Focusing of Paraxial Rays in a Magnetic Dipole Field

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Abstract

It is well known that a magnetic field of axial symmetry has a focusing effect for paraxial orbits. The field of the earth is of this type and it is shown that, if the cosmic rays consist of protons, a bundle of orbits parallel at infinite distance is focused on the surface of the earth if the energy is about 10 GeV. The co-latitude of the bundle at infinite distance is about 2.2 times the co-latitude of the focal point. The asymptote to the orbits is about 22° further to the east than the focal point. There is also an infinite series of lower energies exhibiting focusingproperties.

The calculations refer to paraxial orbits, but a comparison with terrella measurements shows that the focal energy and the above relation between the co-latitudes is appropriate down to about 50° latitude.

I

In a homogeneous magnetic field charged particles move along helical paths. That means that after one gyro period the particles return to the magnetic vector line on which they were originally situated. Because of the symmetry all particles coming from one point, with paths making all the same, arbitrary angle with the magnetic vector lines, again meet after one gyro period. We thus have a focusing effect. If this angle is small compared with one radian trajectories filling a cone with this top angle will give a good approximation to a focus after one turn.

In all fields of cylindrical symmetry we have a similar effect. If the field strength decreases rapidly enough with the radial distance we can also arrive at a parallel beam at infinite distance.

Let us assume a field of rotational symmetry with all its sources at origin. Let us emit particles from a point on the axis in a small cone round the axis. If the energy is infinite all particles will move in straight lines and the cone will persist also at infinite distance. If we now decrease the energy we know from the previous discussion that the beam must sooner or later become parallel at infinite distance (Fig. 1). That means that in this energy interval the solid angle at infinite distance must decrease for decreasing energy when the solid angle at the point of emission is kept constant. When still lowering the energy the solid angle increase again to a maximum and decreases then to zero when we again have a parallel beam. This procedure can be repeated indefinitely. Between two such consecutive zeros there must be a focus at infinite distance.

If the point of emission is off the axis we get a similar effect. The central ray, however, now is bent, and further the double roots to the solid angle-energy curve are split into single roots, because of the astigmatism of the reproduction.

II

The earth has a dipole magnetic field. Let us study the motion of protons in this field near the dipole axis. Fig. 2 gives the ratio

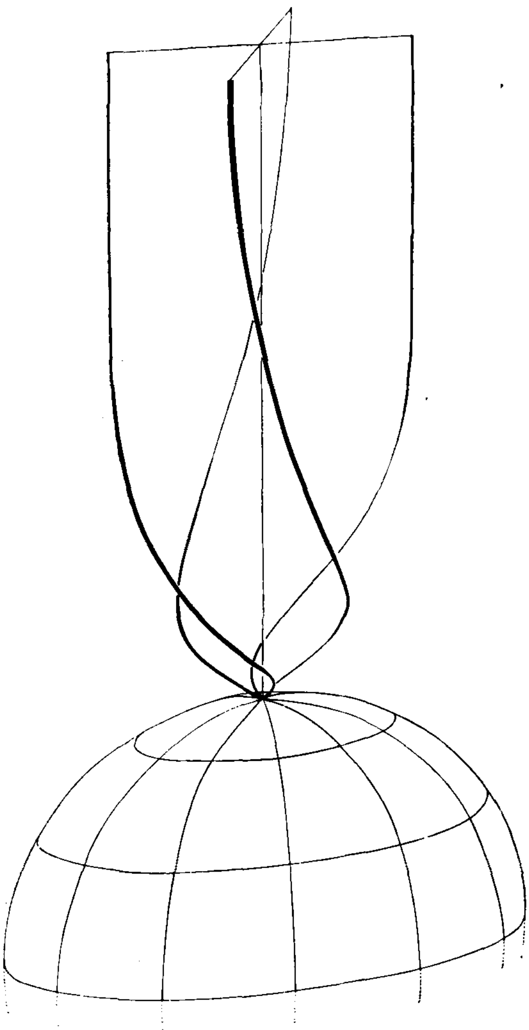

Fig. 1. A beam of protons with the energy of 9.88 GeV arriving at the pole of the earth. The beam is parallel at infinite distance.

of the solid angle at infinite distance to the solid angle at the point of emission as a function of the kinetic energy of the particle. We can see that above 50 GeV the solid angle is roughly unchanged by the magnetic field. Then the mentioned ratio drops rapidly and is zero at about 10 GeV. The important feature is that for energies below this value this ratio never exceeds 0.1. That means that in this whole energy range the top angle of the cone at infinite distance is much smaller than the same angle at the point of emission.

