Electron Orbits in Hyperbolic Magnetic Fields

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(Manuscript received February 1, 1956)

Abstract

Electron motion in the field from cylindrical pole pieces with the section in the shape of a rectangular hyperbola and its conjugate is treated. The accessible region is discussed. For the motion in the asymptote planes to the hyperbola the orbits are characterized by two parameters, e.g. one (k), characterizing the shape, and the other the amplitude. The solution is given in elliptic functions. The orbits may be divided into two groups, one composed of orbits resembling trochoids and the other orbits of a meander type. There exists a periodic orbit similar to the diget 8. The drift is in opposite directions on different sides of this orbit, as characterized by k.

The trochoidal motion is stable if the orbit is not stretched too much. For the meander orbits the criterion used gives a main stability region. This region is bounded by the periodic orbit and the orbit the branches of which just touch each other. For both groups of orbits there exists an infinite number of stable regions.

Introduction

In his theory of aurora VILLARD (1908) discussed the motion of a charged particle in an inhomogenous magnetic field. Some years later GUNN (1929) pointed out that this same motion was of importance in the theory of diamagnetism of an ionized gas and discussed the geophysical and astrophysical applications. He also gave a formula for the drift velocity, which is correct for sufficiently small amplitudes except for a factor $\frac{\pi}{2}$. Four years later THIBAUD (1933) used this type of motion in his recherches on the positron and CARTAN (1934) gave the correct, relativistic formula for the drift velocity. He restricted the discussion to the case when the difference between the maximum and minimum value of the field on a trajectory is only a small fraction of the mean value of the field. In some applications this is a serious limitation, and therefore we shall, in this paper, try to calculate the trajectories for the case when there is a linear gradient in the magnetic field strength but there is no limitation in the range of the amplitude of the velocity. If we require that there shall be no sources of the magnetic field in the region where we study the motion of the charged particles the linearity can hold only in one plane. We shall, therefore, also study the stability of the motion for small deviations of the path from the mentioned plane.

1. Let us examine the motion of charged particles in the magnetic field shown in figure 1. The field is produced by two conjugate, rectangular hyperbolic cylinders, having opposite polarity. The axis of the cylinders coincides with the x_3-axis, and the x_1- and the x_2-axes form the asymptotes. The field thus produced is a Laplace field in infinite space and it is therefore possible to put the pole pieces as far away as we like. In the present investigation we shall think of them as situated at infinite distance. The magnetic vector lines are also rectangular hyperbolas. From Fig. 1 we can see that in the x_1x_3 plane the magnetic

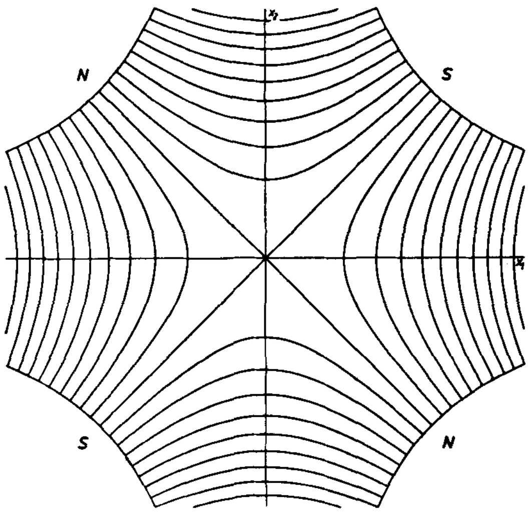

Figure 1. The magnetic field.

field is perpendicular to that plane. Since the magnetic forces act perpendicularly to the magnetic vector lines there exist trajectories entirely confined to the plane $x_2 = 0$.

$$\mathbf{B} = 2\frac{m}{e} a \,|x_2,\ x_1,\ 0| \tag{1}$$

where a is an arbitrary constant, which means that the field strength is a linear function of the x_1-coordinate in the mentioned plane. Thus this motion takes place in the simplest type of an inhomogeneous magnetic field.

We would like to make the discussion relativistically correct but at the same time, as far as possible, keep to the formality used in non-relativistic literature. Then it is suitable to let v_μ be the component of the Minkowski velocity and τ the "eigen-time" (BECKER, 1933). In the low energy region $|v_1,\ v_2,\ v_3|$ and τ will be the common velocity and the ordinary time, respectively. For v (without index) we shall use the term velocity and mean the projection of the Minkowski velocity on a plane perpendicular to the fourth axis. In the non-relativistic region then v is the absolute value of the velocity. Further we ought to remember, that v_4/ic can be put equal to unity in the non-relativistic velocity range.

