Wave Motion in a Rotating Couette Flow of a Viscous Fluid

By SHIH-KUNG KAO, The Johns Hopkins University[1]

(Manuscript received September 27, 1954)

Abstract

With the assumption of non-divergence horizontal flow, a certain class of solutions of the nonlinear hydrodynamic equations are obtained for a viscous Couette flow on an earth whose variation of Coriolis parameter with latitude is constant. These solutions yield waves of transient type, of which the amplitude, wave velocity, and the inclination of the trough and ridge lines vary with time.

For Couette flow with both lateral and vertical velocity-shears, the vertical profile of the amplitude of the waves can either be exponential in z or independent of z, whereas for Couette flow with lateral velocity-shear only it permits to be independent of z, a sinusoidal function, or a hyperbolic sine or cosine of z. These waves will eventually disappear in a Couette flow regardless the fluid is viscous or not, although under certain conditions waves may be amplified within a certain time interval.

Pressure field, geostrophic deviation, meridional transports of energy and westerly momentum are obtained for such a flow system. For reasonable values of the constants for atmospheric systems, geostrophic deviation can be as high as one order magnitude smaller than the actual wind speed.

1. Introduction

The linearized vorticity equation for a rotating inviscid fluid with basic uniform flow was first studied by Rossby (ROSSBY, 1939), who obtained a solution for the motion of sinusoidal waves of infinite lateral extent on an earth whose variation of Coriolis parameter with latitude is constant.

In a recent paper (KAO, 1954) waves solutions of the nonlinear hydrodynamical equations have been obtained for a rotating viscous fluid with uniform (basic) linear velocity for the plane, and with (basic) uniform angular velocity for the sphere. It was shown that under certain conditions these solutions permit to have a (lateral) sinusoidal velocity profile.

[1] This researchw as supported by the Geophysics Research Directorate, Air Force Cambridge Research Center, under Contract AF 19 (604)—916.

The objective of the present paper is to study a certain class of solutions of the nonlinear vorticity equation for a rotating Couette flow of a viscous fluid. On account of the coordinate dependency of the velocity of a Couette flow, the characteristics of the solutions obtained for a Couette flow are significantly different from that for a (basic) uniform flow.

2. Wave Solutions of the Nonlinear Vorticity Equation for a (Rotating, Viscous) Couette Flow with Both Lateral and Vertical Velocity-Shear

If the vertical velocity is neglected, and zero horizontal divergence and constant density (or an autobarotropic condition) are assumed, the vorticity equation for a rotating, viscous fluid on an earth whose variation of Coriolis parameter with latitude is constant is

$$\left\{\frac{\partial}{\partial t}-\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}-\nu\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)\right\}\nabla^2\Psi+ +\beta\frac{\partial \Psi}{\partial x}=0 \qquad (1)$$

where x, y, z are the Cartesian coordinates, positive toward the east, north and zenith respectively; Ψ is the stream function, γ the coefficient of kinematic viscosity, $\beta \approx 2\Omega a^{-1} \cos \varphi$ the variation of the Coriolis parameter with latitude φ, a and Ω respectively the radius and angular velocity of the earth, ∇^2 the horizontal Laplacian operator.

For a Couette flow with velocity shear S and Q respectively with respect to y and z, the stream function may be expressed

$$\Psi(x, y, z, t) = -\left(U_0+\frac{1}{2}Sy+Qz\right)y+ +\psi(x, y, z, t) \qquad (2)$$

where U_0 is the mean zonal velocity at $y = z = 0$. For motions in such a flow system vorticity equation becomes

$$\left\{\frac{\partial t}{\partial}+(U_0+Sy+Qz)\frac{\partial}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}- -\nu\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)\right\}\nabla^2\psi+\beta\frac{\partial\psi}{\partial x}=0 \qquad (3)$$

Let us find the solutions of the above partial differential equation, which belong to the class

$$\nabla^2\psi=J(t)\psi \qquad (4)$$

Using vorticity equation (3) one can show that harmonic waves in the solution of this class is of the form[1]

$$\psi(x, y, z, t) = =F(t)\,G(z)\sin K\{x-h(y, z, t)\} \qquad (5)$$

where $K = 2\pi L^{-1}$ is the wave number, L the wave length.

[1] It can be shown that for flow with uniform (basic) velocity J in (4) becomes independent of time, and waves permit to have finite lateral extent.

