

A Theoretical and Experimental Study of Cellular Convection in Rotating Fluids

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Abstract

The stabilizing and cell-shape changing influence of a basic rotation upon thermally produced cellular circulations are developed in a first order, perturbation theory. The theoretical possibility of a special "overstable" oscillating mode in certain fluids under the influence of rotation is recognized. In this preliminary report, the theory obtained for ordinary convection is compared to results obtained in laboratory experiments with rotating cylinders of water, and the existence of the overstable oscillating mode is demonstrated experimentally. Several possible applications of the theoretical results to atmospheric convection are developed by means of numerical examples, including models that are consistent with the observed association of deep convection with cyclonic relative vorticity and the distribution of clouds in regularly spaced clear and cloudy areas.

Beginning with what was probably the first reported observation of the phenomenon by Weber, who in 1855 described cells with descending currents in their centers which he had seen in a thin layer of alcohol and water under a microscope, cellular convective circulations have been the subject of many investigations from the viewpoints of both experiment and theory. In 1882, James Thomson made the famous observation of "tessellated structure" in a tub of warm, soapy water standing in the yard of an inn. Studies under the controlled conditions of the laboratory began with the classical papers of HENRI BENARD (1900, 1901), the term "Benard cells" often being used in appreciation of his pioneer work. These early investigations plus several of the more recent studies have been summarized by BRUNT (1937, 1951).

LORD RAYLEIGH (1916) contributed the basic theoretical treatment. He obtained an expression determining the degree of instability necessary to overcome the dissipative influences of viscosity and heat conduction in terms of a non-dimensional parameter now known as the Rayleigh number (equation (21) below). As STOMMEL (1947) has pointed out, all subsequent analyses have been essentially refinements and extensions of Rayleigh's original work.

A few writers have considered the theoretical influence of rotation upon cellular convection. JEFFREYS (1928), one of the first, pointed out that "the effect of rotation is to maintain stability" and noted that high viscosities could actually aid instability by preventing the dominance of the stabilizing effect of rotation. Stommel refers to unpublished work in which he apparently obtained expressions similar to



Fig. 1. Top view of convection cells in water driven by evaporation cooling from above. Water 24.3°C ; air 22.5°C ; depth 2.7 cm; rotating 10.2 rpm; Taylor number 2.4×10^6 . The tracer is nigrosine ink introduced into the top layers. (Approx. one third size.)

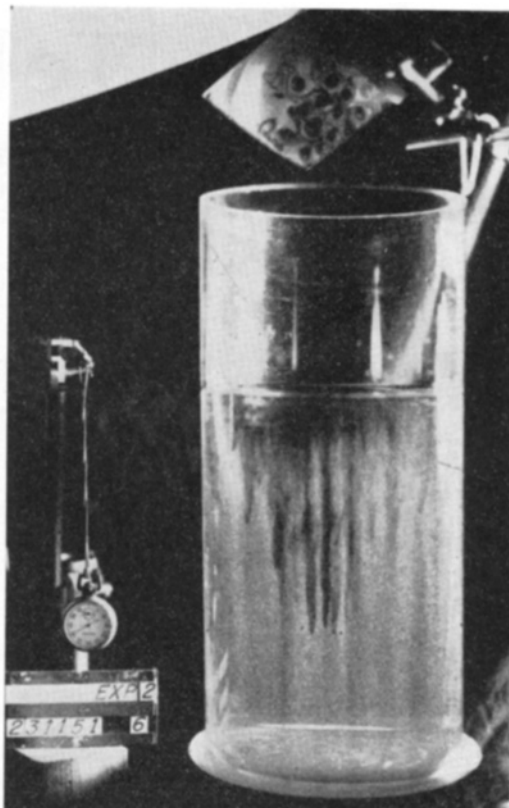


Fig. 2. Side view of convection cells in water driven by evaporation cooling from above. Water 25.7°C ; air 22.7°C ; depth 31 cm; rotating 25 rpm; Taylor number 2.5×10^{11} . Ink descending from the upper layers in the cell cores.

equations (13) and (15) below, for he mentions "an eight order partial differential equation in w and a cubic in n ". He also concluded that rotation inhibits convection and added that the effects are negligible if the depth of the fluid layer is less than $(\nu/\Omega)^{1/2}$, where ν is the kinematic viscosity and Ω is the angular velocity.