Sometimes it is very important to remember that B shall be used in the equation of motion and not $\mu_0 H$. Since we consequently use the MKS system this condition is automatically satisfied. The equation of motion writes, if we take B from (1).

$$\frac{d\mathbf{v}}{d\tau} = 2a\,|-v_3 x_1,\ -v_3 x_2,\ v_1 x_1 - v_2 x_2| \tag{2}$$

The last of these equations can immediately be integrated, giving

$$v_3 = a(x_1^2 - x_2^2) - b \tag{3}$$

where b is a constant of integration.

It is easy to verify that $v_3 = \text{constant}$ is also the equation for the magnetic vector lines. We only have to compare the derivative of (3) for the former lines with the direction of the latter lines, as given by (1). If we now discuss the projection of the trajectory in the $x_1 x_2$ plane, we can say that for one and the same trajectory v_3 has a definite value along each magnetic vector line.

The magnetic vector lines are in whole space situated in the $x_1 x_2$ plane. Equation (3) shows that the velocity v_3 perpendicular to this plane is independent of x_3. If we introduce v_3 from equation (3) into equation (2) then only x_1 and x_2 remain as dependent variables. The problem has thus been reduced from a three-dimensional to a two-dimensional one. Since the electric field strength is zero the velocity remains constant along a trajectory. We, therefore, immediately get

$$v_1^2 + v_2^2 = v^2 - v_1^2 = \{v - b + a\,(x_1^2 - x_2^2)\} \times \\ \times \{v + b - a(x_1^2 - x_2^2)\} \tag{4}$$

The particles are confined to regions in which the r.h.s. is positive. The boundaries of this region are formed by the two hyperbolas

$$x_1^2 - x_2^2 = (b \pm v)/a \tag{5}$$

Since b is the velocity the particle has in the x_3-direction on the asymptotes to these hyperbolas the particles cannot pass these asymptotes when b is numerically greater than v but they can pass when b is numerically smaller than v. In the former case the two limiting hyperbolas are thus situated in the same quadrants and in the latter in different ones. In Fig. 2 the different possibilities are shown. The shadowed regions are the accessible regions. We find