Table 1 gives some of the values of the energy, for which the beam is parallel at infinity. The highest energy for which this happens is approximately 10 GeV. This and a few of the following values are of importance in the applications.

Table 1

χ	V GeV	$\varkappa$	Ψ
5.5618	9.88	2.233	20°.67
11.8122	4.23	2.686	21.60
18.0848	2.50	2.989	21.91
24.360	1.70	3.215	22.06
30.642	1.23	3.404	22.15
36.924	0.937	3.566	22.21
43.205	0.734	3.709	22.25
49.487	0.597	3.837	22.28
55.770	0.492	3.953	22.31

Let us now study the bending of the beam. We are interested in protons which hit the earth perpendicular to its surface. Then the co-latitude for a point at infinite distance is greater than the co-latitude for the point on the earth where the trajectory ends. The ratio between these two quantities, $\varkappa$, is given in table 1. At the highest energy for which a beam, parallel at infinite distance, is focused on the surface of the earth, 10 GeV, $\varkappa$ is about 2.2.

The point at infinite distance on a trajectory is more towards the east than the point on the earth. The difference in longitude, Ψ_E, for these two points is given in table 1, for those

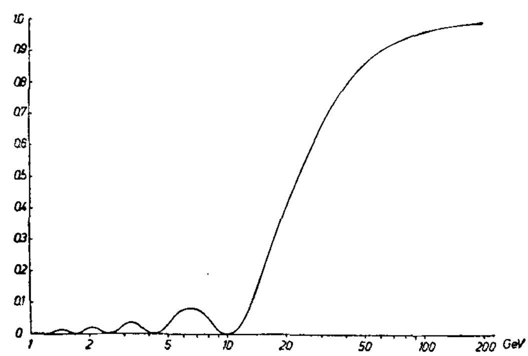

Fig. 2. The ratio of the solid angle at infinite distance to the solid angle at the surface of the earth for a beam of protons. The quantitites refer to the velocity space. Paraxial conditions have been assumed.

values of the energy which give focusing. As we see, it is almost independent of the energy and equal to about 22°, tending towards 22.5 when the energy decreases towards zero.

III

BRUNBERG (1953) and BRUNBERG and DATTNER (1953) have made measurements for energies down to about 10 GeV, and the measured focusing energy is in accordance with the theory. For $\varkappa$ they get 2.2 and further the linear relationship between Φ_N and λ extends to $\lambda = 50°$ (Fig. 3) which is far beyond the limit

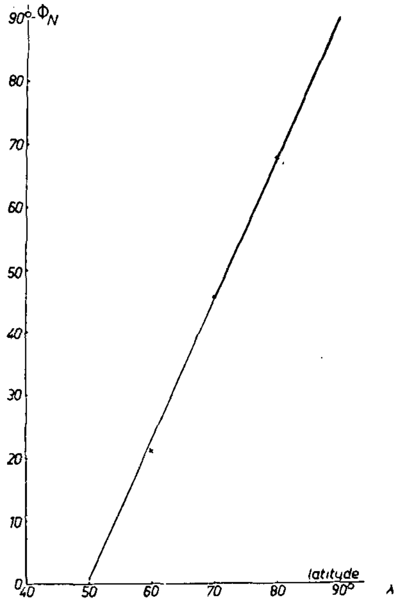

Fig. 3. The latitude Φ_N at infinite distance for a beam of protons with the energy of 9.88 GeV arriving at the surface of the earth at the latitude λ. The crosses refer to values obtained by model experiments made by Brunberg and Dattner.

of our approximation. The values of Ψ_E are about 20°, but the experimental values begin to increase already at fairly high latitudes (Fig. 4).

In the region 1—10 GeV MALMFORS (1945) has made a model experiment. From the measurements the values of the energy giving focusing for paraxial rays cannot be obtained. For $\varkappa$ we get the values 2.3, 3.0, 3.2, and 4.0 for the four highest values of the energy in

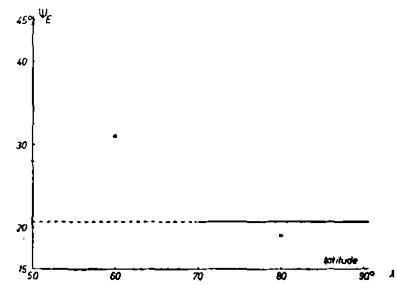

Fig. 4. Eastward longitude at infinite distance measured with reference to the point of arrival on the earth of a beam of protons with the energy of 9.88 GeV. The crosses refer to values obtained by model experiments made by Brunberg and Dattner.

table 1. The three first agree within the probable deviation with the theoretical values. In table 2 some of the values Ψ_E are reproduced. If we remember that the error in the experiment may be about 5° we can conclude that the agreement between theory and experiment is satisfactory.