To satisfy relation (4) one finds that

$$J(t)=-K^2\left\{1+\left(\frac{\partial h}{\partial y}\right)^2\right\}$$

$$\frac{\partial^2 h}{\partial y^2}=0$$

We may therefore express

$$h(y, z, t)=[I(t)+\xi]\,y+R(z, t) \qquad (6)$$

$$J(t)=-K^2\{1+[I(t)+\xi]^2\} \qquad (7)$$

where $I(t)$ and $R(z, t)$ are functions which will be determined later, ξ is a constant to be determined by the initial condition. Substituting from (6) into (5), then into (3) and making use of relation (4), one finds that

$$\frac{1}{J}\frac{dJ}{dt}+\frac{1}{F}\frac{dF}{dt}-\nu\left\{J(t)-K^2\left(\frac{\partial R}{\partial z}\right)^2+ +\frac{1}{G}\frac{d^2G}{dz^2}\right\}=0 \qquad (8)$$

and

$$\frac{\partial h}{\partial t}-\left\{U_0+Sy+Qz+\frac{\beta}{J(t)}\right\}+ +\nu\left\{\frac{2}{G}\frac{dG}{dz}\frac{\partial R}{\partial z}+\frac{\partial^2 R}{\partial z^2}\right\}=0 \qquad (9)$$

By the inspection of (8) one infers that $R(z, t)$ and $G(z)$ are of the following forms:

$$R(z, t)=[H(t)+\eta]\,z+N(t) \qquad (10)$$

$$G(z)=\exp[K\zeta z] \qquad (11)$$

where $H(t)$, $N(t)$ are functions remained to be determined; η and ζ are real constants to be determined respectively by initial and boundary conditions.

Integration of (9) with respect to t, and with the use of (10) and (11) gives

$$h(y, z, t)=(U_0+Sy+Qz)t+\beta\int\frac{dt}{J(t)}+ +2\gamma K\zeta\int[H(t)+\eta]\,dt+M(y, z) \qquad (12)$$

Substituting from (10) into (6), then comparing it with (12) one finds that

$$H(t) = Qt, \qquad I(t) = St$$

$$J(t) = -K^2[1 + (St+\xi)^2]$$

$$M(y, z) = \xi y + \eta z + B$$

$$N(t) = U_0 t - \frac{\beta}{K^2 S} \tan^{-1}(St+\xi) + $$

$$+ \gamma K \zeta (Qt + 2\eta) t \tag{13}$$

If we impose on $h(y, z, t)$ the initial condition

$$\delta = h(0, 0, 0) = -\frac{\beta}{K^2 S} \tan^{-1} \xi + B$$

at $y = z = 0$, we have

$$h(y, z, t) =$$

$$= (U_0 + Sy + Qz)t - \frac{\beta}{K^2 S} [\tan^{-1}(St+\xi) -$$

$$- \tan^{-1}\xi] + \gamma K \zeta (Qt + 2\eta) t + \xi y + \eta z + \delta \tag{14}$$

Integration of (8) with respect to t and with the use of (13) gives

$$F(t) = \frac{-A}{K^2[1 + (St+\xi)^2]} \exp\left\{-\gamma K^2 [(1 + \right.$$

$$\left. + \xi^2 + \eta^2 - \zeta^2)t + (S\xi + Q\eta)t^2 + \frac{1}{3}(S^2 + Q^2)t^3\right\} \tag{15}$$

Stream function (2) becomes therefore

$$\Psi(x, y, z, t) = -\left(U_0 + \frac{1}{2} Sy + Qz\right) y -$$

$$- \frac{AL^2}{4\pi^2[1 + (St+\xi)^2]} \exp\left\{K\zeta z - \gamma K^2 \cdot \right.$$

$$\cdot \left[(1 + \xi^2 + \eta^2 - \zeta^2)t + (S\xi + Q\eta)t^2 + \right.$$

$$\left.\left. + \frac{1}{3}(S^2 + Q^2)t^3\right]\right\} \sin\frac{2\pi}{L}\left\{x - [U_0 + Sy + Qz]t + \right.$$

$$+ \frac{\beta L^2}{4\pi^2 S}[\tan^{-1}(St+\xi) - \tan^{-1}\xi] -$$

$$\left. - \gamma K \zeta (Qt + 2\eta)t - \xi y - \eta z - \delta\right\} \tag{16}$$

The first term of the right hand side of (16) gives the velocity of the Couette flow with lateral velocity-shear s and vertical velocity-shear Q; the last term gives a sinusoidal wave with an amplitude which is exponential in z and varies with time. In (16) $(st + \xi)$ and $(Qt + \eta)$ are respectively the reciprocal of the slope of the trough and ridge lines with respect to y and z. ξ^{-1} and η^{-1} are therefore the initial slope of the trough and ridge lines of the waves.