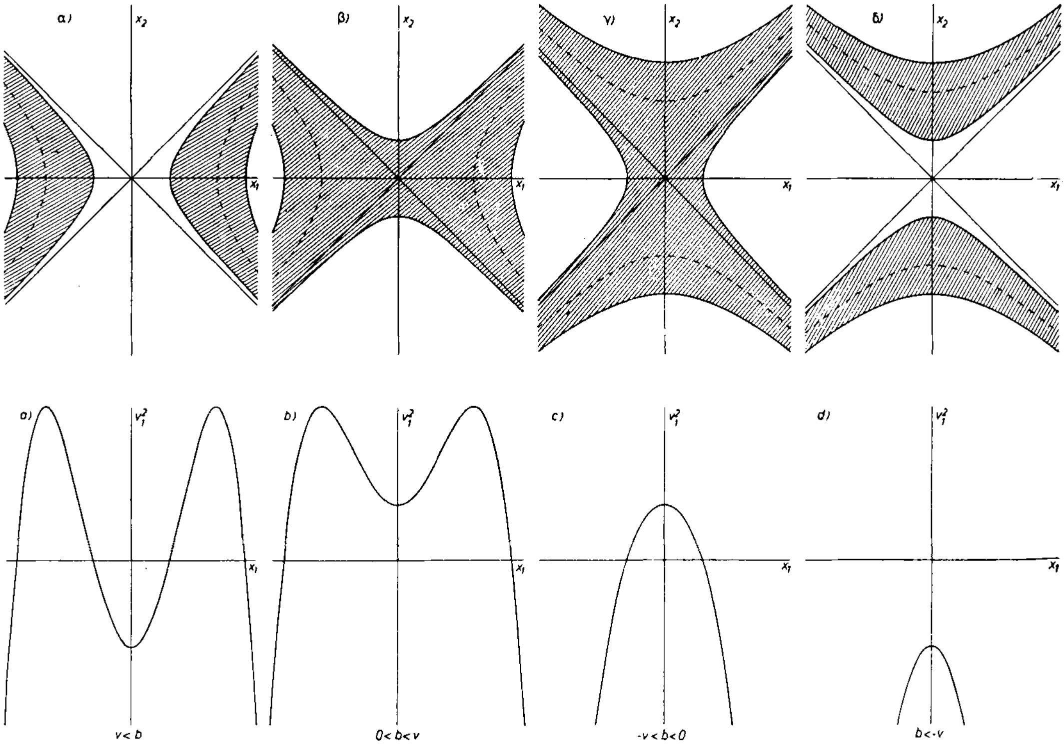

Figure 2. The accessible regions (α—δ) and the velocity $v_1(x_1)$ when $x_2 = 0$ (a—d).

that they extend to infinity along four with distance indefinitely narrowing channels. These latter are parallel to the magnetic vector lines, which fact makes their existence easily understandable. On the dashed lines in Fig. 2 $v_3 = 0$ and on different sides of these dashed lines v_3 has different sign. On the boundaries of the accessible region v_3 has attained its maximum value.

When we discuss the case illustrated in Fig. 2 α we can first think of a homogeneous magnetic field with the magnetic vector lines parallel to the x_2-axis. The trajectory is a circle, if we forget the motion along the vector lines. Let us distort the field so that it varies with the x_1-coordinate, i.e., with a coordinate perpendicular to the field. Then the trajectory get a trochoidal character (Fig. 3 j) with a drift motion in the x_3-direction, i.e., in a direction perpendicular to both the field and the gradient of the energy density of the field. If now the field strength increases with the distance (in both directions) from a certain plane perpendicular to the field the trajectory will oscillate up and down through the mentioned plane.

The field shown in Fig. 1 is, in principle, of that character we have just discussed and the trajectories, in the case illustrated in Fig. 2 α, are, in principle, of the character we have just deduced. We must add 'in principle' since, as we will find from the examination of the stability of the orbit in the plane $x_2 = 0$, there exist cases when our discussion fails to give the correct answer.

Fig. 2 δ is Fig. 2 α turned $\frac{\pi}{2}$ radians in space. In case 2 β and γ the motion in the channels round the asymptotes has still the same character as in the former case. The motion in the central region round the origin, however, generally is quite erratic. For the limiting case between Fig. 2 β and γ we find that $b = 0$ and then equation (3) says that v_3 changes sign when x_1 and x_2 are exchanged for each other. Since, for the motion in the

x_1x_2 plane, the x_1- and the x_2-directions are equivalent we thus may expect the drift velocity in the x_3-direction to be zero. In the x_1x_2 plane the motion is, generally, non-periodic and therefore we may expect the motion in the x_3-direction to be of similar character as the Brownian motion. Is this discussion generally true it fails for the singular case when the motion takes place in the $x_3\,x_1$ or x_3x_2 planes, but these trajectories are unstable for small deviations from the plane.

When the motion takes place in the plane $x_2=0$, v_1 can be plotted as function of the x_1-coordinate as is shown in Fig. 2 a—d. In case *a* motion is possible in two separate regions and the trajectories resemble trochoids, as shown in figure 3 i—k. When $b=v$ the two regions have just become united but it takes infinite time for a particle to pass the point connecting them. That means that the x_3-axis, on which the magnetic field strength is zero, is an asymptote to the trajectory, Fig. 3 h.

In the case illustrated in Fig. 2 b the trajectory passes from the region $x_1>0$, where the magnetic field has a certain direction, over the line $x_1=0$, in which the field strength is zero, to the region $x_1<0$, where the field has opposite direction. The centre of curvature is thus situated on different sides of the trajectory when $x_1>0$ and when $x_1<0$. The path has thus the crossing between the trajectory and the line $x_1=0$ as a centre of symmetry. The actual curves are given in Fig. 3 d—g. There is to be observed that a periodic orbit is possible, Fig. 3 e, and that the drift velocity has different sign on different sides of this orbit.

