Rotor Phenomena in the Lee of Mountains

By PAUL QUENEY, University of Paris, France

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Abstract

The rotors are explained as a simple "cat's eye effect", or transformation of a stationary wave motion into a system of vortices in the vicinity of a level where the basic wind velocity is vanishing. Some other applications of this same effect to phenomena of smaller or larger scale are also given.

1. Introduction

It is well known that the stationary lee-waves produced by a big mountain often break up into turbulent longitudinal whirls or "rotors" in the lower layers of the air flow. The first detailed study of the phenomena was made by KUETTNER in 1939, in the case of the Riesengebirge (central Europe) by means of airplane observations completed by theoretical considerations. According to this author the lower system is mainly gravitational, associated with the free surface of a cold air layer, and the transformation of the waves into rotors occurs when a certain critical amplitude is attained, while the upper waves may be governed by independant factors, either gravitational, dynamical, or both. However, at least in the first stages of their development, the lower waves have the same phase as the upper ones, so that a definite coupling exists between the two systems. Each rotor produces a stationary roll cloud, very different in aspect from the smooth lenticular ones associated with the upper waves.

More recently some extensive investigations were also made in California on the roll clouds frequently observed in the lee of the Sierra Nevada, and which are particularly big and strong, often exceeding the height of the mountain itself.

Whatever may be the factors governing the wave motion produced by the mountain, some explanation has to be found for the breaking of the waves into separate whirls, and this is properly what we may call the "rotor problem". In the following we shall consider it as a particular case of a more general one, namely the problem of the change of a wave motion into a system of vortices in the vicinity of a level where the basic flow has the same velocity as the waves, and it will be convenient to refer to this change as the "cat's eye effect". For the general theory of this effect it is essential to abandon the so-called infinitesimal assumption used in most of previous theories of mountain waves, and also to take account of the wind shear in the layer under consideration.

The suggestions presented in this paper came us about one year ago, in the course of theoretical researches undertaken at Los Angeles (California) with my colleague Professor J. HOLMBOE and his coo-workers, concerning the two-dimensional perturbations of barotropic flows.

2. Theory of the "cat's eye effect"

Since the earth's rotation is negligible at the scale of the rotors we may adopt the usual assumption of a non-divergent stationary motion taking place in an x, z-vertical plane parallel to the wind blowing towards the mountain (basic flow). Taking the z-axis vertical upwards and the origin at the ground level in the basic flow, and denoting by $U(z)$ the velocity of this basic flow, the mountain perturbation is then completely defined by the disturbance $\psi(x, z)$ of the corresponding stream-function $\int Udz$, and if these two functions are given the exact equation for the disturbed streamlines is, in any case,

$$\int Udz + \psi(x, z) = \text{const} \qquad (1)$$

However, the analytical form of $\psi(x, z)$ is not simple in general, since this function must satisfy a certain differential equation depending not only on the velocity profile of the basic flow, but also on gravitational stability and viscosity. Therefore, it is convenient to develop the theory only in a simple case, and then to generalize. This simple case will be defined as follows:

i) The basic velocity is an increasing linear function of z, say

$$U(z) = U_0 + \gamma z = \gamma (z - z_0) \qquad (\gamma > 0;\ z_0 = -U_0/\gamma) \qquad (2)$$

in other words the basic flow is a Couette flow, vanishing at the level $z = z_0$ (negative if U_0 is positive).

ii) There is no gravitational stability in this basic flow.

iii) Viscosity and turbulence are negligible.

iv) The horizontal scale of the perturbation is large compared with the thickness of the flow under consideration.

Of course these assumptions have only to be valid up to 3 or 4 km about, the average height of the rotors.

According to (ii) the motion is barotropic, and since it is also unviscid the vorticity is conservative. On the other hand this vorticity has the constant value γ in the basic flow, therefore its disturbance must be equal to zero, and accordingly the differential equation for ψ is simply the Laplacian equation $\nabla^2 \psi = 0$. Therefore, the horizontal scale of the perturbation is comparable with its vertical scale, and if we then take account of assumption (iv) we see that ψ may also be considered as a linear function of z, say

$$\psi(x, z) = f(x) + zg(x) \qquad (3)$$

and this means that the horizontal component $u = \partial\psi/\partial z$ of the velocity disturbance is practically independant of z and equal to $g(x)$.

