An Improved Barotropic Model and some Aspects of Using the Balance Equation for Three-dimensional Flow[1]

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(Manuscript received November 11, 1955, revised January 24, 1956.)

Abstract

A few suggestions for improving the *barotropic* model presented in a previous article (BOLIN, 1955) have been tested. The results show that replacing the geostrophic relation by a more general balance equation improves the forecasts over 24 hours in certain respects. Furthermore the incorporation of the effects of a stratosphere in a very approximate way gives a better description of the behaviour of the largest scales of motion in the atmosphere.

The application of the balance equation in numerical forecasting is extended to three-dimensional *baroclinic* flow. It is shown that the criterion of integrability of the balance equation to obtain the stream function from a knowledge of the pressure field has to be satisfied in the course of the computations to maintain elliptic character of the equations.

1. Introduction

In a recent article by the author (BOLIN, 1955, hereafter denoted by III) a number of forecasts for 24, 48 and 72 hours with the barotropic model were presented. The results showed that this model has a definite prognostic value up to and possibly beyond three days. Some difficulties appeared, however, when trying to extend the forecast period in this way. For example large anticyclones (and to some extent also cyclones) were forecast, which had little correspondence in reality and it was noticed that in general phenomena of middle scale were better forecast than those of a planetary scale. An activation of very small-scale waves in the course of the computations was often observed partly being a result of the non-linear character of the forecast equation, partly depending upon truncation errors. These as well as other errors were apparently not due to the neglect of baroclinic effects and led us to the conclusion that unsolved fundamental problems still remained within the framework of the barotropic model and that some further improvements of the forecasts still might be obtained without incorporating the baroclinicity of the atmosphere. Furthermore, such difficulties as the effect of the boundary assumptions are preferably studied with the aid of the simplest model. A number of modifications of the barotropic model were discussed in this article. We shall here report on the results of incorporating some of these modifications one by one into the simple barotropic model.

The attempts to eliminate the geostrophic approximation resulted in a more general balance equation between the pressure and wind fields (cf. CHARNEY 1955; BOLIN 1955). Recently THOMPSON (1955) has generalized this balance equation to apply to the more general three-dimensional flow of the atmosphere. The author will give a somewhat different analysis of this problem in the last section of this paper.

[1] The research reported in this document has been sponsored in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, *Air Research and Development Command*, United States Air Force, under contract No. AF 61 (514)-648-C, through the European Office ARDC.

2. The use of the balance equation between wind and pressure in the non-divergent barotropic model

It was suggested in III that replacing the geostrophic relation by a more general balance equation might improve the barotropic forecasts, in particular in areas with large cyclonic or anticyclonic vorticity or in regions with a strong deformation field. Representing the (non-divergent) wind field by the stream function ψ and the pressure field by ϕ ($=gz$, z being the height of a pressure surface) we arrived at

$$f\nabla^2\psi+2\left(\psi_{xx}\psi_{yy}-\psi_{xy}^2\right)+\nabla\psi\cdot\nabla f=\nabla^2\phi \quad (1)$$

where f is the Coriolis parameter and ∇ denotes the two-dimensional (horizontal) nabla operator. It is now our problem to determine ψ and ϕ so that equation (1) is satisfied and the observed wind and pressure fields are described in the best possible way. It was considered that the pressure field is quite well described by the contour field of a constant pressure map in spite of the fact that the observed winds are used as a guide in the analysis. We can then determine the corresponding stream function field with the aid of (1). It was shown that this equation is of the elliptic type if

$$\frac{\nabla^2\phi}{f}-\frac{1}{f}\nabla f\cdot\nabla\psi>-\frac{f}{2} \quad (2)$$

Prescribing some boundary values for ψ we can in such cases obtain the interior distribution of ψ by an iterative method similar to Liebmann's scheme for solving a Poisson equation (cf. III).

One important modification in the scheme outlined in III turned out to be necessary. The non-linear term in (1) should be evaluated by the finite difference expression (m is a scale factor, cf BOLIN, 1955)

$$\begin{aligned}\psi_{xx}\,\psi_{yy}-\psi_{xy}^2 &\approx\\ \approx\frac{m^4}{4}\Big[&\left(\psi_{i+1,j+1}+\psi_{i-1,j-1}-2\psi_{i,j}\right)\times\\ &\times\left(\psi_{i+1,j-1}+\psi_{i-1,j+1}-2\psi_{i,j}\right)-\\ &-\left(\psi_{i,j+1}+\psi_{i,j-1}-\psi_{i+1,j}-\psi_{i-1,j}\right)^2\Big]\end{aligned}$$

Otherwise a systematic error is introduced due to the fact that the first and second term are evaluated as finite differences with a different gridsize.

It was first decided to investigate the importance of the non-linear terms in (1) and neglect the variation of the Coriolis parameter. This is, however, not permissible. It is true that the simplified relation

$$\nabla^2\psi=\frac{1}{f}\nabla^2\phi$$

is used for evaluating the vorticity, when applying the geostrophic approximation, and that both the non-linear terms and the effect of the variation of the Coriolis term are neglected here. The corresponding ψ-field is, however, not used for obtaining the wind. Instead the geostrophic wind $\mathbf{v}_g=-f^{-1}\nabla\phi\times\mathbf{k}$ is applied. Denoting the stream function of the non-divergent part of the geostrophic wind by $\psi^\star$ we notice that

$$f\nabla^2\psi^\star+\frac{1}{f}\nabla f\cdot\nabla\phi=\nabla^2\phi$$

Comparing this equation with (1) we find that they are very similar except for the non-linear terms. The geostrophic approximation has merely been used in evaluating the term in (1) depending upon the variation of the Coriolis parameter. In view of the fact that $\nabla f\cdot\nabla\psi\approx f^{-1}\cdot\nabla f\cdot\nabla\phi$ is proportional to the zonal velocity and thus predominantly positive it is clear that a systematic and large error is introduced if the variation of the Coriolis parameter is neglected in (1) when determining ψ and the wind field.

Having obtained the initial ψ-field, the forecasting proceeds in the usual manner with the aid of the vorticity equation in the form

$$\frac{\partial}{\partial t}\nabla^2\psi=J\left(\nabla^2\psi+f,\ \psi\right) \quad (3)$$

Here J denotes the Jacobi operator. At the end of the forecast ϕ can be evaluated with the aid of equation (1), which merely means solving a Poisson equation with proper boundary conditions. The procedure thus briefly described has been tested and the results are reported here.

The 500-mb surface is used in barotropic forecasts approximately to depict the mean motion of the atmosphere. For the same reason the 500-mb surface will be used in these considerations[1]. It was found that the criterion (2) usually is not fulfilled everywhere on an ordinary 500-mb map. Regions where (2) is not satisfied on an average constitute 3—5 % of the total area in middle or high latitudes (based on maps from September and October 1954). The iterative method given in III does not converge if (2) is not satisfied, which was readily seen by actually attempting a solution. Some modification has to be introduced. Since very little is known about methods for solving a differential equation which is elliptic in some regions and hyperbolic in others, it was decided to change the ϕ-field slightly to eliminate the hyperbolic areas. By this we also prescribe that $\zeta > -f$ everywhere. Our main purpose here is to get a preliminary idea of the importance of using the balance equation (1) instead of the geostrophic approximation. This slight modification of the data field is probably not important for getting a first answer to this question; certainly not in areas of strong *cyclonic* vorticity. In more than 50 % of the points it merely meant smoothing out small-scale irregularities due to imperfect interpolation in reading off the data from the map. In no point a height value had to be changed by more than 15 m, usually only 5 m or less.