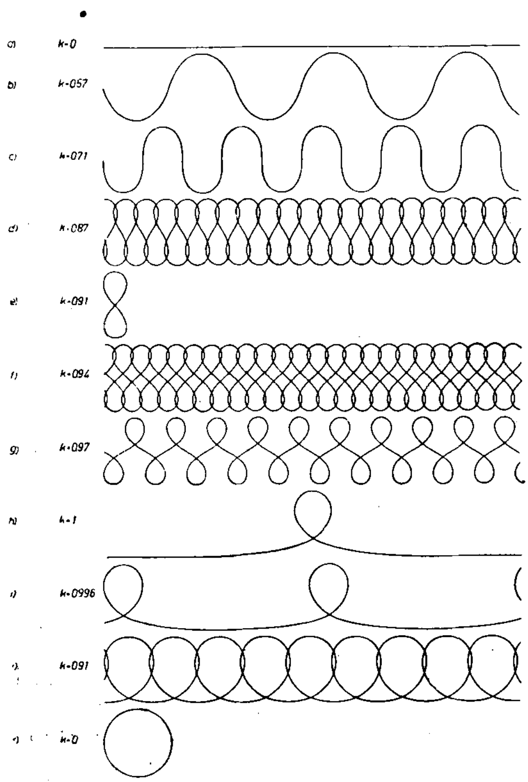

Figure 3. The possible types of trajectories.

When $b=0$ we have the symmetric case we have discussed above. Then the two maxima in Fig. 2 b have merged into one maximum with the value $v_1=v$. Consequently the trajectory will become that one shown in Fig. 3 c. In the case illustrated by Fig. 2 c the maximum value of v_1 is below v, and the trajectory will become that shown in Fig. 3 b. When $b=-v$ the trajectory is a straight line along the line on which the magnetic field strength vanishes. Finally, when $b<-v$, no motion is possible in the plane $x_2=0$.

2. For the quantitative discussion of the motion in the plane $x_2=0$ we may distinguish between three distinct cases, namely

1) Four real roots exist to the equation $v_1=0$ (Fig. 2 a).
2) Two real roots exist to the equation $v_1=0$ (Fig. 2 b and c).
3) No real roots exist to the equation $v_1=0$ (Fig. 2 d).

This third case is of no physical interest since the velocity is purely imaginary.

Case I.

Let us first examine the case when the path resembles a trochoid. In that case we require four real roots to the equation $v_1=0$. Since v is the magnitude of the velocity it is a positive quantity. Our requirement then can be written $v<b$. Let us then introduce the parameter $k=(2v/v+b)^{1/2}$. To simplify the notation it is convenient to introduce also k' defined by $k^2+k'^2=1$. With the new variables

$$z=k(a/2v)^{1/2}x_1 \qquad (6)$$
$$y=k^{-1}(2av)^{1/2}\tau$$

our differential equation (4) can now be written

$$\left(\frac{dz}{dy}\right)^2=(1-z^2)(z^2-k'^2) \qquad (7)$$

The solution is the elliptic function (JAHNKE and EMDE, 1948) $z = dn\ y$, i.e.

$$x_1 = k^{-1}(2v/a)^{1/2} dny \quad (8)$$

Then equation (4) gives

$$v_3 = v + (2v/k^2)\,(dn^2 y - 1) \quad (9)$$

After having introduced dx_1/dy instead of $dx_1/d\tau$ we may integrate;
We get

$$x_3 = k^{-1}\left(\frac{2v}{a}\right)^{1/2}\left(zny - \frac{k^4 \mathsf{C}}{2\mathsf{K}}\,y\right) \quad (10)$$

where C and K are elliptic integrals (JAHNKE and EMDE, 1948).
The Jacobian Zeta-function $zn\ y$ is periodic with the period 2K. Disregarding the drift the motion is periodic and the period, measured in seconds, is

$$T = (v_4/ic)\,2k(2av)^{-1/2}\,\mathsf{K} \quad (11)$$

The quantity v_4/ic appears because the relation between time t and the quantity τ is $t = v_4\tau/ic$. In non-relativistic cases v_4/ic is to be put equal to 1.

Upon this periodic motion a drift, represented by the last term in equation (10), is superimposed, and the velocity, measured in meters per second, is

$$u_{\text{drift}} = v(ic/v_4)k^2\mathsf{C}/\mathsf{K} = uk^2\mathsf{C}/\mathsf{K} \quad (12)$$

The factor v_4/ic enters for the same reason as above. If we exchange v for the classical velocity u the factor v_4/ic disappears but the theory is still relativistically correct.