If (2) and (3) are used the equation (1) then gives, for the streamlines,

$$\gamma^2 z^2 + 2[U_0 + g(x)]\gamma z + 2\gamma f(x) = C \qquad (4)$$

where C is a constant along each streamline, and if we solve for γz we get

$$\gamma z = -[U_0 + g(x)] \pm \pm \sqrt{[U_0 + g(x)]^2 - 2\gamma f(x) + C} \qquad (5)$$

It is easy to see that if ψ is small this equation represents a system of unlimited streamlines in the entire flow, except in the vicinity of $z = z_0$ where it gives a system of closed streamlines bounded by particular streamlines intersecting each other: we thus get actually a cat's eye pattern in the vicinity of the level $z = z_0$.

Let us now consider more particularly what happens in the lee of the mountain.

If we assume the existence of lee-waves with a moderate amplitude, we may take for ψ a sinusoidal function of x, and this function must vanish at the ground level $z = 0$. Therefore we can write, if we choose a proper x-origin and denote the wave length by $2\pi/k$,

$$f(x) = 0;\ g(x) = a \sin kx \ (a > 0) \qquad (6)$$

so that we get for the streamlines

$$\gamma z = -(U_0 + a \sin kx) \pm \pm \sqrt{(U_0 + a \sin kx)^2 + C} \qquad (7)$$

The complete patterns of streamlines given by this equation are shown on fig. 1 (including the virtual portions for $z < 0$, drawn in thin dotted line), they differ according to the value of U_0 as compared to a. For their description it will be convenient to put

$$M = (|U_0| + a)^2;\ M' = (|U_0| - a)^2 \qquad (8)$$

A) If $U_0 < -a$ (fig. 1 A) there is one row of equal cat's eyes located entirely above the ground level, and bounded by the two particular streamlines corresponding to $C = -M'$. The closed streamlines are obtained for $-M < C < -M'$, they are reduced to points for $C = -M$, and their axis is the sine curve