In applying the iterative scheme outlined in III it is not sufficient that (2) is fulfilled, but the approximate solutions $\psi^{(n)}$ obtained in the course of the relaxation must always be such that $\zeta^{(n)} = \nabla^2\psi^{(n)} > -f$ in order to secure convergence. This was continuously checked during the computations which is particularly important if over-relaxation is used. As a first guess we used $\psi^{(n)}$ given by

$$\nabla^2\psi^{(1)} = \frac{1}{f}\nabla^2\phi \tag{4}$$

$\psi^{(1)}$ was obtained by solving (4) applying the same boundary conditions as for (1) (cf. below). It is obvious that $\psi^{(1)}$ satisfies the criterion $\nabla^2\psi^{(1)} > -f$ if (2) is satisfied.[1]

ψ must be prescribed on the boundary of the region in order to solve (1) for ψ in the interior, i.e. the flow across the boundary should be given. We shall here apply the geostrophic relation in the following way. Let us assume that

$$\frac{\partial\psi}{\partial s} = \frac{1}{f}\cdot\frac{\partial\phi}{\partial s} + \gamma \tag{5}$$

where s denotes a coordinate along the boundary and γ is a constant, which we determine from the additional requirement that $\nabla\cdot(\mathbf{k}\times\nabla\psi) = 0$. Thus

$$\gamma = -\frac{1}{S}\int\frac{1}{f}\cdot\frac{\partial\phi}{\partial s}\,ds \tag{6}$$

S denotes the total length of the boundary. Having determined γ from (6) we can integrate (5) and obtain ψ except for an irrelevant constant. Usually γ is very small and the procedure in effect means an assumption of geostrophic flow across the boundary. The boundary assumptions during the following forecast will therefore be practically the same as in the forecasts made with the geostrophic approximation. It should be remarked here that this procedure is somewhat inconsistant due to the variation of the Coriolis parameter as can be seen by integrating the basic equations of motion over the area. This will be left for a more detailed discussion by Dr J. Adem at the International Institute of Meteorology in Stockholm.

Computations have been made for an area 21×24 gridpoints with a gridsize of 300 km at 50° lat. (a stereographic map projection was used). Because of the fact that the criterion of ellipticity (2) is barely satisfied in some regions the rate of convergence is quite slow when solving (1). The maximum residual decreases by a factor of two in 25—40 iterations (no

[1] In justifying the use of the 500-mb surface in the barotropic model the idea of an equivalent barotropic atmosphere is introduced in which the wind (or pressure-field) varies linearly with height. A similar discussion can be carried through for the balance equation.

[1] It turned out that a better first approximation would have been obtained by using

$$\nabla^2\psi^{(1)} = \frac{1}{f}\nabla^2\phi - \frac{1}{f}\nabla f\cdot\nabla\phi$$

since then the major part of the first harmonic of the final solution is obtained in $\psi^{(1)}$, whereby the rate of convergence of the following relaxation becomes better.

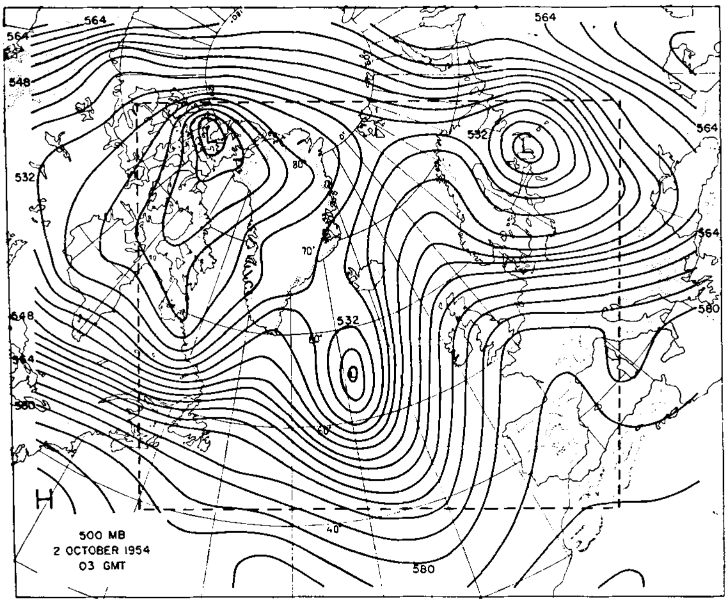

Fig. 1 a. 500 mb contours on October 2, 0300 GMT, 1954. The heights are given in decameter.

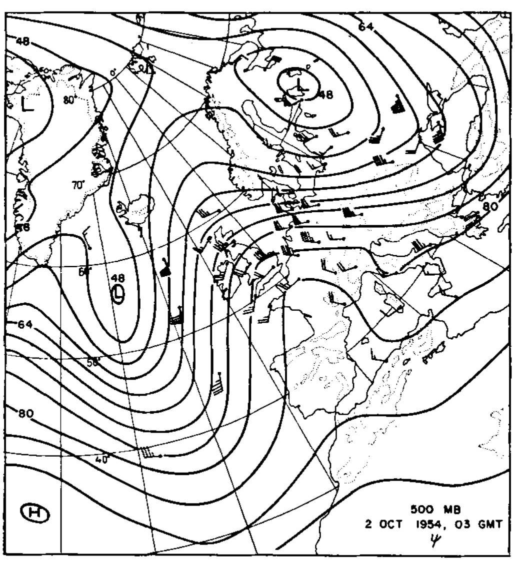

Fig. 1 b. The streamfunction in unit 1.28×10^5 m^2 sec^{-1} as derived with eq. (1) from the contour field given in fig. 1 a. The observed winds are given in conventional notations: a full barb denotes 10 knots and a triangle 50 knots.

over-relaxation) and it therefore was necessary to perform 100—150 iterations to obtain satisfactory accuracy. This takes 40 minutes with the Swedish Computor BESK. It is quite important not to discontinue the iterations too early. Already small errors will have an appreciable effect in the following forecast, since they will effect the first harmonic and give rise to an erroneous circulation around the area. Since the rate of convergence decreases quite rapidly when extending the area, the present scheme is not suitable for routine forecasting for an area of the size used in III and also in sec. 3 of this article. For the limited number of forecasts made here the procedure worked satisfactorily.