Since 1 and k' are the maximum and minimum values of the Delta amplitudinis function, respectively, we also find that

$$x_{1\max} - x_{1\min} = (1 - k')k^{-1}(2v/a)^{1/2} \quad (13)$$

Our calculation shows that in the trochoidal case we exactly have

$$x_{1\max} - x_{1\min} = 2mv/e\,\frac{1}{2}\,(B_{\max} + B_{\min}) \quad (14)$$

where $B_{\max}$ and $B_{\min}$ are the maximum and minimum values of the magnetic field strength on the trajectory we examine. That means that the corresponding formula, which is derived for a homogeneous magnetic field, can be used also here, if we introduce the magnetic field strength in a point halfways between $x_{1\max}$ and $x_{1\min}$. For the further discussion it is useful to observe that $k' = B_{\min}/B_{\max}$ is a measure for the inhomogeneity in the magnetic field, as seen by the charged particle. When $k' = 1$ the particle moves in a circle (when the field is inhomogeneous the path has degenerated to a point: the particle is at rest) and when $k' = 0$ the trochoidal motion has degenerated to a single loop, as illustrated by Fig. 3 h. For the point x_1, where $v_3 = 0$, we find

$$\frac{x_1 - x_{1\min}}{x_{1\max} - x_{1\min}} = \frac{1}{2}\,\frac{1 + k'}{\sqrt{\frac{1}{2}(1 + k'^2) + k'}} \quad (15)$$

For the circular motion this expression has the value 1/2 as it ought to have for that case. For increasing value of k' the point where $v_3 = 0$ moves towards $x_{\max}$. For the limit of stable orbits ($k' = 0.51$) the last expression has become 0.58 and the limit is $2^{-1/2}$.

For the drift velocity we get the following approximate formula

$$u_{\text{drift}} \approx \frac{ic}{v_4}\,\frac{mv^2}{2e}\,\frac{\partial B}{\partial x_1}\left(\frac{B_{\max} + B_{\min}}{2}\right)^{-2} \quad (16)$$

Since the field varies linearly with the coordinate the value of the magnetic field in the denominator is the field we have halfways between $x_{1\max}$ and $x_{1\min}$. Our formula gives a value of the drift velocity less than about 1 % lower than the correct value for the main stable region, and even when $B_{\min}/B_{\max} = 0.2$ the value is only about 4 % too low. It thus seems to be a fairly good estimate of the drift velocity.

The period is given by

$$T = \frac{v_4}{ic}\,2\pi\left(\frac{e}{m}\,\frac{B_{\max} + B_{\min}}{2}\right)^{-1}\frac{1 + k'}{2}\,\frac{2}{\pi}\,\mathsf{K} \quad (17)$$

This formula can be divided into two parts; the first is the period in a homogeneous field with the strength we have halfways between $x_{1\max}$ and $x_{1\min}$ and the latter part is a correction factor. We find that the correction factor is below 1.03 in the main stability region but increases rapidly with further decreasing values of $B_{\min}/B_{\max}$. When the latter quantity has the value 0.2 the correction factor has become 1,13, i.e., the error, when dropping this term, is greater than 10 %.

We have found that the drift velocity is perpendicular to the magnetic field and also perpendicular to the gradient of the intensity of the magnetic field. We can use the gradient of the energy density of the field since its direction is the same as that of the previously mentioned gradient. Therefore we write

$$\mathbf{u}_{\mathrm{drift}} = \frac{ic}{v_4}\frac{mv^2}{4eB^4}\mathbf{B} \times \nabla B^2 \tag{18}$$

For a medium with the permeability equal to unity this expression can be brought to the form previously given by ALFVÉN (1950).

Case II.

In this case the path crosses the line on which the magnetic field vanishes. The crossing points are centres of symmetry to the path. It is suitable to introduce the parameter $k = (v + b/2v)^{1/2}$ and also k', the latter with the previously given definition. With the new variables

$$\begin{aligned} z &= k^{-1}(a/2v)^{1/2}x_1 \\ y &= (2av)^{1/2}\tau \end{aligned} \tag{19}$$

the differential equation (4) then can be written in the form

$$\left(\frac{dz}{dy}\right)^2 = (1 - z^2)(k'^2 + k^2 z^2) \tag{20}$$

The solution is the elliptic function $z = cn\ y$,

i.e.,
$$x_1 = k(2v/a)^{1/2} cn\ y \tag{21}$$

Equation (3) gives

$$v_3 = v(2\ dn^2 y - 1) \tag{22}$$

After changing the variable τ to y, we can as in the previous case then integrate, and become

$$x_3 = (v/2a)^{1/2}\left(2zny + \frac{2\mathsf{E} - \mathsf{K}}{\mathsf{K}}y\right) \tag{23}$$