Inspection of (16) reveals that the wave velocity and the slope of the trough and ridge lines vary with time. These are two characteristic features of wave motions in a Couette flow, whereas there is no change in the wave velocity and the inclination of trough and ridge lines in a (basic) uniform flow.

It is of importance to note that no stationary or quasi-stationary waves are possible to occur in a rotating Couette flow, since waves in such a system move with a speed,

$$C = U_0 + Sy + Qz - \frac{\beta L^2}{4\pi^2[1 + (St+\xi)^2]} +$$

$$+ \frac{4\pi\gamma\zeta}{L}(Qt + \eta) \tag{17}$$

which depends not only on the velocity of the Couette flow, wave length, latitude, but also on the instantaneous slope of the trough and ridge lines, the coefficient of kinematic viscosity and the exponential power of the amplitude of the velocity profile in z.

Examining frequency equation (17) one finds that term containing β has an extreme value at $t = -s^{-1}\xi$, which is possible only if $S\xi < 0$, and tends to zero as time tends to infinity, whereas the last term associating with viscosity, exponential power in z and the vertical slope of the trough and ridge lines decreases or increases with time according to the negative or positive sign of $Q\zeta$. It is readily seen that the last term of (17) disappears if the fluid is inviscid, or the velocity profile is independent of z, or when the trough and ridge lines are perpendicular to the zonal current. It is also readily seen that for uniform flow of an inviscid fluid frequency equation (17) reduces to that of Rossby's wave (ROSSBY, 1939),

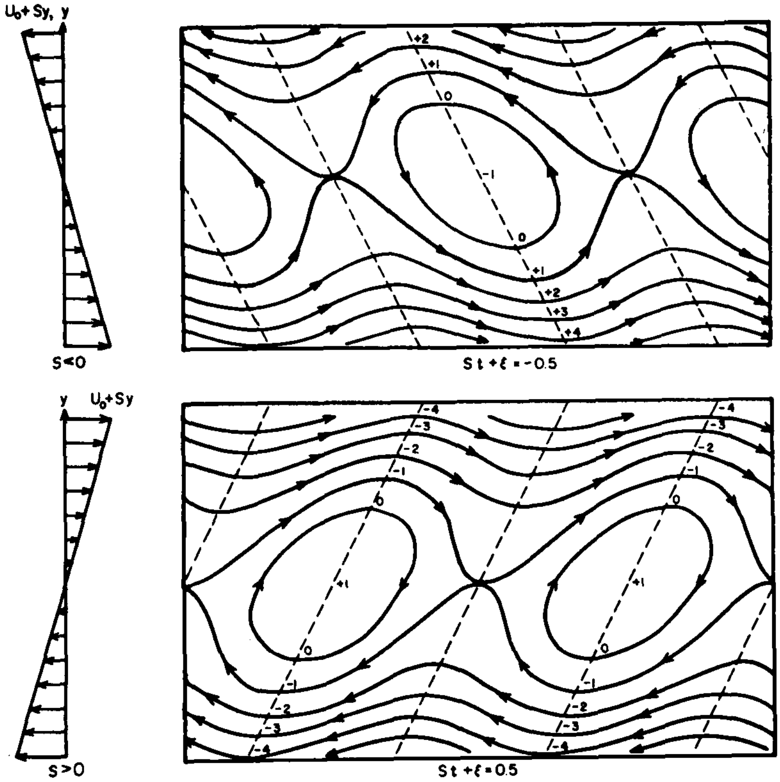

Fig. 1. Distributions of stream function (solid lines), ridge and trough lines (dashed lines).
a. For $S < 0$, $st + \xi = -0.5$, $\gamma = z = 0$
b. For $S > 0$, $st + \xi = 0.5$, $\gamma = z = 0$

$$C = U_0 - \frac{\beta L^2}{4\pi^2}$$

The amplification and damping of the waves depend on the behavior of the time dependency of the amplitude of the waves. By the inspection of (15) one finds that, for sufficient large value of t, $|F(t)|$ is a monotonic decreasing function of time. Thus *waves will eventually disappear in a Couette flow regardless the fluid is viscous or not.* However, under certain conditions waves may be amplified within a certain time interval. Waves may be damped, unchanged or amplified depending respectively on

$$\frac{d}{dt}\ln F(t) \gtreqless 0$$

or

$$S(St+\xi) + 2\pi^2\gamma L^{-2}[1 + (St+\xi)^2] \cdot \cdot [1 - \zeta^2 + (St+\xi)^2 + (Qt+\eta)^2] \gtreqless 0 \quad (18)$$