Three 24-hour forecasts have been made using the procedure outlined above. They were based on maps from 28 and 30 September and 2 October, 03 GMT, 1954, and covered an area somewhat larger than the one shown in fig. 1 b. We shall denote these forecasts by BB in the following discussion. Ordinary barotropic forecasts using the geostrophic

Table 1

Date	r	r_B	ε m	ε_B m	σ_y m	σ_B m	σ m	$\bar{y}$ m	$\bar{y}_B$ m	$\bar{x}$ m
24^h-forecasts										
28 Sept.-54, 03	0.91	0.96	47	39	87	93	104	3	— 8	10
30 Sept.-54, 03	0.96	0.92	30	43	61	52	80	5	—22	—3
2 Oct.-54, 03	0.89	0.96	49	38	108	93	98	19	0	26

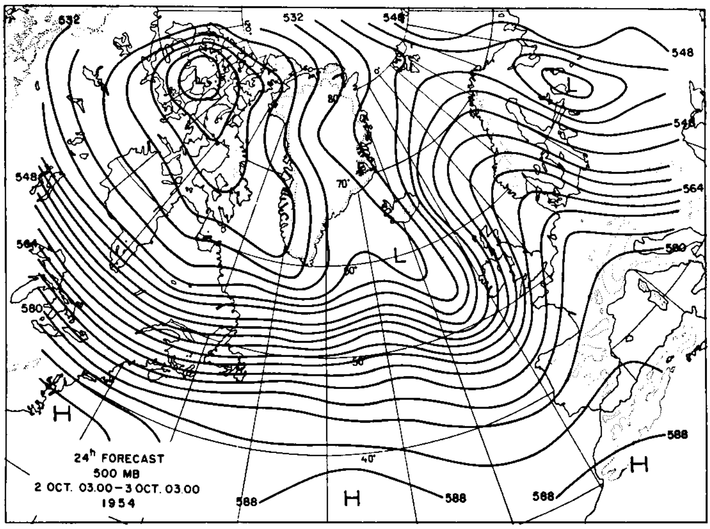

Fig. 2 a. 24-hour barotropic forecast of 500 mb contour field from October 2, 0300 GMT to October 3, 0300 GMT, 1954 using the geostrophic approximation.

approximation (denoted by B) were also prepared for this area. These forecasts were exceptionally good (cf. III). This is of some importance, since we thereby can study the effect of the present modification without serious interference with other sources of errors. The verification was made inside an area covered by 10 × 12 gridpoints and somewhat shifted towards the eastern portion of the area.

The results are shown in table 1. Here r denotes the correlation coefficient between observed and computed changes using the geostrophic approximation, while r_B indicates the corresponding coefficient for the changes computed with the aid of the balance equation. ε and ε_B denote the errors in the B and BB forecasts respectively and σ_y and σ_B the root mean squares of the computed changes, while σ is the root mean square of

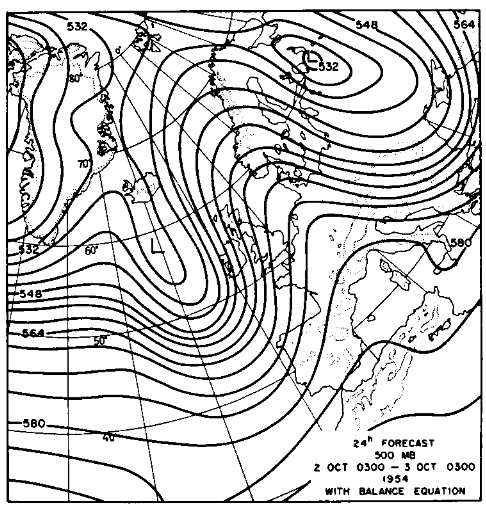

Fig. 2 b. 24-hour barotropic forecast of 500 mb contour field from October 2, 0300 GMT to October 3, 0300 GMT, using the streamfunction given in fig. 1 b.

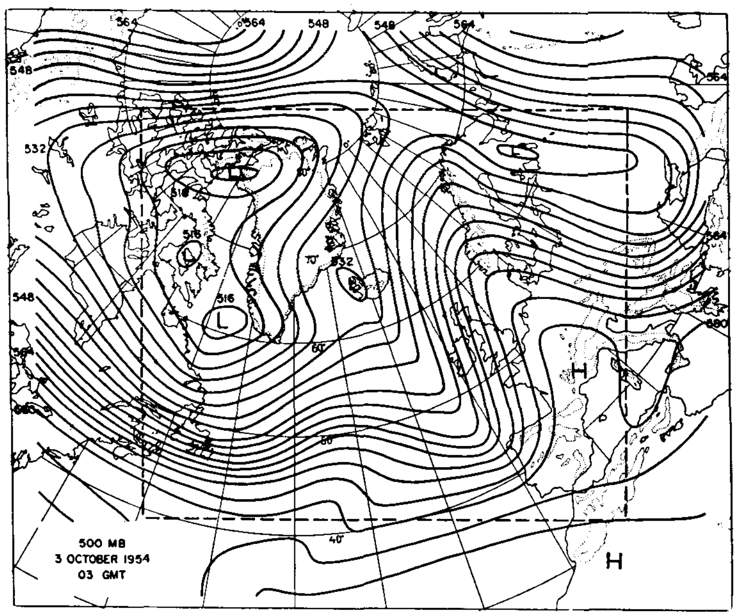

Fig. 2 c. 500 mb contours on October 3, 0300 GMT, 1954.

the observed changes. $\bar{y}$ and $\bar{y}_B$ denote the average changes in the two forecasts and $\bar{x}$, finally the observed change.

We observe:

a) The small scale features of the map (900—1,500 km) are better forecast using the balance equation. This may be quite important in some cases. An interesting example is shown in the forecast from October 2. Figure 1 a shows the initial ϕ-field and figure 1 b the corresponding stream-function ψ obtained from eq. (1). Notice here in particular the overestimation of the winds in the vicinity of the low in the Atlantic if using the geostrophic approximation. In the B-forecast for 24 hours (fig. 2 a) this caused the development of comparatively high ϕ-values just to the east of Iceland. The ridge becomes unsymmetrical. The BB-forecast (fig. 2 b) shows considerably better agreement with reality (fig. 2 c). It is of some interest to point out that this error in the B-forecast developed still more during the following 48 hours (cf. III) and gave quite a bad forecast in most parts of the Norwegian Sea. The details of the forecasts from September 28 and 30 were also better forecast using ψ, than was obtained with the geostrophic approximation, which is shown by considering the changes of the *wind.*

b) The large-scale flow pattern is quite sensitive to good accuracy in the solution. Systematic errors due to for example an inaccurate evaluation of the non-linear term in (1) may sometimes cause significant errors (as was found by actual computations). This would be still more the case if forecasting over a larger area and also if extending the forecasting period because of the activation of small scale features in the flow.

We may conclude that the use of the balance equation instead of the geostrophic relation between wind and pressure, improves the forecast of certain details of the map that may be important. It seems therefore desirable to develop a more rapidly converging method of solving the balance equation. Furthermore the boundary conditions need further consideration as well as the question of the importance of the divergence (cf. sec. 3).