The x_1 coordinate is given in the cosinus amplitudinis function, which has the period 4K. That is the period of the phenomenon, since x_1 has half that period. Measured in seconds it is

$$T = (v_4/ic)\, 4(2av)^{-1/2}\, \mathsf{K} \tag{24}$$

Also in this case the particle drifts in the x_3-direction and the drift velocity measured in meters per second is

$$u_{\mathrm{drift}} = v\frac{ic}{v_4}\frac{2\mathsf{E} - \mathsf{K}}{\mathsf{K}} = u\frac{2\mathsf{E} - \mathsf{K}}{\mathsf{K}} \tag{25}$$

where u is the velocity, in meters per second, of the particle. Since the function $cn\ y$ is confined between $+1$ and -1 we get

$$x_{1\max} - x_{1\min} = 2k(2v/a)^{1/2} \tag{26}$$

3. Up to now we have discussed motion of particles in a mathematical plane. In applications deviation from this conditions is unavoidable. We are therefore interested to know under which conditions the trajectories are stable for small perturbations out of the mentioned plane. In the present paper we shall take the term stability in the mathematical sense. Then we get regions in which the amplitude after infinite time is still finite. In the applications we always deal with trajectories of finite length and we also are interested in those cases in which the amplitude in this interval stays below the specified level. But those intervals are, in general, greater than the mathematically stable regions.

For getting an estimate of the main stability zone we shall use a criterion deduced by KREĬN (1951). He defines his equation as

$$\frac{d^2x_2}{dy^2} + \lambda p(y)x_2 = 0; \tag{27}$$

In § 1 and § 2 we have discussed the motion in the plane $x_2 = 0$. Let us now allow the particle to deviate from this plane, but so little that we can assume that the projection on the plane $x_2 = 0$ of the trajectory is still what we have just calculated.

Case I.

Then the second component equation of (2) with x_1 taken from equation (8) takes the form (27) with $\lambda = 1$, if we put

$$p = k^2\,(2\ sn^2 y - 1) \tag{28}$$

Kreĭn then defines a new function

$$p_+(y)=\begin{cases} p(y) & \text{when } p(y)>0 \\ O & \text{when } p(y)<0 \end{cases}$$

and defines

$$M=\int_0^l p_+(y)dy=(2-k^2)\times$$
$$\times\left[\mathrm{K}-F\left(\frac{\pi}{4}\right)\right]-2\left[\mathrm{E}-E\left(\frac{\pi}{4}\right)\right]$$

where the period of p is $l=2\mathrm{K}$. The maximum value of p is $H=k^2$. Now Kriěn states that we have stability, if

$$O<\lambda<\frac{4}{Ml}\frac{\chi(t)}{t} \text{ with } t=\frac{M}{Hl} \qquad (29)$$

The function χ is tabulated by SVARCMAN (1954). We find that $\lambda=1$ falls within the stable region when $k<0.858$.

Case II.

For the non-trochoidal motion x_1, obtained from equation (21), has to be put into equation (2). Equation (27), with $\lambda=1$, is then obtained, if we put

$$p=1-2\ dn^2\ y \qquad (30)$$

p is negative for all values of the argument when $k<2^{-1/2}$. The motion is thus unstable for these values of k. The Kreĭn criterion for stability requires that the mean value of p over one period shall be positive. This mean value, $1/2-\mathrm{E}/\mathrm{K}$, is positive when $k>0.9089$. The Kreĭn criterion now gives stability when $k<0.9068$. Since there are no values of k satisfying the two last unequalities at the same time this criterion fails to give any part of the stable region.

Both in case I and II the differential equation we have to solve is of Lamé type. Since I have not found any general discussion about the stability of this equation and since the criteria of stability, I have seen, are only partly useful, we shall try to approximate our equations to Mathieu equations. We may observe that $\left(\frac{2}{3}\right)^{1/2}(\mathrm{BD}+\mathrm{EC})^{1/2}\,\mathrm{K}^{-1}\cos\frac{\pi y}{\mathrm{K}}$ has the same period, linear and square mean value as $sn^2 y-\frac{\mathrm{D}}{\mathrm{K}}$. I think that a substitution on these grounds will give the best approximation to a Mathieu equation. If we write the latter in the usual form (McLACHLAN, 1947).