Table 2

λ	Ψ_E			
	9.9	4.2	2.5	1.7 GeV
80°	20°	23	24	25
75	22	27	27	29
70	23	29	31	37

IV

If we apply these results to the measurements of cosmic radiation we thus find that for a discrete series of energies particles arriving in the zenith direction in a cone of finite value of the solid angle come from a well defined direction at infinite distance. That means that with a measuring equipment measuring over a fairly large solid angle we can measure the radiation from a well defined direction in space. Further, the direction in longitude is practically the same for all focal energies, which is suitable, but the latitude depends on the energy, which means that we have to restrict the measurement to a certain energy interval, if we require a fairly well defined direction at infinite distance. Figure 1 shows that for practical purpose we can accept the direction of a trajectory at a distance of a few times the radius of the earth to be the same as at infinite distance.

Let us now study paraxial rays in a magnetic dipole field somewhat more in detail.

In a magnetic field of strength B the equation of motion of a particle of charge e and rest mass m is

$$m\ddot{\mathbf{r}} = e\dot{\mathbf{r}} \times \mathbf{B} \tag{1}$$

In order to secure validity also in the relativistic region a dot represents differentiation with respect to the eigen time (BECKER, 1933).

Let the axis of the dipole field be along the z-axis. In order to obtain equations containing only one dependent variable it has been found suitable to introduce a frame rotating round the z-axis (GLASER, 1952). Let us at the same time take the radius of the earth, R, as the unit of length

$$\left.\begin{aligned} R\xi &= x\cos\Theta + y\sin\Theta \\ R\eta &= -x\sin\Theta + y\cos\Theta \end{aligned}\right\} \tag{2}$$

Here Θ is a function of time.

On the axis the radial component of the magnetic field is zero and an approximate value near the axis is easily obtained by integrating div $\mathbf{B} = 0$, while assuming B_z depending only on z. We get

$$B_r \approx -\frac{r}{2}\frac{\partial B_z}{\partial z}$$

where B is the component of the field along the axis. With these notations our equations write

$$\left.\begin{aligned} &\ddot{\xi} - \dot{\Theta}\left(\dot{\Theta} + \frac{eB_z}{m}\right)\xi - \left(2\dot{\Theta} + \frac{eB_z}{m}\right)\dot{\eta} - \\ &\quad - \left(\ddot{\Theta} + \frac{1}{2}\frac{e}{m}\frac{\partial B_z}{\partial z}\dot{z}\right)\eta = 0 \\ &\ddot{\eta} - \dot{\Theta}\left(\dot{\Theta} + \frac{eB_z}{m}\right)\eta + \left(2\dot{\Theta} + \frac{eB_z}{m}\right)\dot{\xi} + \\ &\quad + \left(\ddot{\Theta} + \frac{1}{2}\frac{e}{m}\frac{\partial B_z}{\partial z}\dot{z}\right)\xi = 0 \end{aligned}\right\} \tag{3}$$

Let us put the coefficient for $\dot{\xi}$ and $\dot{\eta}$ equal to zero, i.e.

$$\dot{\Theta} = -\frac{1}{2}\frac{eB_z}{m} \tag{4}$$

By differentiating this relation, remembering that B_z is static and since we keep to points near the axis, in the first approximation independent of the radial distance, we find that also the last term in the equation (3) vanishes. We have thus arrived at identical equations for ξ and η, namely

$$\ddot{\xi} + \left(\frac{1}{2}\frac{e}{m}B_z\right)^2 \xi = 0 \tag{5}$$

In a dipole field, with the dipole strength a, $B_z = \frac{\mu_0}{4\pi}\frac{2a}{z^3}$. Since now the coefficient in our equation is a function of z we would like to have z as independent variable. We discuss paraxial rays. Then z is almost equivalent to the magnitude of the velocity of the particle (which we define according to Minkowski), i.e., we can use the relation $z \approx v\tau$ where τ is the eigen time. Further, the radius of the earth, R, is a suitable unit for the length, and consequently the new independent variable is $\zeta = z/R$. Let a prim denote a differentiation with respect to ζ. To shorten the writing we introduce the parameter