3. An approximate way of incorporating the effect of the stratosphere on the mean motion of the troposphere

One of the most obvious defects of all models for numerical forecasting advanced so far is the very crude description of the vertical structure of the atmosphere. For example, the existence of a tropopause is usually completely neglected. It is well-known that the high vertical stability aloft has a marked influence upon the processes in the troposphere. We indicated in III how the effect of such a stable stratosphere could be incorporated into the barotropic or one-parameter model in a very simple way. On top of the homogeneous fluid, which is supposed to represent mean conditions in the troposphere, we place an infinitely deep fluid of somewhat smaller density. The ratio ($\varkappa'$) of the density difference between these two layers and the density of the lower fluid is chosen in such a way that the variations in the height of the interface depicts the variations in the height of the tropopause in reality. For simplicity the upper fluid is assumed to be in rest. By this last assumption the problem remains one-parametric. It is obvious that this model is extremely crude, but it at least permits an average convergence or divergence in the troposphere, which is of great importance for the study of the longest waves.

A homogeneous fluid with a *free* surface has previously been studied by ROSSBY (1945) and YEH (1949), and the procedure outlined by these authors can be followed in detail in order to arrive at the present forecast equations (cf. III). We obtain the following formula:

$$(\nabla^2 - \lambda^2)\frac{\partial D}{\partial t} = J(\eta, D) \tag{7}$$

where

$$\lambda^2 = \frac{f^2}{\varkappa' g D_0} \tag{8}$$

Here D is the depth of the bottom fluid and D_0 is the average value of D. The model studied by Yeh (model B, free surface) is obtained as a special case by putting $\varkappa' = 1$ and the ordinary barotropic model corresponds to $\varkappa' = \infty$. To depict actual height variations of the tropopause we shall put $\varkappa' = 1/8$.

Before presenting the results of numerical forecasts with this model, we shall discuss some principle features of it with the aid of the corresponding linearized equations. We assume a basic current U, neglect variations in the north-south (y) direction and denote the perturbations of the interface by D'. (This is obviously somewhat inconsistant. $U = - g \cdot f^{-1} \cdot \partial D/\partial y$ and is finite and D must therefore become zero if going sufficiently far north. Permitting a finite extension of the wave in the north-south direction would not change the results in principle; cf. CHARNEY and ELIASSEN, 1949.) We get

$$\frac{\partial^3 D'}{\partial t \partial x^2} + U\frac{\partial^3 D'}{\partial x^3} + \beta\frac{\partial D'}{\partial x} - \lambda^2\frac{\partial D'}{\partial t} = 0 \tag{9}$$

The phase velocity c and the group velocity c_g are given by

$$c = \frac{U - \beta/k^2}{1 + \lambda^2/k^2} \tag{10}$$

$$c_g = \frac{U + \beta/k^2 + 2\lambda^2 c/k^2}{1 + \lambda^2/k^2} \tag{11}$$

$\beta = \partial f/\partial y$ and $k = 2\pi/L$, where L is the wave length. c and c_g are given as functions L in fig. 3. The following values of the parameters have been used: $U = 15$ m/sec, $\varkappa' = 1/8$, $D_0 = 8{,}000$ m, $g = 9.81$ m/sec^{-2}, f and β are evaluated at lat. 45°. For comparison c and c_g for waves in a zonal current governed by the

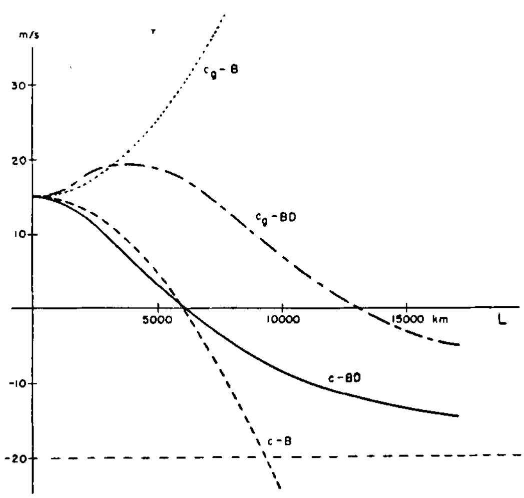

Fig. 3. Phase velocity c and group velocity c_g as a function ot wave length for divergent and non-divergent model.

non-divergent barotropic model are also shown. It is clear that only waves larger than some 3,000 or 4,000 km are essentially influenced by the divergence term. The values obtained by Yeh and those given here differ considerably. We obtain for example $|c_g| < 20$ m/sec which is very reasonable in comparison with the values given by Yeh.

Yeh also studied the behaviour of a solitary wave in course of time to simulate a blocking ridge. We have here recomputed some of his pattern with $\varkappa' = 1/8$. The initial solitary wave represented by $D'(x, 0) = A \exp(-x^2/l^2)$, where l was chosen in such a way that $D'(10^6 \cdot \sqrt{2}, 0) = 1/2 \; D'(0, 0)$. Fig. 4 shows the character of the wave after 18 and 36 hours and also the changes of such a wave using the non-divergent barotropic model ($\varkappa \to \infty$). It is clear that the dispersion has been decreased considerably in our present model. Qualitatively Yeh obtained the same result but because of the large value of $\varkappa$ the effect was only noticeable for very long waves, indeed. — It is interesting to compare these results with those obtained by numerical forecasting with the aid of the non-linear equation (7).

In using (7) for numerical computations we notice that $\partial D/\partial t$ and D can be replaced by $\partial z/\partial t$ and z, where z is the height of any pressure surface in the lower layer of our model. We shall again assume conditions at 500 mb to be representative for mean conditions in the troposphere and use them as initial conditions for our computations. We then proceed as follows:

Let i and j denote the gridpoints in a rectangular grid covering the area over which a forecast is going to be made. z^τ_{ij} and η^τ_{ij} are given at time τ as well as one time step (Δt) earlier, ($\tau - 1$). Here

$$\eta^\tau_{ij} = g\frac{m^2_{ij}}{f_{ij}} \nabla^2_{ij} z^\tau + f_{ij} \tag{11}$$

m_{ij} is a scale factor of the map projection used and ∇^2_{ij} denotes the finite difference Laplace operator. Then $J_{ij}(\eta^\tau, z^\tau)$ is evaluated at all interior points, J_{ij} being the finite difference form of the Jacobi operator. Transforming equation (7) into finite difference form gives

$$\nabla^2_{ij}\frac{\partial z^\tau}{\partial t} - \frac{\lambda^2_{ij}}{m^2_{ij}} \cdot \frac{\partial z^\tau_{ij}}{\partial t} = \frac{1}{4} J_{ij}(\eta^\tau, z^\tau) \tag{12}$$

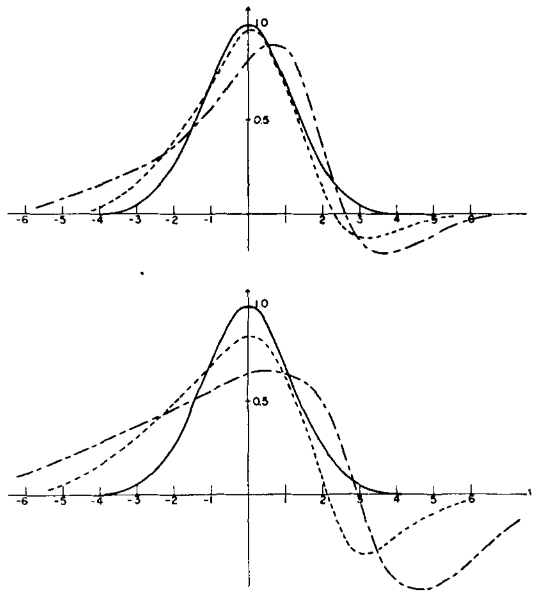

Fig. 4. The development of a solitary wave in an infinite basic current incorporating the effect of the divergence above and according to a non-divergent model below.