Let us consider first the wave motion in a Couette flow of an inviscid fluid. For this case only the term $s(st + \xi)$ remains in (18). Waves will be amplified when the slope of the trough and ridge lines is in the opposite sign to the lateral velocity-shear of the Couette flow, i.e. if $S\xi < 0$ and when $t < -S^{-1}\xi$. Wave amplitude reaches its maximum when the trough and ridge lines are perpendicular to the zonal current ($t = -s^{-1}\xi$), and drops when the slope of the trough and ridge lines is of the same sign as the lateral velocity-shear. It is interesting to note that for the inviscid case maximum wave amplitude occurs when the extreme wave velocity is reached.

Now we take into account of the frictional effect and consider the wave motion in a simple Couette flow with $\zeta = 0$. For this case the viscosity term in (18) is always positive. Viscosity tends therefore to suppress the amplification effect and bring the maximum wave amplitude to an earlier moment than in the case of an inviscid Couette flow. Amplification occurs if $S\xi < 0$ and $\frac{d}{dt}\ln F(t) > 0$. Waves are always damped when the slope of the trough and ridge lines are of the same sign as the lateral velocity-shear of the Couette flow, i.e. $S\xi > 0$.

The negative sign in front of ζ^2 in (15) and (18) indicates that the frictional effect resulted from the vertical exponential profile of the wave amplitude tends to amplify the wave. This is due to the growth of kinetic energy of the system resulted from such an exponential velocity profile. A discussion of the growth of kinetic energy of a system in relation to different velocity profiles may be found in a previous paper (KAO, 1954).

The distribution of the stream functions for different sign of lateral velocity-shears are shown in Fig. 1 a for $s < 0$, $st + \xi = -0.5$, $y = z = 0$, and in Fig. 1 b for $s > 0$, $st + \xi = 0.5$, $y = z = 0$. It is to be noted that the closed cyclonic streamlines occur around zero basic velocity with negative velocity-shear, anticyclonic streamlines around basic velocity with positive velocity-shear.

The pressure field corresponding to a stream function (16) may be obtained by the integration of the equations of motion for the horizontal directions

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+\gamma\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)u,$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-\frac{1}{\varrho}\frac{\partial p}{\partial y}+\gamma\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)v \qquad (19)$$

where f is the Coriolis parameter, p the pressure, ϱ the density. The result is given by the expression

$$\frac{p}{\varrho}=-\Bigg\{\frac{\beta L(St+\xi)}{2\pi[1+(St+\xi)^2]}\cos\frac{2\pi}{L}\cdot\left(x-[U_0+Sy+Qz]t+\frac{\beta L^2}{4\pi^2 S}[\tan^{-1}(St+\xi)-\tan^{-1}\xi]-\gamma K\zeta[Qt+2\eta]t-\xi y-\eta z-\delta\right)+$$
$$+\left(f_0+\beta y-\frac{2S}{[1+(St+\xi)^2]}\right)\sin\frac{2\pi}{L}\left(x-[U_0+Sy+Qz]t+\frac{\beta L^2}{4\pi^2 S}[\tan^{-1}(St+\xi)-\tan^{-1}\xi]-\gamma K\zeta[Qt+2\eta]t-\xi y-\eta z-\delta\right)\Bigg\}\cdot$$
$$\cdot\frac{AL^2}{4\pi^2[1+(St+\xi)^2]}\exp\left\{K\zeta z-\gamma K^2\cdot\left[(1+\xi^2+\eta^2-\zeta^2)t+(S\xi+Q\eta)t^2+\frac{1}{3}(S^2+Q^2)t^3\right]\right\}-\left(f_0+\frac{1}{2}\beta y\right)\cdot$$
$$\cdot\left(U_0+\frac{1}{2}Sy+Qz\right)y-\frac{1}{12}\beta Sy^3+E(z,t) \qquad (20)$$

in which f_0 is the value of the Coriolis parameter at $y = 0$, and $E(z, t)$ is an arbitrary constant or function of z and t. Equation (20) will be used for the computation of the part of meridional transport of energy due to the work done by the pressure field.