$$\chi = \frac{\mu_0}{4\pi}\frac{ea}{mvR^2} \tag{6}$$

Then our equations writes

$$\xi'' + \frac{\chi^2}{\zeta^6}\xi = 0 \tag{7}$$

and the solution is

$$\xi = A\zeta^{1/2} J_{1/4}\left(\frac{\chi}{2\zeta^2}\right) + B\zeta^{1/2} J_{-1/4}\left(\frac{\chi}{2\zeta^2}\right) \tag{8}$$

where A and B are arbitrary constants. Since we have the same solution for η we can generally write

$$\begin{aligned} \xi &= Au(\zeta) + Bw(\zeta) \\ \eta &= Cu \quad\;\, + Dw \end{aligned} \tag{9}$$

where u and w are two independent solutions.

We are interested in the behaviour at infinite distance of orbits passing a certain arbitrary point on the unit sphere. Since we discuss points near the axis x, y, x', and y' are given at $\zeta=1$. This enables us to calculate A, B, C, and D.

$$\left.\begin{aligned} kA &= xw' - (x' + \Theta' y)\, w, \\ kC &= yw' - (y' - \Theta' x)\, w, \\ kB &= -xu' + (x' + \Theta' y)\, u, \\ kD &= -yu' + (y' - \Theta' x)\, u \end{aligned}\right\} \quad (10)$$

with $k = uw' - u'w$.

We are now specially interested in the direction of the orbits when $z \to \infty$. But lim $(\xi\Theta') \to 0$ when $\zeta \to \infty$ whatever value we attribute to A and B, which means that we can neglect the rotation of the $\xi\eta$-frame at that distance. Then is

$$\begin{aligned} kx'(\infty) &= \{u'(\infty)w' - u'w'(\infty)\}\,(x\cos\Theta - \\ &- y\sin\Theta) - \{u'(\infty)w - uw'(\infty)\}\,\{\Theta'(x\sin\Theta + \\ &+ y\cos\Theta) + (x'\cos\Theta - y'\sin\Theta)\} \\ ky'(\infty) &= \{u'(\infty)w' - u'w'(\infty)\}\,(x\sin\Theta + \\ &+ y\cos\Theta) - \{u'(\infty)w - uw'(\infty)\} \times \\ &\times \{\Theta'(-x\cos\Theta + y\sin\Theta) + (x'\sin\Theta + y'\cos\Theta)\} \end{aligned} \quad (11)$$

In these formulae the value of Θ refers to $\zeta=\infty$, but all the other quantities without explicitly given arguments refer to $\zeta = 1$.

Let us look at the trajectories inside a cone with the top angle $2\sigma \ll 1$ radian. The solid angle is then approximately $\pi\sigma^2$. At infinite distance the boundaries of the cone, in the velocity space, is given by

$$\begin{aligned} kx'(\infty) &= a - \{u'(\infty)w - uw'(\infty)\}\sigma\cos(\Theta + \varphi) \\ ky'(\infty) &= b - \{u'(\infty)w - uw'(\infty)\}\sigma\sin(\Theta + \varphi) \end{aligned} \quad (12)$$

where a and b are constants of no interest in this connection and φ is a parameter. The solid angle is now $\vartheta\pi\sigma^2$ where

$$\vartheta = k^{-2}\{u'(\infty)w - uw'(\infty)\}^2 = \left\{\Lambda_{1/4}\left(\frac{1}{2}\chi\right)\right\}^2 \quad (13)$$

To simplify the notation we have introduced Λ_p from Jahnke and Emde (1948), defined by

$$J_p(x) = \frac{\left(\frac{1}{2}x\right)^2}{p!}\,\Lambda_p(x)$$

Consequently ϑ is the ratio of the solid angle at infinite distance to the solid angle at the distance $\zeta=1$. For infinite energy of the particle $\chi=0$ and then is $\vartheta=1$, i.e., the trajectories are not influenced by the magnetic field. The equation $\vartheta=0$ has an infinite number of roots, giving the energies for which the bundle of rays, we study, is parallel at infinite distance. We get merely double roots, which means that all rays really have the same direction and are not merely laying in one plane. The further we proceed from the polar region the more astigmatic will the reproduction become, which means that the double roots are split into single roots.