which can be solved by Liebmann's method of iteration using $\partial z^{\tau-1}_{ij}/\partial t$ as a first approximation. On the boundaries we assume $\partial z^\tau/\partial t = 0$ in accordance with the assumptions for the ordinary barotropic model. Extrapolation in time yields

$$z^{\tau+1}_{ij} = z^{\tau-1}_{ij} + 2\Delta t \cdot \frac{\partial z^\tau_{ij}}{\partial t} \tag{13}$$

$\eta^{\tau+1}_{ij}$ is finally evaluated from $z^{\tau+1}_{ij}$ with the aid of (11) in all interior points, while $\eta^{\tau+1}_{ij} = \eta^\tau_{ij}$ on the boundaries. We are then ready for the next time step. At the initial time $\tau = 0$, we have to use uncentered differences in the time extrapolation. To get better accuracy, the first time step was only half of the value used in the following steps.

We notice that the new vorticity field at each instant is evaluated from z (eq. 11), which in turn has been obtained by solving (12). In ordinary barotropic forecasts, on the other hand, the vorticity change is directly given by the vorticity advection. To avoid cumulative errors in the present procedure it is therefore necessary to have somewhat more

Table 2

Date	r	r_D	ε m	ε_D m	σ_y m	σ_D m	σ m	y m	$\bar{y}_D$ m	$\bar{x}$ m
24^h-forecasts										
3 Dec.-54, 03	0.84	0.73	144	62	212	83	102	150	44	34
8 Dec.-54, 03	0.86	0.81	75	101	139	105	124	—4	6	—29
13 Dec.-51, 03	0.75	0.84	92	57	130	95	100	77	8	20
17 Febr.-48, 03	0.54	0.60	106	92	109	73	116	—34	—6	—17
Mean	0.75	0.75	104	78	148	90	110	—	—	—
Mean error	—	—	—	—	—	—	—	66	18	—
48^h-forecasts										
3 Dec.-54, 03	0.67	0.60	207	137	292	152	172	153	83	55
8 Dec.-54, 03	0.80	0.63	134	144	211	153	179	54	8	—33
13 Dec.-51, 03	0.58	0.73	212	112	250	123	143	153	73	19
17 Feb.-48, 03	0.35	0.59	169	120	170	118	143	—60	17	15
Mean	0.60	0.64	176	128	231	137	159	—	—	—
Mean error	—	—	—	—	—	—	—	101	37	—

severe demands on the accuracy of the solution of (12) than on the solution of the Poisson equation in the ordinary barotropic model. In solving (12) the maximum residual was not permitted to be more than half of the value previously used. On the other hand the rate of convergence is more rapid for a Helmholz equation than for a Poisson equation. The gridsize used in these computations was 300 km in a polar stereographic projection at 50° lat. and a time step (Δt) was one hour.

Four forecasts have been made for 24 and 48 hours covering an area of the same size as in III. Two of those were chosen to illustrate the large scale behaviour of the atmosphere (February 17, 1948 and December 13, 1951). The results are summarized in table 2, where also the results of ordinary barotropic forecasts for the same area are given. r denotes the correlation coefficient between observed and computed changes with the ordinary barotropic model (again denoted by B), r_D is the corresponding coefficient for the forecasts with the divergence included (in the following discussion denoted by BD). ε and ε_D are the mean errors in these two forecasts and σ_y and σ_D are the root mean squares of the changes respectively. σ denotes the root mean square of the observed changes. $\bar{y}$ and $\bar{y}_D$ are finally the average changes over the verification area with the two models and $\bar{x}$ is the corresponding observed value. The verification has been made inside the dashed line as shown on fig. 1 in III. In all these cases $\varkappa' = 1/10$. Later the forecasts for December 8, 1954 were repeated with $\varkappa' = 1/8$.

It is quite clear that a significant improvement of the forecasts has been obtained with this modification of the barotropic model. A significant difference of the correlation coefficient is hardly obtained on an average, but the average errors are considerably smaller if using the BD model. Also the root mean square of the changes are in much better accord with reality as obtained with the BD-model. We observe, for example, that the ratio σ_D/σ_x is almost constant, while σ_y/σ_x, varies considerably. σ_D/σ_x is, however, less than one indicating that the value for $\varkappa'$ used in these computations is somewhat too small. Repeating the forecast with $\varkappa' = 1/8$ for one case brought σ_D in close agreement with σ_x but did not change r_D and ε_D appreciably. Too few tests have been made to determine the best value of $\varkappa'$, but $\varkappa' = 1/8$ is probably close to the optimum value. The mean change over the verification area, finally, is considerably better forecast with the BD model. All this clearly indicates that essentially the largest scales of motion are influenced by this modification of the barotropic model as is also obvious from inspecting the change pattern themselves.

It is of interest to study some of these forecasts in detail in particular in view of the discussion of dispersion for the linearized model that has been given above.

The synoptic situation of February 17, 1948,

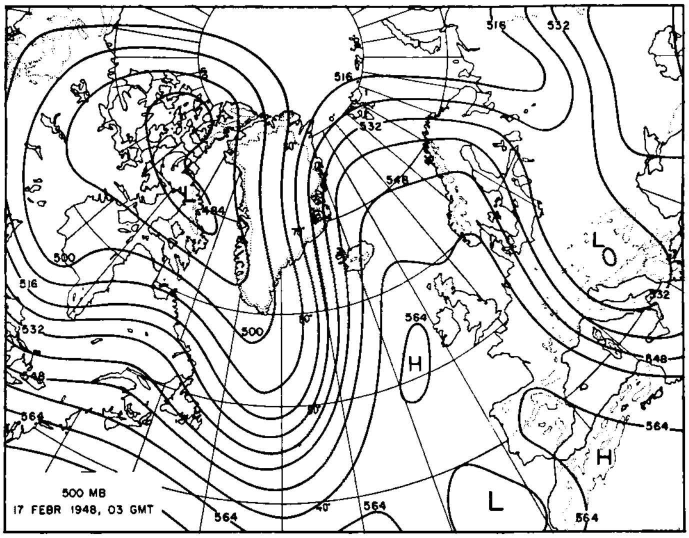

Fig. 5 a. 500 mb contours on February 17, 03 GMT, 1948.