$$z = z_0 - \frac{a}{\gamma} \sin kx \qquad (9)$$

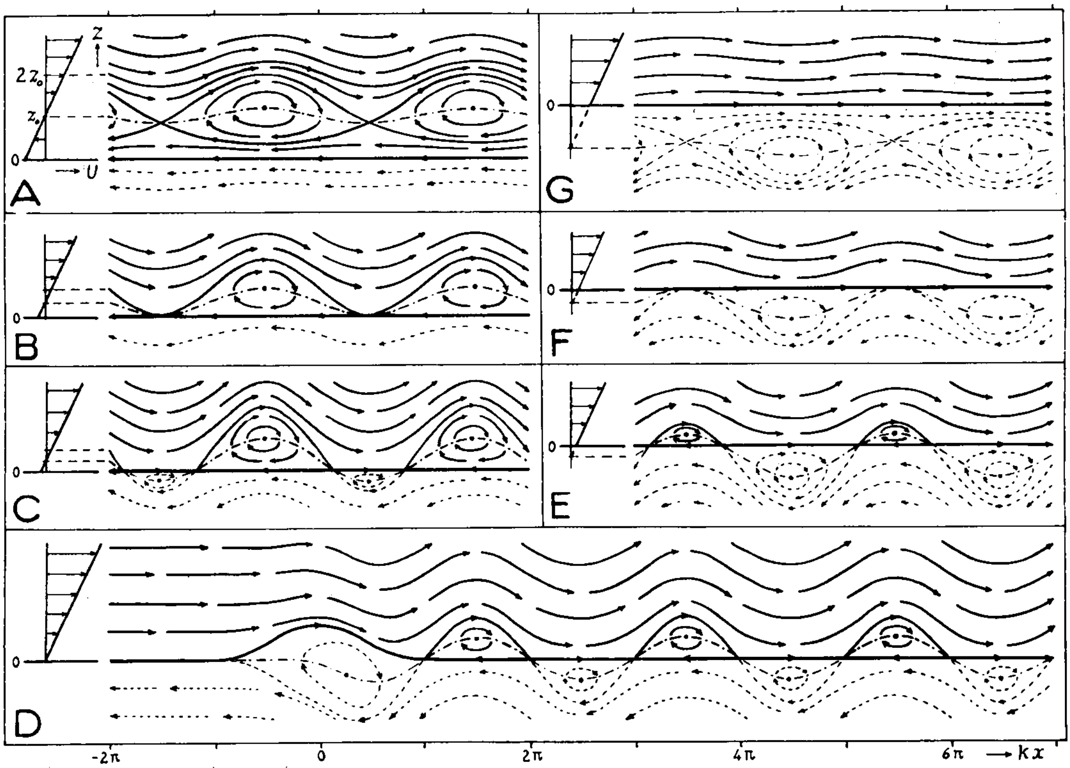

Fig. 1. Stationary wavelike mountain perturbation in a barotropic Couette flow: streamlines corresponding to various wind profiles.

For $C = 0$ we get $z = 0$ as one particular streamline, at the same time as the sinusoidal streamline

$$z = 2\left(z_0 - \frac{a}{\gamma}\sin kx\right) \tag{10}$$

All the other streamlines are wavelike, and practically sinusoidal for $C > 0$.

B) If $U_0 = -a$ (fig. 1 B), since $M' = 0$ the boundary of the cat's eyes is obtained for $C = 0$, and it is formed by the x-axis and the sinusoidal streamline (10) which are now tangent to each other. The axis (9) is also tangent to $z = 0$. Accordingly the vortices are just in contact with the ground.

C) If $-a < U_0 < 0$ (fig. 1 C), the boundary of the cat's eyes is again obtained for $C = 0$ and formed by the x-axis and the sine curve (10), but these lines are now intersecting each other. Accordingly the cat's eyes are alternately located on the positive and negative side of the x-axis, the axis of the whole system being still formed by the sine curve (9). The positive cat's eyes are larger than the negative one's, their closed streamlines being obtained for $-M < C < 0$ whereas those of the latter are obtained for $-M' < C < 0$.

D) If $U_0 = 0$ (fig. 1 D), we have the same situation but the positive and negative vortices are equal, the pattern being skew-symmetrical with respect to the x-axis. The closed streamlines are obtained for $-a^2 < C < 0$.

E) If $0 < U_0 < a$ (fig. 1 E), we have again a similar situation, but with the negative vortices larger than the positive ones.

F) If $U_0 = a$ (fig. 1 F), the pattern is symmetrical of that of fig. 1 B with respect to the x-axis, so that the cat's eye system is entirely virtual.

G) Similarly, if finally $U_0 > a$ (fig. 1 G) the pattern is symmetrical of that of fig. 1 A, and the cat's eyes are again entirely virtual.

In the region between the basic flow and the lee-waves the streamlines can be computed by formula (5) if the mountain profile is given together with the function $g(x)$, for instance, and this formula gives at the same time the virtual streamlines forming the continuation of those described above. In particular the equation for the axis of the closed streamlines is

$$\gamma z = -[U_0 + g(x)] \qquad (11)$$

The result of this computation is represented on fig. 1 D for $U_0 = 0$, in the case of a symmetrical bell-shaped mountain profile: we see that there is an additional virtual cat's eye under the mountain, and the same thing necessarily occurs with any mountain profile.

Remark. It is important to note that the infinitesimal assumption would give no closed streamlines at all: according to this assumption the equation of the disturbed streamline corresponding to a basic streamline $z = h$ would be

$$z = h - \frac{\psi(x, h)}{U(h)} \qquad (12)$$

therefore all the lee-streamlines would be sinusoidal. This formula (12) is actually correct when $|U(h)|$ is large compared to a, in other words when $|h - z_0|$ is large compared to a/γ.

Now we have to examine how the preceding results, derived in the ideal case of an unviscid barotropic Couette flow and a perturbation with a large horizontal scale, can be generalized.