CHANDRASEKHAR (1953) has covered the theoretical problem of the influence of rotation upon convection quite thoroughly. His recent work, containing in part the solution included in this paper, came to the authors' attention after the present analysis had been completed by Nakagawa. Chandrasekhar gives explicit expressions for the critical Rayleigh numbers necessary for convection in the case of two free bounding surfaces and approximate critical Rayleigh numbers for the case of two rigid

and for the case of one free and one rigid boundary.

In the past, considerable speculation in cellular convection studies has been concerned with the observed directions of the vertical circulations. Early observations seemed to indicate that cells in liquids formed with ascending cores (with the exception of Weber, above), whereas cells in gases universally exhibited descending central cores. This last considerably damaged attempts to incorporate Benard cells in meteorological convection theory. Explanations based on the opposite variations of viscosity with temperature in liquids and gases were advanced for this apparent difference between convection cells in the two media.

Two investigators, Chandra and Dassanayake (summarized by BRUNT, 1937, 1951) separately

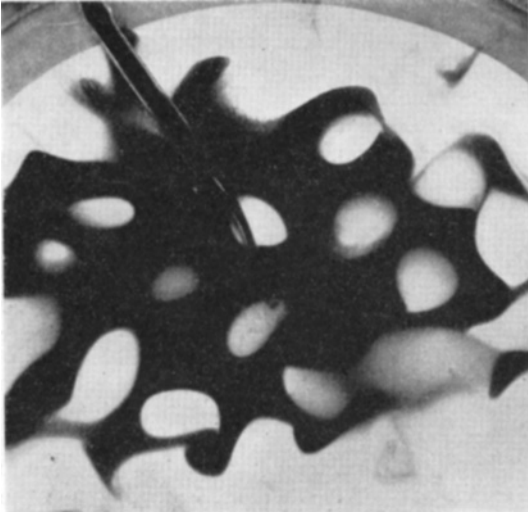


Fig. 3. Top view of holes punched in a pool of ink on the bottom of a cylinder of water by descending cores of cells driven by evaporation cooling. Water 24.3°C ; air 23.5°C ; depth 19 cm; rotating 10.0 rpm; Taylor number 5.7×10^9 . (Approx. one third size.)

demonstrated a special form of convection in gases in which ascending central currents did occur; but this "columnar type" was never observed in gaseous layers deeper than eight millimeters. Further, measurements of the vertical temperature gradient in these shallow layers indicated that the circulation was fundamentally different from the usual Benard-Rayleigh type of convection. A much smaller vertical temperature gradient was required to initiate the ascending columnar circulations than that called for by the Rayleigh criterion. SUTTON (1950) has suggested that this special mode is the result of instability in the shallow boundary layer above the heated surface and that it becomes the principal mode when the overall fluid depth is of the order of the boundary layer depth.

Convection cells in rotating laboratory models were first observed at almost the same time by two individuals working independently at widely separated locations, by D. Fultz at the University of Chicago and by Y. Nakagawa at the University of Tokyo. In the Chicago experiments, fields of cells with descending central cores were seen to occur in rotating cylinders of water whose temperature was very close to that of the room (Figs 1 and

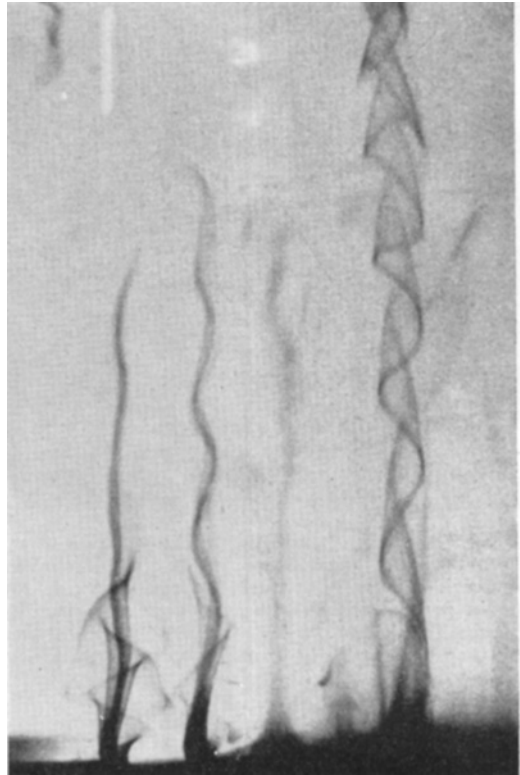


Fig. 4. Side view of ink rising from the bottom of a cylinder of water in the ascending cores of cells driven by heating from below. Water 23.5°C ; air 24.0°C ; depth 18 cm; rotating 10.9 rpm; Taylor number 5.5×10^9 ; heating plate input 5 watts. (Approx. full size.)