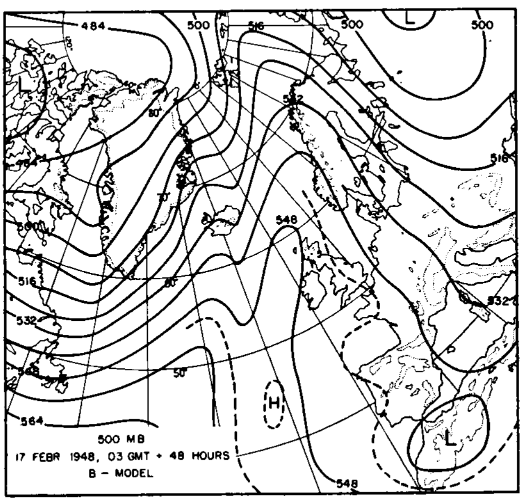

Fig. 5 b. 48-hour B-forecast of 500 mb contour field from February 17, 03 GMT to February 19. 03 GMT, 1948.

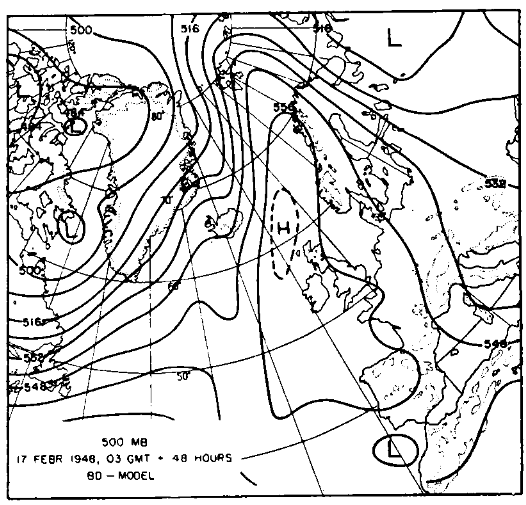

Fig. 5 c. 48-hour BD-forecast of 500 mb contour field from February 17, 03 GMT to February 19, 03 GMT, 1948.

has previously been described in detail by the author (BERGGREN, BOLIN, ROSSBY, 1949). A very intense blocking ridge had been formed over and to the west of the British Isles and was still intensifying on the initial day of the forecast. The initial map used in the computations is reproduced in fig. 5 a. Of course, the few radiosondes over the Atlantic makes the analysis somewhat uncertain in particular in the area of the trough at long. 40° W. The conclusions to be drawn are based upon the difference of the forecasts made with the B-model and the BD-model and are presumably valid even if the initial map is changed somewhat. Fig. 5 b shows the 48-hour B-forecast and fig. 5 c the corresponding BD-

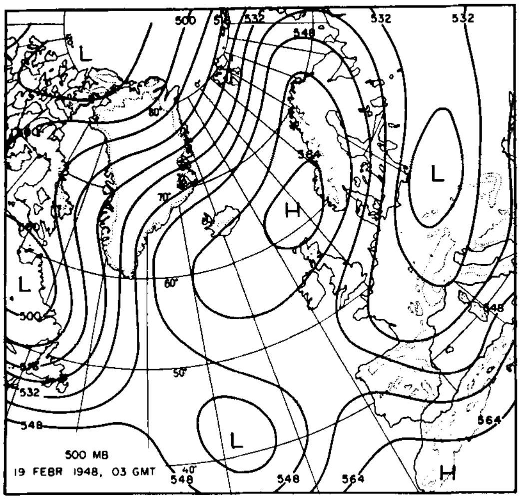

Fig. 5 d. 500 mb contour field on February 19, 03 GMT, 1948.

forecast. The actual flow on February 19, 03 is given in fig. 5 d. It is obvious that the neglect of baroclinic processes causes considerable errors in both forecasts. Thus the intensification of the trough, that is located over Labrador on February 19, is completely missing. This in turns may be the reason for the fact that the anticyclogenesis between Iceland and Southern Greenland is not forecast to be as intense as in reality. Also over middle Europe some errors very likely depend on the neglect of baroclinic processes. A comparison of how the blocking ridge is forecast by the two models is, however, interesting. A marked improvement is here obtained with the BD-model. The contour value in the centre is forecast to decrease by about 30 m, while it remains unchanged in reality. The B-forecast, on the other hand, has errors of more than 200 m in this area. The difference can be interpreted in terms of different rates of dispersion as discussed above for the linearized model. The scale of the solitary wave was chosen to be about the same as the size of the ridge on February 17. A definite difference in the rate of dispersion was obtained with the two models. A closer agreement between the one-dimensional linear case and the two-dimensional non-linear treatment is not to be expected. The similarity is, however, suggestive.

Also in the forecast from December 13, 1952 a similar difference in the prognosis of a blocking ridge was obtained, but not as pronounced as in the case studied above.

The barotropic forecasts from December 3 and 8, 1954 were characterized by strong anticyclogenesis in the southern part of the area which had little counterpart in reality. This is the most disastrous error when forecasting with the barotropic model and often makes it impossible to extend the forecasts beyond 48 or 72 hours. Undoubtedly the fact that constant values of ψ and ζ are imposed as boundary conditions is one main source of these errors (cf. III). Some experiments in which the values prescribed on the boundaries were altered and attempts to forecast for the whole northern hemisphere north of the subtropic anticyclones (GATES, personal communication) show that the boundary assumptions are not solely responsible for these errors. Forecasts with the BD-model show a marked improvement in this respect. Partly this may be the effect of a more correct description of the dynamics of large scale phenomena in the atmosphere, partly it probably is the result of the smaller speed of propagation of influences of large-scale systems that is valid for this model and that prevents the large scale boundary effects to spread rapidly into the centre of the forecast area. Indeed, the improvement may be very large. For example the maximum height of the subtropic anticyclone on December 3, 1954 was forecast to increase from 5,920 m to 6,180 m two days later with the B-model, while the actual height was 5,940 m. A maximum height of 5,981 m was forecast with the BD-model.

From the discussion above it seems highly desirable to extend the forecasts to the whole northern hemisphere and thereby to a large extent eliminate the effect of erroneous assumptions on the boundaries. However, already from the few forecasts made so far it seems justifiable to draw the following conclusion. *It seems necessary to include the effect of a tropopause and its variations into any model for numerical forecasting to be able to forecast phenomena of a planetary scale satisfactorily.* The method used here is admittedly very approximate, but still seems to give an important correction in the right direction.

4. On the generalization of the balance equation between wind and pressure fields to three-dimensional flow

We have previously studied the balance equation between the wind and pressure fields for two-dimensional non-divergent flow in some detail. It is possible to show that a similar balance equation is valid also in the three-dimensional case (cf. THOMPSON, 1955). The underlying hypothesis is that gravity-inertia waves are not of any interest in the study of the large scale behaviour of the atmosphere and should be eliminated. Thompson then has shown that it is necessary and sufficient to omit the total derivative of the horizontal divergence in the divergence equation to filter out these oscillations. This was shown for a linearized model of the atmosphere. The same is true in the non-linear case without any further approximations as will be shown below. However, the question still remains, if it is desirable to eliminate these oscillations under all circumstances.

