiabatically because of the fall in pressure; secondly, the air which is entrained is mixed at constant pressure with the cloud air and thirdly, always at constant pressure, the liquid water held in the cloud is evaporated until saturation is reached. The formula which he obtains (STOMMEL, 1947, formula 4) is inexact. This is due to the fact that he considers in the first transformation that the decrease of of temperature follows a dry adiabatic, whereas it follows a saturated adiabatic while the cloud contains liquid vapour. On taking account of this fact one arrives at the formula (22) by following Stommel's statement. The calculations are long but present no special difficulties.

Taking up Stommel's idea, AUSTIN (1948) has first studied the problem graphically, supposing that the transformation we have referred to is saturated. Later (1948) he established

[1] This is the first time, to our knowledge, that the equation of the adiabatic transformations of an open system has been used in meteorology.

formula (22), in collaboration with Fleisher, by means of an intuitive thermodynamical reasoning which is simple but rather dangerous. In reality, it is not formula (22) that these authors have established but that which can be deduced by supposing that the air is in hydrostatic equilibrium in the cloud, that is to say that

$$\frac{dp}{dz} = -\varrho' g \tag{23}$$

where z is the altitude, reckoned positive towards the zenith, g the acceleration under gravity and ϱ' the density. The use of relation (23) is not admissible in this case because equally one has

$$\frac{dp}{dz} = -\varrho'' g \tag{24}$$

where ϱ'' is the density of surrounding air, and it therefore cannot be considered that equation (23) gives the relation between pressure and altitudes unless $\varrho' = \varrho''$, which is not the case. On the other hand, it is preferable to avoid making this hypothesis even in the classical case of a system of the same constitution as the prime system but which is closed instead of being open, because this hypothesis does not satisfy the equation of continuity.

We now return to equation (22). Observation seems to indicate that the rate of mixing of dry air is proportional to the mass of the cloud, that is to say that

$$dm_a'' \simeq k' m_a' dp. \tag{25}$$

Taking account of equation (25), the equation (22) can be written

$$\frac{dT'}{dp} = \frac{\dfrac{R_a T'}{c_{pa} p}\left(1 + \dfrac{\tau_v' L_v'}{R_a T'}\right) + k\left[(T' - T'') + \dfrac{L_v'}{c_{pa}}(\tau_v' - \tau_v'')\right]}{1 + \dfrac{\varepsilon \tau_v' (L_v')^2}{c_{pa} R_a (T')^2}}. \tag{26}$$

When $k = 0$, that is to say when there is no mixing, the equation (26) takes the classical form of the rate of variation of temperature with pressure during a saturated adiabatic transformation:

$$\frac{dT'}{dp} = \frac{\dfrac{R_a T'}{c_{pa} p}\left(1 + \dfrac{\tau_v' L_v'}{R_a T'}\right)}{1 + \dfrac{\varepsilon \tau_v' (L_v')^2}{c_{pa} R_a (T')^2}}. \tag{27}$$

During the formation of a cumulus, observation shows that it is colder air which is introduced into the cloud, so that $T' - T'' > 0$ and, in consequence, $\tau_v' - \tau_v'' > 0$. The rate of mixing of dry air dm_a''/dp being positive, it arises that the second term of the second member of (26) is positive and is the bigger, the larger are $T' - T''$, $\tau_a' - \tau_v''$ and dm_a''/dp. Theoretically, the entrainment of the air into a cumulus therefore increases the rate of change of the temperature in a cloud with respect to pressure, as is also indicated by the observations, which show that this rate is greater than that given by relation (27).

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