2). Evaporation cooling at the top surface provided the small degree of thermal instability necessary to drive the cells. They did not form in water more than a degree or two cooler than the room, while motions in water several degrees warmer than the air resembled the long wave and jet patterns observed in the "dishpan" experiments (CORN and FULTZ, 1954; FULTZ, 1952, 1954), because of horizontal temperature gradients set up by cooling at the walls of the container. In Tokyo, Nakagawa found cells with ascending central cores in rotating cylinders of water gently heated from below.

In every experiment run during the series reported in detail later, convection cells in rotating cylinders of water driven by cooling from above, either by evaporation or by contact with an external heat sink, exhibited

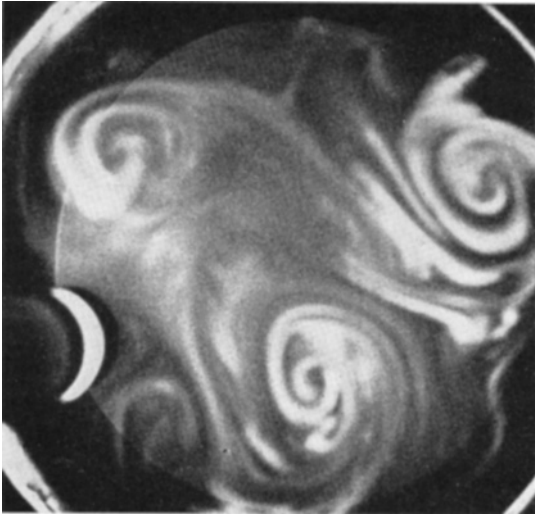


Fig. 5. Top view of an illuminated cross section 15 cm from the bottom of a rotating cylinder of air. The smoke tracer is converging and rotating cyclonically in the lower portions of cells with ascending cores driven by heating from below. Temperature 27°C ; depth 51 cm; rotating 5.3 rpm; Taylor number 2.9×10^8 ; heating plate input 1 watt. (Approx. one third size.)

descending central cores, while all of those driven by heating from below had ascending cores. The direction of the horizontal circulation at the top of each cell was that appropriate to the conservation of angular momentum in the divergent or convergent field, i.e., cyclonic for cells with descending cores and anticyclonic for those with ascending cores. The difference is strikingly illustrated in Figs 3 and 4. Here, the conditions of the two experiments are almost identical; the Taylor number (see eqn. (22)) for both experiments is approximately 5.6×10^9 . The slightly shallower depth in Fig. 4 (necessitated by the introduction of an aluminum plate bottom, in order to lift the base of the cells clear of the heating panel frame for a profile view) was compensated by an increase in the rate of rotation. In Fig. 3 evaporative cooling at the top surface has produced cells with descending cores that have punched holes in a pool of ink at the bottom, while in Fig. 4, a similar pool of ink feeds the ascending cores of cells driven by heating from below.

A series of unique observations of Benard-Rayleigh cells with ascending cores in air were made in a fifty-one centimeter deep, rotating cylinder heated from below. Cross sections of

the circulation at selected levels were photographed by illuminating the tracer smoke with an intense, horizontal slit of light. At the fifteen centimeter level (Fig. 5) cyclonic spirals are visible in a convergent field feeding the ascending cores. By the time the smoke has reached the upper levels of the cylinder it is quite diffuse and only the general outlines of the cells can be seen (Fig. 6). But visual observations determined that here the smoke in the large vortices was rotating anticyclonically; therefore the field of motion was divergent. Descending cores were observed in air cells when the rotating cylinder was cooled at the top by contact with a water bath. Relatively clearer air descending from the cooled top surface punched small holes down through the smoke at regular intervals. In a few cases the descending cores could be seen extending almost to the bottom of the fifty centimeter cylinder, but they were too indistinct to be photographed.