3. Wave Solutions of the Nonlinear Vorticity Equation for a (Rotating, Viscous) Couette Flow with Lateral Velocity-Shear

For a Couette flow with lateral velocity-shear only the vorticity equation may be obtained by putting $Q = 0$ in (3), the stream function and the corresponding pressure field for waves with vertically tilted trough and ridge lines may be obtained by putting $Q = 0$ respectively in (16) and (20). However, for waves without vertical tilt of trough and ridge lines $G(z)$ in (5) becomes less restricted than in a Couette flow with both lateral and vertical velocity-shear. It can be shown that in the former case functions R and h in (9) and (10) become independent of z, and $G(z)$ permits to be a sinusoidal function of z, or a hyperbolic sine or cosine of z, or of course independent of z.

If one follows the similar procedures as in the previous section, one may obtain for this case the stream function

$$\Psi(x, y, z, t) = -\left(U_0 + \frac{1}{2} Sy\right) y -$$

$$-\frac{AL^2}{4\pi^2[1+(St+\xi)^2]} \exp\left\{-\gamma K^2\left[(1+\xi^2 \pm \zeta^3)t + \right.\right.$$

$$\left.\left. + S\xi t^2 + \frac{1}{3} S^2 t^3\right]\right\} \begin{Bmatrix} \cos K(\zeta z + \varepsilon) \\ \cosh K(\zeta z + \sigma) \end{Bmatrix} \sin \frac{2\pi}{L} \cdot$$

$$\cdot \left\{x - [U_0 + Sy]t + \frac{\beta L^2}{4\pi^2 S} [\tan^{-1}(St + \xi) - \right.$$

$$\left. - \tan^{-1}\xi] - \xi y + \delta \right\} \qquad (21 \text{ a, b})$$

and wave velocity

$$C = U_0 + Sy - \frac{\beta L^2}{4\pi^2[1 + (St+\xi)^2]} \qquad (22)$$

where the positive sign in front of ζ^2 in (21) is associated with the vertical sinusoidal profile of the stream function, the minus sign with the vertical hyperbolic profile; ε and σ are two arbitrary constants. It is worth noticing that for this case the wave velocity is not affected by friction, and is completely determined by the velocity of the Couette flow, wave length, latitude, and the instantaneous slope of the trough and ridge lines. Like the wave motion in an inviscid Couette flow wave velocity reaches its extreme value when the trough and ridge lines are perpendicular to the zonal current.

It is to be noted that for waves with vertical sinusoidal amplitude-profile viscosity tends to damp out the waves, whereas for waves with vertical exponential amplitude-profile viscosity tends to amplify the waves.

For waves in such a Couette flow system the criterion of amplification becomes

$$S(St+\xi) + \frac{\gamma K^2}{2} [1 + (St+\xi)^2] \cdot$$

$$\cdot [1 \pm \zeta^2 + (St+\xi)^2] \gtreqless 0 \qquad (23 \text{ a, b})$$

corresponding to waves to be damped, unchanged and amplified. One may find that for waves with vertical exponential amplitude-profile the amplification of waves is similar to that in the Couette flow with both lateral and vertical vorticity-shears. For waves with vertical sinusoidal amplitude-profile viscosity always tends to suppress the amplification effect and bring the maximum wave amplitude to an earlier moment than that in the case of an inviscid Couette flow, although wave velocity in such a system is equal to that in an inviscid Couette flow.

The pressure fields corresponding to the stream functions (21 a, b) can be shown to be

$$\frac{p}{\varrho} = -\begin{Bmatrix} \cos K(\zeta z + \varepsilon) \\ \cosh K(\zeta z + \sigma) \end{Bmatrix} \frac{A}{K^2(1 + (St+\xi)^2]} \cdot$$

$$\cdot \left\{ \frac{\beta(St+\xi)}{K[1+[St+\xi)^2]} \cos \frac{2\pi}{L} \left(x - [U_0 + Sy]t + \right.\right.$$

$$\left. + \frac{\beta}{K^2 S} [\tan^{-1}(St+\xi) - \tan^{-1}\xi] - \xi y - \delta\right) +$$

$$+ \left(f_0 + \beta y - \frac{2S}{[1+(St+\xi)^2]}\right) \sin \frac{2\pi}{L} \cdot$$