The period of gravity-inertia oscillations is equal to or less than twelve pendulum hours, i.e. 12—24 hours to the north of lat. 30° N (in the northern hemisphere). This is about one order of magnitude less than the period of the large scale planetary waves in middle latitudes. On the other hand, one frequently observes wave-cyclones with a characteristic time-scale of a little more than 24 hours (cf. part I, BERGGREN, BOLIN, ROSSBY, 1949). Furthermore, the speed of propagation of the zero mode of the gravity-inertia waves is about 300 m/sec., which is at least one order of magnitude greater than that of the weather systems. The *internal* waves, however, move considerably slower (cf. BOLIN, 1953) and their speed is again only a few times larger than for example the speed of waves in the westerlies. These comparisons indicate that the elimination of gravity-inertia waves probably is well justified for the zero (or barotropic) mode and for the largest scales of motion in the atmosphere. On the other hand, when studying phenomena of the scale of ordinary cyclones with the aid of multiple-parameter models, in particular in cases of rapid development, it is more questionable if the internal gravity-inertia waves are irrevalent. With these reservations in mind we shall for the time being accept the hypothesis that the possibility for gravity-inertia oscillation to occur in our equations may be excluded.

Let us use a semi-Lagrangian coordinate system with Θ, the potential temperature, as our vertical coordinate. In this way we automatically incorporate the assumption of adiabatic flow. This transformation is possible provided hydrostatic balance exists, i.e. the atmosphere should have a stable stratification. The equations of motion then are

$$\frac{\partial u}{\partial t}+\mathbf{v}\cdot\nabla u-fv=-\frac{\partial\Psi}{\partial x} \tag{14}$$

$$\frac{\partial v}{\partial t}+\mathbf{v}\cdot\nabla v+fu=-\frac{\partial\Psi}{\partial y} \tag{15}$$

where

$$\Psi=c_pT+gz \tag{16}$$

the so-called Montgomery's streamfunction. The hydrostatic equation is

$$\frac{\partial\Psi}{\partial\Theta}=\frac{1}{\varkappa}p^{\varkappa} \tag{17}$$

and the continuity equation becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\Theta}\right)+\nabla\cdot\left(\frac{\partial p}{\partial\Theta}\mathbf{v}\right)=0 \tag{18}$$

u and v are the two components in the x and y-axes respectively of the wind vector $\mathbf{v}$ in an isentropic surface, f is the Coriolis parameter, T is temperature, g is acceleration of gravity, p is pressure, z is the height of an isentropic surface, $\varkappa=c_p/c_v$ where c_p and c_v are the specific heat of air at constant pressure and constant volume, respectively. All differentiations with respect to t, x and y are made keeping Θ constant and ∇ denotes the two-dimensional nabla operator in an isentropic surface. Let us decompose the wind vector into one rotational and one divergent field

$$\mathbf{v}=-\nabla\psi\times\mathbf{k}+\nabla\alpha \tag{19}$$

Thus

$$\begin{cases}\zeta=\nabla^2\psi & (20)\\ \nabla\cdot\mathbf{v}=\nabla^2\alpha & (21)\end{cases}$$

ζ is the relative vorticity. The two components of the deformation field A and B are defined as

$$\left\{\begin{aligned} A &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ B &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right. \tag{22}$$

They are functions of both ψ and α.

Equations (14) and (15) can be replaced by the divergence equation and the vorticity equation

$$\frac{d}{dt}(\nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{v})^2 + 2\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}\right) + \\ + f_y u - f_x v - f\zeta = -\nabla^2\Psi \tag{23}$$

$$\frac{d}{dt}(\zeta+f) + (\zeta+f)\nabla\cdot\mathbf{v} = 0 \tag{24}$$

Subscript indicates differentiation. In the linear case it was necessary and sufficient to neglect the first term in equation (23) in order to eliminate gravity-inertia oscillations from our system of equations: (17), (18), (23) and (24). We shall see that this is also possible in the non-linear case. It is clear from (23) that this approximation is permissible if

$$\nabla\cdot\mathbf{v} < \zeta \tag{25}$$

where the inequality sign means at least one order of magnitude difference. To the extent this inequality is fulfilled gravity-inertia oscillations are unimportant. Already CHARNEY (1948) pointed out that this is an important feature of the large-scale motion of the atmosphere. With the further assumption that $\zeta \ll f$ Charney could show that the geostrophic approximation is permissible in relating the wind and pressure fields to each other. This is, however, a more serious restriction and is usually not satisfied in moderate or intense cyclones and anticyclones. Only eliminating gravity-inertia oscillations, i.e. applying (25) transforms (23) into the following relation

$$f\zeta - \frac{1}{2}(A^2 + B^2 - \zeta^2) - f_y u + f_x v = \nabla^2\Psi \tag{26}$$

The equations (17), (18), (24) and (26) now constitute a complete set of equations in x, y, Θ and t with the dependent variables ψ, α, Ψ and p. Our problem is to design the proper boundary and initial conditions for solving this system of equations and devise a feasible (numerical) method for this solution. It is furthermore of interest to show that inertia-gravity waves are excluded from this set of equations. Thereby the maximum speed of propagation of waves is presumably reduced permitting larger time steps in the finite difference time extrapolation.[1]

It is interesting to notice that the state of the atmosphere is determined with relatively good accuracy by the three quantities ψ, Ψ and p, which can be determined by observations, while α only means a comparatively small correction to the major part of the wind field given by ψ and is thus of the same order of magnitude as the errors of the observations. This does not mean, however, that the α-field and thus the $(\nabla\cdot\mathbf{v})$-field is unimportant for determining the changes of the state of the atmosphere as can be seen from our set of equations. Hence first we are faced with problem of determining the initial field of α from the quantities that are observed. Knowing ψ^τ, Ψ^τ, p^τ and α^τ at a given instant (τ) we can obtain $(\partial\zeta/\partial t)^\tau$ and thus $(\partial\psi/\partial t)^\tau$ from (24). Extrapolation in time yields $\psi^{\tau+1}$. An approximate value of $\Psi^{\tau+1}$ can be obtained from (26), if neglecting α in evaluating the left side of the equation, by solving a Poisson equation with proper boundary conditions. $p^{\tau+1}$ is then evaluated with (17) and we are back to the problem of determining α from a knowledge of the three other variables. To do this one can proceed as follows:

Let us introduce the vertical velocity $\omega = dp/dt$ as a new variable instead of $\partial p/\partial t$. Equation (18) can then be written

$$\nabla\cdot\mathbf{v} = \nabla^2\alpha = \left(\frac{\partial p}{\partial\Theta}\right)^{-1}\left[\frac{\partial\mathbf{v}}{\partial\Theta}\cdot\nabla p - \frac{\partial\omega}{\partial\Theta}\right] \tag{27}$$

Thus a knowledge of a ω will permit us to evaluate $\nabla^2\alpha$ and by proper integration α. For simplicity we shall in the present discussion neglect the small effect of the variation of the Coriolis parameter in (26) and thus use

$$f\zeta - \frac{1}{2}(A^2 + B^2 - \zeta^2) = \nabla^2\Psi \tag{28}$$

[1] Charney has suggested to solve the simultaneous system of equations directly. The question of integrability as discussed below is still of interest.