Let us study a certain trajectory and its intersection with two spheres with the radius equal to ∞ and 1, respectively. The corresponding latitudes may be called Φ_N and λ and the difference in longitude Ψ_E. The latter quantity is positive when the point at infinite distance is eastwards with respect to the other point. Then (11) shows that the co-latitudes are proportional to each other. At those values χ_n giving $\vartheta=0$ we easily get

$$\frac{\pi}{2} - \Phi_N = \varkappa_n\left(\frac{\pi}{2} - \lambda\right)$$

with

$$\varkappa_n = \Lambda_{-3/4}\left(\frac{1}{2}\chi_n\right)$$

For the calculation of Ψ_E we need to know Θ. For paraxial rays v_z is approximately constant and we can immediately integrate (4) to

$$\Theta(\infty) - \Theta(1) = \mp\frac{1}{2}\chi \quad (15)$$

with the upper sign for positively and the lower sign for negatively charged particles. Θ gives the angle the frame has rotated, but it is not, in general, the angle a particle has moved in the fixed frame. Let us, therefore, now turn our attention to the motion of the particle in

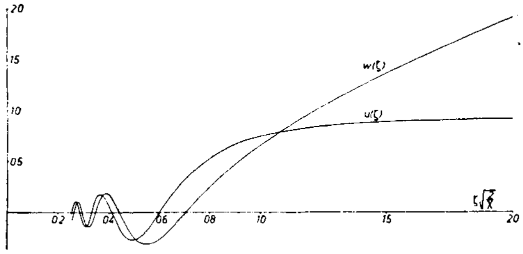

Fig. 5.

the $\xi\eta$-frame. We wish the particle to move normal to the sphere of radius 1 at its surface. Let us thus put $y' = y = 0$ at $\zeta = 1$. Since we are specially interested in the case when the beam is focused, i.e., $\vartheta = 0$ we can also put $u(1) = 0$. Then (9) and (10) give

$$\left.\begin{aligned} k\xi\ (\zeta) &= (xw' - x'w)\ u\ (\zeta) - xu'w(\zeta) \\ k\eta\ (\zeta) &= x\Theta' w\ u\ (\zeta) \end{aligned}\right\} \quad (16)$$

Let us in the $\xi\eta$-frame draw the line $w = 0$ and the line $u = 0$ and use these as the u- and w-axes, respectively, of a reference system. Since the roots of u and w are alternating (Fig. 5) equation (16) shows that the point in the uw-frame rotates round origin when ζ goes from 1 to ∞. Since there are a finite number of roots above every value ζ which is different from zero the number of turns are finite. The particle starts at $\zeta = 1$ on the u-axis and since $u(\infty)$ is finite, but w increases indefinitely with ζ, then the limiting direction is parallel to the u-axis. Let us now number the roots of $u(s) = 0$ so that there are no roots for $s > s_1$, and so that $s_1 > s_2 > s_3$, etc. Then $w = 0$ has n roots in the region $s_n < s < \infty$. That means that if we choose the scale (i.e., the value of χ) so that $s_n = 1$, then the beam has rotated $\frac{1}{2}\, n$ turns when ζ goes from 1 to ∞. Since the u-axis falls along the ξ-axis the number of turns is the same in the $\xi\eta$-frame.

u and w have the same sign above the highest root of $w(s) = 0$. That means that u' and w have different sign at the roots of $u(s) = 0$. From that follows that our point in the uw-frame rotates in positive direction if $\Theta'(1)$ is negative and vice versa. But the frame itself rotates with the angular velocity Θ, which has the same sign along the whole path. We conclude that the absolute rotation (in the xy-frame) of the particle is thus the difference of these contributions.

The cosmic ray particles are mainly positively charged and come from large distance towards the earth, but in the present paper we have discussed particles moving in the opposite direction. If we change the direction of the magnetic field of the earth the cosmic ray particles can follow the former paths but in opposite direction. In applying our results to the cosmic rays we thus have to put $a > 0$ and $e > 0$. Then we get

$$\Psi_E = n\pi - \frac{1}{2}\chi_n \quad (17)$$

In cosmic ray physics the energy of a particle is usually given in electron volts. Since the energy of a particle accelerated from rest through the potential V is $eV + mc^2 = -imcv_4$ and $v_s^2 + v_4^2 + c^2 = 1$ we get, by introducing v from equation (6), the following relation between V and χ

$$V = \left\{\left(\frac{mc^2}{e}\right)^2 + \left(\frac{\mu_0}{4\pi}\frac{ac}{R^2}\frac{1}{\chi}\right)^2\right\}^{1/2} - \frac{mc^2}{e} \quad (18)$$

In our calculations we have for the dipole momentum of the magnetic field of the earth used the value $a = 0.811 \cdot 10^{23}\ Am^2$.

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