$$\cdot \left(x - [U_0 + Sy]t + \frac{\beta}{K^2 S} [\tan^{-1}(St+\xi) - \tan^{-1}\xi] - \right.$$

$$\left.\left. - \xi y - \delta\right)\right\} \exp\left\{-\gamma K^2[(1 + \xi^2 \pm \zeta^2)t + S\xi t^2 + \right.$$

$$\left. + \frac{1}{3} S^2 t^3]\right\} - \left(f_0 + \frac{1}{2}\beta y\right)\left(U_0 + \frac{1}{2} Sy\right) y -$$

$$- \frac{1}{12} \beta S y^3 + P(z, t) \qquad (24 \text{ a, b})$$

where $P(z, t)$ is an arbitrary constant or function of z and t.

4. Physical Characteristics of Wave Motion in a (Rotating, Viscous) Couette Flow

In the previous sections we have studied the wave solutions of the nonlinear vorticity equation for a rotating Couette flow of a viscous fluid. It may be interesting to investigate the physical characteristics of these waves such as the northward transports of westerly momentum, vorticity and energy, and the geostrophic deviation.

(i) The rate of the northward transport of westerly momentum per wave length in unit height is given by

$$M_T = \int_0^L \varrho u v \, dx$$

where ϱu is the linear westerly momentum per unit volume and the multiplication by v

provides the northward transport of this momentum. By substituting for $u = -\partial\Psi/\partial y$, $v = \partial\Psi/\partial x$ from (16) and (21 a, b) one finds that

$$M_T = \frac{\varrho A^2 L^3 (St+\xi)}{8\pi^2[1+(St+\xi)^2]} \begin{bmatrix} \exp\left\{-2\gamma K^2\left[(1+\xi^2+\eta^2-\zeta^2)t+(S\xi+Q\eta)t^2+\frac{1}{3}(S^2+Q^2)t^3\right]\right\} \\ \exp\left\{+2\gamma K^2\left[(1+\xi^2\pm\zeta^2)t+S\xi\, t^3+\frac{1}{3}S^2t^3\right]\right\} \end{bmatrix} \tag{25 a, b, c}$$

corresponding respectively for waves represented by the stream functions (16) and (21 a, b). The sign of the transport depends solely on the instantaneous inclination of the trough and ridge lines; positive transport is associated with the northeast—southwest orientation of the trough and ridge lines, negative transport with the northwest—southeast orientation of the trough and ridge lines. Zero transport occurs when the trough and ridge lines are perpendicular to the zonal current, and the transport tends to zero as the trough and ridge lines tend to coincide with the zonal current, i.e. as $t \to \infty$.

It may be interesting to compare the rate of northward transport of westerly momentum due to true wind and the geostrophic wind, and compute the error produced by the geostrophic approximation. Let us denote the rate of the northward geostrophic transport of westerly geostrophic momentum per wave length in unit height by

$$(M_g)_T = \int_0^L \varrho\, u_g v_g\, dx$$

where u_g and v_g are the geostrophic wind components respectively in the x- and y-directions. These are obtained from (19) by setting the acceleration and frictional terms equal to zero, and by the use of (20) and (24 a, b). It can be shown that

$$(M_g)_T = \left\{\left(1-\frac{2S}{f[1+(St+\xi)^2]}\right)^2 - \left(\frac{\beta}{fK[1+(St+\xi)^2]}\right)^2\right\}\frac{\varrho A^2L^3(St+\xi)}{8\pi^2[1+(St+\xi)^2]} \tag{26}$$

For the convenience of comparing the actual transport and the geostrophic transport we shall neglect the frictional terms in (25), since the frictionel effect is generally small in the atmosphere. The percentage error produced by the geostrophic approximation is thus

$$100\left\{1-\frac{(M_g)_T}{M_T}\right\} = \frac{400}{[1+(St+\xi)^2]}\frac{S}{f}\cdot \left\{1-\frac{1}{1+(St+\xi)^2]}\frac{S}{f}\left(1-\frac{\beta^2}{4S^2K^2}\right)\right\} \tag{27}$$