Obviously the transformation of the wave motion into vortices is due to the fact that around $z = z_0$ the basic velocity is overcome by the horizontal component u of the velocity disturbance, and therefore the same must occur with any basic flow provided u is not itself vanishing in the vicinity of $z = z_0$, that is, provided ψ is not maximum or minimum with respect to z. This is certainly the case, whatever may be the scale of the perturbation, if the basic flow remains an unviscid barotropic Couette flow, since ψ is then a solution of the Laplacian equation. Furthermore if the basic flow is modified continuously the streamlines' pattern must also change continuously, and therefore it is undoubtful that the general aspect of the streamlines cannot change as long as such factors as gravitational stability, turbulent viscosity, or vertical variation of the wind shear, are not too much predominant. This restricted generalization will be sufficient for the following applications.

It is also to be noted that this theory of the cat's eye effect may be extended to any two-dimensional basic flow upon which a wave perturbation is superposed, if this perturbation has a definite phase velocity: the waves change into vortices in the vicinity of any basic streamline along which the basic fluid velocity equals this phase velocity.

3. Application to the rotors

The wind soundings made in the Sierra Nevada area have shown that in most of rotor situations the vertical wind shear is roughly independant of height up to more than 5 km, so that the use of a Couette model appears as quite justified at least in this case. In addition the basic wind at the ground level is always very small if not exactly equal to zero. Therefore we may conclude that, unless the gravitational stability or turbulence is an essential factor in the lower layers, the pattern of streamlines associated with a system of lee-waves must be approximately that of fig. 1 D, or eventually that of fig. 1 E. In both cases we have near to the ground a succession of separate equal vorticies, each vortex corresponding with a wave crest: obviously we thus get the explanation of the rotors with their essential properties, and this explanation is very simple indeed, and quite independant of the factors governing the waves themselves. More precisely, the rotors simply appear as a necessary consequence of the fact that the basic wind is vanishing or very small at the ground level, and rapidly increasing upwards.

If we assume $U_0 = 0$ the theoretical height of the rotors, deduced from formula (10), is

$$H_r = \frac{2a}{\gamma} \qquad (13)$$

therefore it is proportional to the amplitude of the lee-waves for a given basic wind. On the other hand if we adopt the model of fig. 1 D, where the mountain crest is at $x = 0$ and the value of $g(x)$ is very small at the same point, the formula (5) gives for the height H of the mountain the approximate value

$$H = \frac{1}{\gamma}\sqrt{-2\gamma f(0)} \qquad (14)$$

whence
$$f(0) = -\frac{\gamma}{2} H^2 \tag{15}$$

Now since ψ satisfies a linear differential equation the amplitude of the lee-waves is roughly proportional to $f(0)$ for a given basic wind and a fixed width of the mountain, and we can therefore conclude that H_r is approximately proportional to H^2. Accordingly the relative importance of the rotors as compared to the mountain increases with the height of this mountain, and we can thus understand why big rotors are only observed with high mountains, and why no definite rotors are to be found in the lee of low mountains.

4. Application to similar phenomena

i) Besides the rotors, the same theory explains more generally all the stationary vortices and wavelike clouds frequently observed around a level where the wind changes sign, whatever may be the origin of the responsible wave motion: if we look at fig. 1 A we note that the vertical displacement of the air is much larger inside of the vortices than outside, and this simple remark may explain why condensation occurs at the upper part of each vortex, in the form of a stationary cloud.

ii) At a smaller scale the many wavelike features commonly seen at the ground surface, such as the sand ripples, the dunes of the deserts, and even the water waves formed by the wind, may also be considered as a simple effect of the transformation of some wave perturbation into stationary vortices as a result of the vanishing of the wind at the ground level.

iii) At the synoptic scale also there are various wave motions that may be considered as two-dimensional at a first approximation (they are mainly horizontal instead of vertical), and therefore the general theory of the cat's eye effect again applies to them. Accordingly a system of vortices is to be expected everywhere the phase velocity equals the fluid velocity. We thus get the explanation of subtropical anticyclones between westerlies and easterlies, and it is possible that the cyclones themselves are essentially nothing but the result of a similar transformation of a wave motion into vortices.

We thus see how so many meteorological phenomena, apparently very different, may be finally related to only one very simple hydrodynamical effect.