The results of these qualitative experiments can be summarized as follows: Under the stabilizing influence of rotation, convection cells in a fluid that is initially isothermal and finally approximately critical in vertical lapse rate occurred with their central cores moving away from the horizontal surface near which

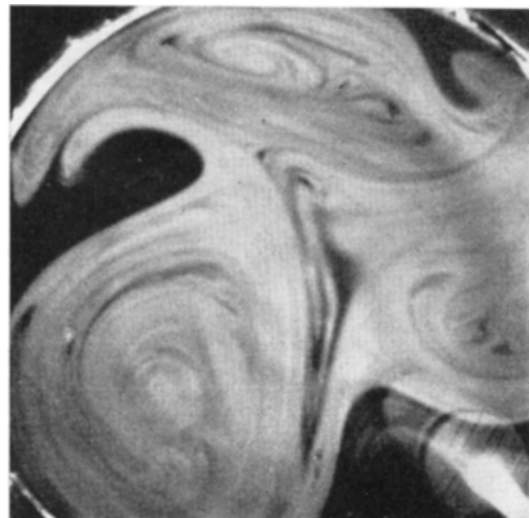


Fig. 6. Top view of a cross section 37 cm from the bottom of a cylinder of air with conditions similar to those of Fig. 5. The smoke tracer is diverging and rotating anticyclonically in the upper portions of the ascending cell cores. (Approx. one third size.)

the lapse rate had first attained the critical value. As we shall see, the theory suggests that the role of the rotation in this process is to increase the required critical lapse rate and to make it improbable that it be first attained anywhere other than near a surface where heat is being actively added or removed.

It is to be noted that this statement applies to lapse rate changes as they have been observed to occur in the experiments and not as they are assumed to occur in the theoretical discussions, below. With the relatively high rates of heating or cooling of the experiments, the critical lapse rate builds up or down from the appropriate surface, leaving the remaining isothermal layers undisturbed. In the theoretical analysis the lapse rate is assumed to increase uniformly, maintaining a linear character and finally establishing the critical lapse rate simultaneously throughout the depth of the fluid. Experimentally, with a low enough heating rate, adequate side insulation, and a provision to remove heat from the top at the same slow rate, this linear assumption might be fulfilled. If all the levels of an undisturbed fluid layer were brought to instability simultaneously in this ideal manner, the direction of the cells' vertical circulation would very probably either be completely arbitrary, or would be determined by some smaller effect such as the variation of viscosity with temperature,

Theoretical Analysis

As Jeffreys and others have pointed out, the effect of a basic rotation upon convection is a stabilizing one. There are two other, less apparent effects: 1) a special "overstable" oscillation appears as the first mode of instability in certain fluids under large rotational influence; and 2) the diameters of convection cells vary inversely with the rate of rotation, the ratio of the depth to the diameters of cells increasing markedly. The effects can be shown by extending Rayleigh's treatment to rotating systems.

For the analysis, the following equations apply.

The Eulerian form of the Navier-Stokes equations of motion for a fluid system rotating around the vertical z axis with angular velocity Ω :

$$\frac{Du}{Dt} - 2\Omega v = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \Omega^2 x + \frac{\nu}{3} \frac{\partial X}{\partial x} \quad (1)$$

Tellus VII (1955), 1

$$\frac{Dv}{Dt} + 2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v + \Omega^2 y + \frac{\nu}{3} \frac{\partial X}{\partial y} \quad (2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w - g + \frac{\nu}{3} \frac{\partial X}{\partial z} \quad (3)$$

where ν is the kinematic viscosity,

$$X = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right),$$

and the other symbols have their familiar meanings.

The equation of continuity:

$$\frac{D\rho}{Dt} + \rho(X) = 0 \quad (4)$$

The energy equation (GOLDSTEIN, 1943):

$$\frac{D\theta}{Dt} + \frac{p}{\rho J c_v} (X) = \kappa \nabla^2 \theta + \frac{\Phi}{\rho J c_v} \quad (5)$$

where θ is the temperature, κ the thermometric conductivity, J the mechanical equivalent of heat, and Φ is the "dissipation function". The equation of thermal expansion:

$$\rho = \rho_1 (1 - \alpha \Delta\theta) \quad (6)$$

where $\Delta\theta$ is the difference between the temperature at which the density is ρ and the temperature at which the density is ρ_1 ; α is the coefficient of thermal expansion.