For middle latitudes $f \sim 10^{-4}$ sec^{-1}, $\beta \sim 10^{-13}$ cm^{-1} sec^{-1}. If we let $L \sim 5\times10^8$ cm and $S \sim 5\times10^{-6}$ sec^{-1} which corresponds to a velocity shear of 20 m sec^{-1} over a latitudinal distance of 40 degrees, thus $S/f \sim 5\times10^{-2}$, $\beta/2\,SK \sim 1$. Since the first term of the right hand side of (27) is generally one to two order of magnitude larger than the other two terms, we may use the first term to estimate the error of transport produced by the geostrophic approximation. The error is about twenty per cent of the true transport for[1] $(st+\xi) = 10^{-1}$ about ten per cent for $(st+\xi)=1$, the latter corresponds to a 45° inclination of the trough and ridge lines with the x-axis. *The error increases with increasing lateral velocity-shear and with decreasing latitude.*

For waves of tilted trough and ridge lines in a uniform (basic) flow, i.e. $S = 0$ in (27), the error of the northward transport of westerly momentum is only about 0.5 per cent of the actual transport for $\xi = 10^{-1}$, and two-tenths of a per cent for $\xi = 1$. It is significant that the error of transport by geostrophic approximation in a Couette flow is more than ten times larger than that in a uniform flow.

(ii) It can be shown that the rate of the northward transport of mass vorticity is equal

[1] For $(st+\xi) = 0$, $M_T = (Mg)_T = 0$.

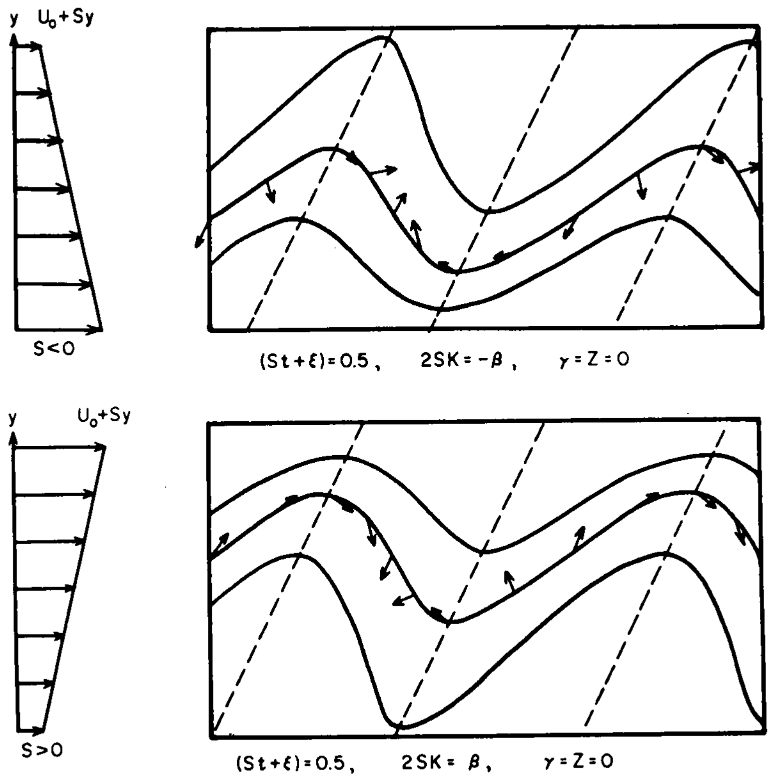

Fig. 2. Distributions of stream function (solid lines), geostrophic deviation (vectors), ridge and trough lines (dashed lines)
a. For $S < 0$, $2SK = -\beta$, $st + \xi = 0.5$, $\gamma = z = 0$
b. For $S > 0$, $2SK = \beta$, $st + \xi = 0.5$, $\gamma = z = 0$

to the rate of the meridional convergence of the northward transport of westerly momentum. Since waves in a Couette flow provide for an equal rate of the northward transport of westerly momentum at all latitudes, no net northward transport of mass vorticity is possible in a Couette flow.

(iii) Harmonic waves with tilted trough and ridge lines produce a north—south transport of energy. Since the motions are assumed horizontal with no net flow of fluid northward, the transport of energy is thus due to advection of kinetic energy and work done by pressure force. The meridional transport of energy integrated over one wave length per unit height is given by

$$E_T = \int_0^L \frac{1}{2}\varrho(u^2+v^2)v\,dx + \int_0^L pv\,dz$$

After substitution from the previous equations and integration, one finds that

$$E_T = \left\{U_0 + Sy + Qz - \frac{\beta L^2}{4\pi^2[1+(St+\xi)^2]}\right\}M_T \quad (28)$$

Thus the rate of meridional transport of energy is equal to the product of the wave velocity and the rate of the meridional transport of westerly momentum. Eastward displacement of waves and northeast—southwest tilting are associated with northward transport of energy, whereas westward displacement and northeast—southwest tilting are associated with southward transport of energy. Other combinations may also occur.