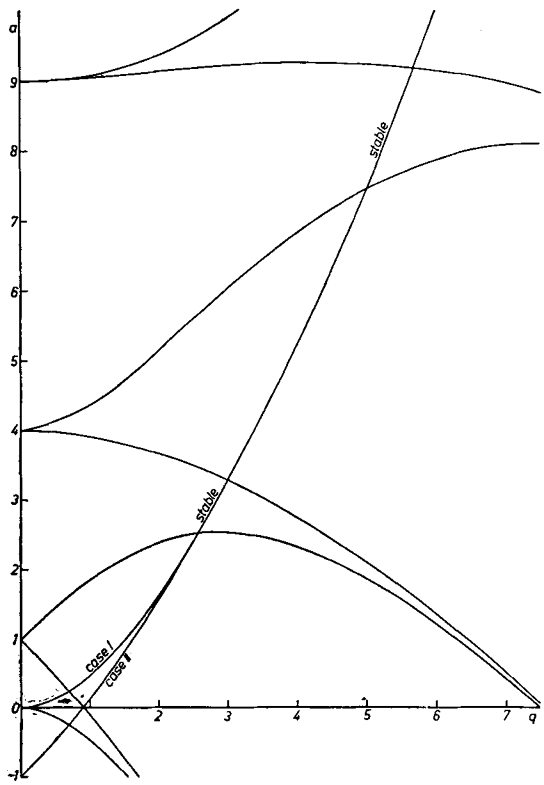

Figure 4. The stability chart for a Mathieu equation with the relation between q and a for our approximate equation introduced.

$$\frac{d^2x_2}{dz^2}+(a-2q\cos 2z)x_2=0 \qquad (31)$$

We will find for the constants

$$\left.\begin{aligned} q&=\frac{4}{\pi^2}\left(\frac{2}{3}\right)^{1/2}k^2\mathrm{K}(\mathrm{BD}+\mathrm{EC})^{1/2} \\ x&=\pi y/2\mathrm{K} \\ a&=\frac{4}{\pi^2}k^4\mathrm{CK} \qquad \text{(case I)} \\ a&=\frac{4}{\pi^2}\mathrm{K}(2\mathrm{E}-\mathrm{K}) \qquad \text{(case II)} \end{aligned}\right\} \qquad (32)$$

In the stability chart, Fig. 4, we have drawn the line representing the relation between q and a as defined by (32). We find that for case I we have stability when $0 < k < 0.86$. $k = 0$ represents a circle and $k = 0.86$ is a trochoid with $B_{\min}/B_{\max} = k' = 0.51$. The next stability region appears when $0.990 < k < 0.998$. In this case the ratio of the minimum to the maximum magnetic field is about 0.1. Then comes an infinite number of stable and unstable regions which, however, seem to be of minor practical importance.

When we come to case II we find that, as long as $k < 2^{-1/2}$ the motion is unstable. If we look at the Mathieu equation we find that the unstable region extends to $k = 0.86$. The limiting curve is approximately one, which just crosses itself (compare Fig. 3 d). The stable region is confined to $0.86 < k < 0.91$. This latter curve has approximately zero drift velocity and resembles the figure ∞. For larger values on k there are an infinite number of stable and unstable regions, and they appear at approximatively the same values for k as in case I.

The last method does not make it possible to tell anything about the reliability of the regions of stability we have calculated. When k tends to zero the approximate value of p in equation (27) tends to the exact value and the approximate solution tends towards the exact one. The greater k becomes, the greater the difference in shape of the approximate p and the exact value becomes and the less accurate the calculations become.

A comparison of the results obtained by the two methods we have used for the main stability region in case I shows that the agreement is excellent. For the main region in case II the two methods give the same value for one limit of the interval. Even if the Kreĭn criterion is not formally applicable at that point, I do not think that it is by chance we have got coincidence between the two methods. I think that we can take this coincidence as an indication on the reliability of the method.

For the regions of stability of higher order I have not found any criterion of any value but, since these regions are of minor value, we may be satisfied with the result the Mathieu equation has given.

This investigation has been sponsored by Statens Naturvetenskapliga Forskningsråd.

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