Superposing small perturbations upon an initial state of relative rest (solid rotation) and neglecting terms of higher order involving the products of small quantities, a set of perturbation equations of motion is obtained. These are further simplified through a special incompressibility assumption first applied to the cellular convection problem by Rayleigh¹: the fluid is considered incompressible, the density constant, except in so far as thermal expansion affects the vertical equation of motion through the gravity term. In addition, the horizontal forces arising from centrifugal accelerations are considered small enough to be neglected. This essentially amounts to regarding the expansion coefficient α as a small quantity of the first order in the perturbation analysis. With

¹ Rayleigh credits the introduction of this assumption to BOUSSINESQ, *Theorie Analytique de la Chaleur*, vol. II, p. 172 (1903).

these considerations the perturbation equations of motion reduce to

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + v \nabla^2 u \quad (7)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + v \nabla^2 v \quad (8)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + v \nabla^2 w + \alpha \theta g \quad (9)$$

where the usual primes have been omitted and u , v , w , p , and θ now represent perturbation quantities.

Under the Rayleigh-Boussinesq assumption, the perturbation equation of continuity becomes

$$\nabla \cdot \mathbf{u} = 0 \quad (10)$$

where we have again treated α as a small quantity of the first order.

Assuming an initial state of relative rest, terms of the energy equation (5) involving velocity components or their derivatives disappear. Assuming in addition an initial steady state temperature distribution (conductive equilibrium) and considering the temperature field uniform in the horizontal, we have

$$\theta_0 = \Theta - \beta z \quad (11)$$

where $\beta = -\frac{\partial \theta_0}{\partial z}$ and Θ is the temperature at $z=0$. Substituting (11) in the reduced energy equation and applying the usual perturbation method we obtain

$$\frac{\partial \theta}{\partial t} - \beta w = \kappa \nabla^2 \theta \quad (12)$$

for the perturbation energy equation.

Equations (7), (8), (9), (10), and (12) are solved simultaneously to eliminate u , v , θ , and p .¹ An eighth order partial differential equation in w results:

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \left(\frac{\partial}{\partial t} - v \nabla^2 \right)^2 \nabla^2 w - \alpha \beta g \left(\frac{\partial}{\partial t} - v \nabla^2 \right) \cdot \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 4\Omega^2 \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \frac{\partial^2 w}{\partial z^2} = 0 \quad (13)$$

¹ If u , v , w , and p are eliminated, a similar eighth order expression results, but for θ in place of w . Thus the field of perturbation vertical velocity and the perturbation temperature are of similar configuration.

The solution of (13) is assumed to be separable in the form

$$w = f(x, y) g(z) h(t)$$

and is required to meet the following conditions: 1) it must have a cellular distribution in the horizontal, i.e., $\nabla_h^2 f(x, y) \propto f(x, y)$; 2) it must fit the boundary conditions appropriate to free surfaces at the top and bottom, i.e., $w=0$, $\theta=0$, $\partial \zeta / \partial z = 0$, and $\partial^2 w / \partial z^2 = 0$. A form meeting these requirements is

$$w = W_0 \cos(k_x x) \cos(k_y y) \cos[(\pi/D)z] \exp(nt) \quad (14)$$

where $k_j = 2\pi/L_j$, the wave number in the direction of j with corresponding wave length L_j , for $j=x, y$; W_0 is an arbitrary constant. The solution represents a rectangular cell of horizontal dimensions L_x and L_y with free top ($z = +D/2$) and bottom ($z = -D/2$) surfaces. For stability considerations, a time dependent exponential factor has been included in the usual manner.