(iv) The difference between the actual and geostrophic wind vectors is called the geostrophic deviation. The components of this deviation are given by

$$u' = u - u_g; \quad v' = v - v_g$$

It can be shown that

$$u' = \frac{2SK}{f[1+(St+\xi)^2]} \frac{AG(z)}{J(t)} \left\{ (st+\xi) \cos K \cdot \right.$$

$$\left. \cdot [x - h(y, z, t)] + \frac{\beta}{2SK} \sin K [x - h(y, z, t)] \right\} \quad (29\text{ a})$$

$$v' = \frac{2SK}{f[1+(St+\xi)^2]} \frac{AG(z)}{J(t)} \cdot$$

$$\cdot \left\{ \cos K[x - h(y, z, t)] - \frac{\beta}{2SK}(St+\xi) \sin K \cdot \right.$$

$$\left. [x - h(y, z, t)] \right\} \quad (29\text{ b})$$

where $J(t)$, $G(z)$ and $h(y, z, t)$ have the same meanings as before, the latter two functions may be different for Couette flows with only lateral velocity-shear, and with both lateral and vertical velocity-shears. The geostrophic deviations depend on velocity-shear, wave length, latitude and the instantaneous slope of the trough and ridge lines. It can easily be shown that for the previous assigned values of the constants for atmospheric systems the geostrophic deviations can be as high as one order of magnitude smaller than that of the actual wind speed. It is particularly to be stressed that different signs of velocity-shear give different geostrophic deviations. Such differences are illustrated in Figs. 2 a and 2 b. Fig. 2 a shows the distributions of stream function (solid curves), geostrophic deviation (vectors), and the trough and ridge lines (dashed lines) for the case: $s < 0$, $st + \xi = 0.5$, $2SK = -\beta$ and $y = z = 0$; Fig. 2 b shows the distributions of the same quantities for $S > 0$, $2SK = \beta$, $st + \xi = 0.5$ and $y = z = 0$. It is to be remarked that these figures do not represent the typical distribution of geostrophic deviations, since the distribution of geostrophic deviation can be quite different for different values of velocity-shear, wave length, latitude and the inclination of trough and ridge lines.

5. Conclusions

It has been shown that the solutions of the nonlinear vorticity equation for a rotating Couette flow of viscous fluid yield waves of transient type, of which the amplitude, wave velocity, and the inclination of the trough and ridge lines vary with time. These waves will eventually disappear in the Couette flow regardless the fluid is viscous or not, although under certain conditions waves may be amplified within a certain time interval.

For Couette flow with both lateral and vertical velocity-shears the vertical profile of the amplitude of the waves can either be exponential in z or independent of z, whereas Couette flow with lateral velocity-shear only permits it to be independent of z, a sinusoidal function, or a hyperbolic sine or cosine of z.

For inviscid Couette flow waves will be amplified when the slope of the trough and ridge lines is in the opposite sign to that of the velocity-shear, but will be damped when they have the same sign. The maximum wave amplitude and the extreme wave velocity occur when the trough and ridge lines are perpendicular to the x-axis. For waves with vertical sinusoidal amplitude-profile viscosity tends to damp out the waves, whereas for waves with vertical exponential amplitude-profile viscosity tends to amplify the waves.

Pressure field, geostrophic deviation, meridional transport of energy and westerly momentum are obtained for such a flow system. It has been shown that for reasonable values of the constants for atmospheric systems the geostrophic deviation and the error resulted from the geostrophic approximation to the northward transport of westerly momentum can be as high as one order of magnitude smaller than the actual values.

Acknowledgment

The writer wishes to thank Drs George S. Benton and Robert R. Long for their critical review of this paper.

REFERENCES

KAO, S. K., 1954: Harmonic Wave Solutions of the Nonlinear Vorticity Equation for a Rotating Viscous Fluid. *J. Meteor*, **11**, No. 5 373—379.

ROSSBY, C.-G., 1939: Relation between Variation in the Displacement of the Zonal Circulation of the Atmosphere and the Displacement of the Semipermanent Centers of Action. *J. Marine Res.* (Sears Foundation), **2**, 38—55.