This is not necessary but merely done for convenience. For the following derivation it is useful to replace the vorticity equation (24) in our system by another equation also derived from the two basic equations (14) and (15). By proper differentiation and making use of the definitions of A and B according to (22) we obtain.

$$\frac{\partial A}{\partial t}+\mathbf{v}\cdot\nabla A-fB+A\nabla\cdot\mathbf{v}-f_x v-f_y u+ + \frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi}{\partial y^2}=0 \qquad (29)$$

$$\frac{\partial B}{\partial t}+\mathbf{v}\cdot\nabla B+fA+B\nabla\cdot\mathbf{v}-f_y v+ +f_x u+2\frac{\partial^2\Psi}{\partial x\partial y}=0 \qquad (30)$$

Multiplying (24), (29), and (30) by $(\zeta+f)$, $-A$ and $-B$ respectively and adding we obtain with the aid of (28)

$$\frac{\partial G}{\partial t}+\mathbf{v}\cdot\nabla G+2G\nabla\cdot\mathbf{v}+F_1+F_2=0 \qquad (31)$$

where

$$G=\frac{1}{2}[(\zeta+f)^2-A^2-B^2]=\nabla^2\Psi+\frac{1}{2}f^2 \qquad (32)$$

$$F_1=-A\left(\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi}{\partial y^2}\right)-2\frac{\partial^2\Psi}{\partial x\partial y}B \qquad (33)$$

and

$$-F_2=v^2\nabla f\cdot\nabla\frac{u}{v}+J(f,K) \qquad (34)$$

J denotes the Jacobi operator and $K={}^1/_2\,(u^2+v^2)$ being the kinetic energy per unit mass. With the aid of (27) and (32) we can transform (31) into

$$\nabla^2\frac{\partial\Psi}{\partial t}-2G\left(\frac{\partial p}{\partial\Theta}\right)^{-1}\frac{\partial\omega}{\partial\Theta}= -\mathbf{v}\cdot\nabla G-F_1-F_2- -2G\left(\frac{\partial p}{\partial\Theta}\right)^{-1}\frac{\partial\mathbf{v}}{\partial\Theta}\cdot\nabla p=H \qquad (35)$$

Furthermore we differentiate (17) with respect to t and obtain

$$\frac{\partial}{\partial\Theta}\cdot\frac{\partial\Psi}{\partial t}-p^{\varkappa-1}\omega=-\mathbf{v}\cdot\nabla\frac{\partial\Psi}{\partial\Theta} \qquad (36)$$

We can now eliminate ω or $\partial\Psi/\partial t$ between these two equations and obtain

$$\nabla^2\frac{\partial\Psi}{\partial t}-2G\left(\frac{\partial p}{\partial\Theta}\right)^{-1}\frac{\partial}{\partial\Theta}\left(p^{1-\varkappa}\frac{\partial}{\partial\Theta}\frac{\partial\Psi}{\partial t}\right)= =H+2G\cdot\left(\frac{\partial p}{\partial\Theta}\right)^{-1}\frac{\partial}{\partial\Theta}\left(p^{1-\varkappa}\mathbf{v}\cdot\nabla\frac{\partial\Psi}{\partial\Theta}\right) \qquad (37)$$

and

$$\nabla^2 p^{\varkappa-1}\omega-\frac{\partial}{\partial\Theta}\left[2G\left(\frac{\partial p}{\partial\Theta}\right)^{-1}\frac{\partial\omega}{\partial\Theta}\right]= =\frac{\partial H}{\partial\Theta}+\nabla^2\left(\mathbf{v}\cdot\nabla\frac{\partial\Psi}{\partial\Theta}\right) \qquad (38)$$

We shall restrict the following discussion to equation (38) since a knowledge of ω will enable us to evaluate α with the aid of (27). Equation (38) is a second order differential equation in ω. The right side is a function of Ψ, ψ and p, but also of α, which is related to ω by equation (27). One can, however, verify that no other terms are of second order, than those retained on the left side. Thus the character of the equation is directly determined from (38). We find that it is elliptic provided

$$G\left(\frac{\partial p}{\partial\Theta}\right)^{-1}<0 \qquad (39)$$

We have already assumed that the atmosphere should be stable, i.e. $\partial p/\partial\Theta<0$ and in that case the criterion of ellipticity becomes

$$\frac{1}{f}\nabla^2\Psi>-\frac{f}{2} \qquad (40)$$

in view of (32). This is the same criterion as the one that has to be satisfied in order to solve (28) for ψ as a boundary value problem, when Ψ is known and α is neglected in evaluating A and B (cf. CHARNEY, 1955 and BOLIN, 1955). It is easy to show that (40) implies that $\zeta>-f$ but *not* vice versa.

It was already pointed out that the right side of equation (38) is a function of α. Now

$|\nabla\alpha| \ll |\nabla\psi|$ as implied by (25), and thus the terms depending upon α are small compared to those only being functions of ψ, Ψ and p. Since the character of the equation is determined by the left side it should therefore be possible to determine ω by successive approximations. First the right side is evaluated by putting $\alpha = 0$ and we solve the equation (38) for ω. The corresponding α-field can be used for a better estimate of the right side, etc. Probably already the first approximation is sufficient for our purpose.

Equation (38) is a differential equation for ω that does not contain any derivatives of time. It is well-known that the vertical velocities cannot be obtained by merely knowing the mass and windfields at a given instant if gravity-inertia waves can occur in the system, but it is necessary to know also the rate of change of these fields. The very fact that we can get an equation for ω of the type (38) proves that gravity-inertia waves are excluded from our system of equations (cf. BOLIN, 1953).

It was pointed out in section 2 that (40) is usually not satisfied all through the atmosphere in middle and high latitudes, in particularly not at higher levels. Thus equation (38) is hyperbolic in such regions. It is well-known that boundary conditions cannot be prescribed for a hyperbolic equation. In this case, however, the equation is hyperbolic only within small areas and it is not known to the author in which way the boundary conditions can be prescribed. Furthermore, even if (40) is satisfied everywhere at one instant the question arises whether it will remain so in the course of time[1]. It is clearly seen from (31) that G may become negative even if being positive everywhere at one instant because of the existence of the two terms F_1 and F_2. Thus it also follows that *hyperbolic regions may develop*. This is a basic difficulty inherent in using the balance equation (26) in numerical forecasting. The physical meaning of the criterion (40) is not clear but it is interesting to notice that in the linearized case or in the case of a circular vortex it is similar to the criterion of inertia (or dynamic) stability.

It might be worth while to attempt an integration and force the criterion (40) to be satisfied in the course of the computations. It seems to the author, however, that we need a better understanding of the physical processes involved to use the balance equation in treating the three-dimensional flow of the atmosphere.

Acknowledgement

The author is indebted to the staff at the International Institute of Meteorology in Stockholm, for many fruitful discussions and also to Dr. J. Charney, Institute for Advanced Study, Princeton, N.J., for a stimulating discussion on the subject of section 4 of this article. Mr A. Bring and Miss B. Lindén have coded the computations for the Computor BESK and Mrs B. Dahlborg has prepared the figures. Their help is gratefully acknowledged.

[1] This was pointed out to the author by Dr. J. Charney.

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