Substituting (14) in (13) we obtain a third degree frequency equation for n :

$$n^3 + A_1 n^2 + A_2 n + A_3 = 0 \quad (15)$$

where

$$A_1 = \frac{(\kappa + 2\nu)(a^2 + \pi^2)}{D^2} \quad (16)$$

$$A_2 = \frac{\nu(2\kappa + \nu)(a^2 + \pi^2)^2}{D^4} - \frac{\alpha \beta g a^2}{(a^2 + \pi^2)} + \left\{ \frac{4\Omega^2 \pi^2}{(a^2 + \pi^2)} \right\} \quad (17)$$

$$A_3 = \frac{\kappa \nu^2 (a^2 + \pi^2)^3}{D^6} - \frac{\alpha \beta g a^2 \nu}{D^2} + \frac{4\Omega^2 \pi^2 \kappa}{D^2} \quad (18)$$

in which

$$a^2 = D^2(k_x^2 + k_y^2) \quad (19)$$

the two dimensional wave number (cf. PELLEW and SOUTHWELL, 1940). The roots, n , of this frequency equation, appearing as they do in the time dependent factor $\exp(nt)$ in (14), determine the stability condition of the system. The possible roots of the cubic, the criteria for their appearance in terms of the coefficients of the frequency equation (cf. VON KARMAN and BIOT, 1940), and the physical interpretation of

Table 1

Real root n	$n < 0$	$A_3 > 0$	damped; perturbation decreases	
	$n = 0$	$A_3 = 0$	neutral; perturbation steady	
	$n > 0$	$A_3 < 0$	amplified; perturbation increases; ordinary convective instability	
Complex root $m + ib$	$m < 0$	$A_1 A_2 - A_3 > 0$	perturbation causes oscillation	damped; decreasing amplitude
	$m = 0$	$A_1 A_2 - A_3 = 0$		neutral; steady amplitude
	$m > 0$	$A_1 A_2 - A_3 < 0$		amplified; increasing amplitude; "overstable" oscillation

Criteria for roots of the frequency equation and the corresponding physical behavior of the perturbation. (Provided $A_1 > 0$.)

the behavior of the imposed perturbation are summarized in Table 1. Two kinds of instability are indicated, cells which grow monotonically and cells which oscillate with increasing amplitude.

It is assumed that a system initially stable and proceeding toward instability will pass through a state of neutral equilibrium on the way (the principle of "marginal stability"). Therefore, $A_3 = 0$ or $A_1 A_2 - A_3 = 0$ will preface the appearance of one or the other kind of instability. An evaluation of each of these criteria shows the range of conditions for the first appearance of either kind in preference to the other.

Ordinary Convective Instability

Rewriting the condition $A_3 = 0$ from (18) we obtain

$$(a^2 + \pi^2)^3 - R a^2 + T \pi^2 = 0 \quad (20)$$

where the Rayleigh number

$$R = \frac{\alpha \beta g}{\kappa \nu} D^4 \quad (21)$$

and the Taylor number¹

$$T = \frac{4 \Omega^2}{\nu^2} D^4 \quad (22)$$

Since $R = f(a^2, T)$, for a specific T we may write $R = f(a^2)$. Fig. 7 plots three such relations (heavy solid line) for selected Taylor numbers. At any particular value of T there is a minimum

point designating the first occurrence of instability in a system proceeding from stability; the magnitudes of both the minimum Rayleigh number for convection and the wave number of the cells that first occur increase with increasing values of T .

From (20) we find $\partial R / \partial (a^2)$ and set it equal to zero to obtain

$$R_c = 3(a_c^2 + \pi^2)^2 \quad (23)$$

where the subscript c refers to the critical value of these parameters. Substituting (23) for R in (20) we have

$$(a_c^2 + \pi^2)^3 - 3(a_c^2 + \pi^2)^2 a_c^2 + T \pi^2 = 0 \quad (24)$$

The only real root of this equation is

$$a_c^2 = \pi^2 \cosh \frac{\phi}{3} - \frac{\pi^2}{2} \quad (25)$$

where

$$\phi = \cosh^{-1} \left(1 + \frac{2T}{\pi^4} \right) \quad (26)$$

By substituting (25) in (23), the relation between T and R_c for ordinary convective instability is obtained:

$$R_c = 3\pi^4 \left(\frac{1}{4} + \cosh \frac{\phi}{3} + \cosh^2 \frac{\phi}{3} \right) \quad (27)$$

This expression is plotted in Fig. 8 (heavy solid line). Beyond $T = 10^4$ the curve approximately parallels $R_c = T^{3/4}$ on a log-log plot. At $T = 0$, $R_c = 27\pi^4/4$ and $a_c^2 = \pi^2/2$, values which are identical to those obtained by others for the non-rotating, two free boundary case.

¹ This quantity, originally used in this particular form by Chandrasekhar readily derives from a parameter first introduced by G. I. TAYLOR.

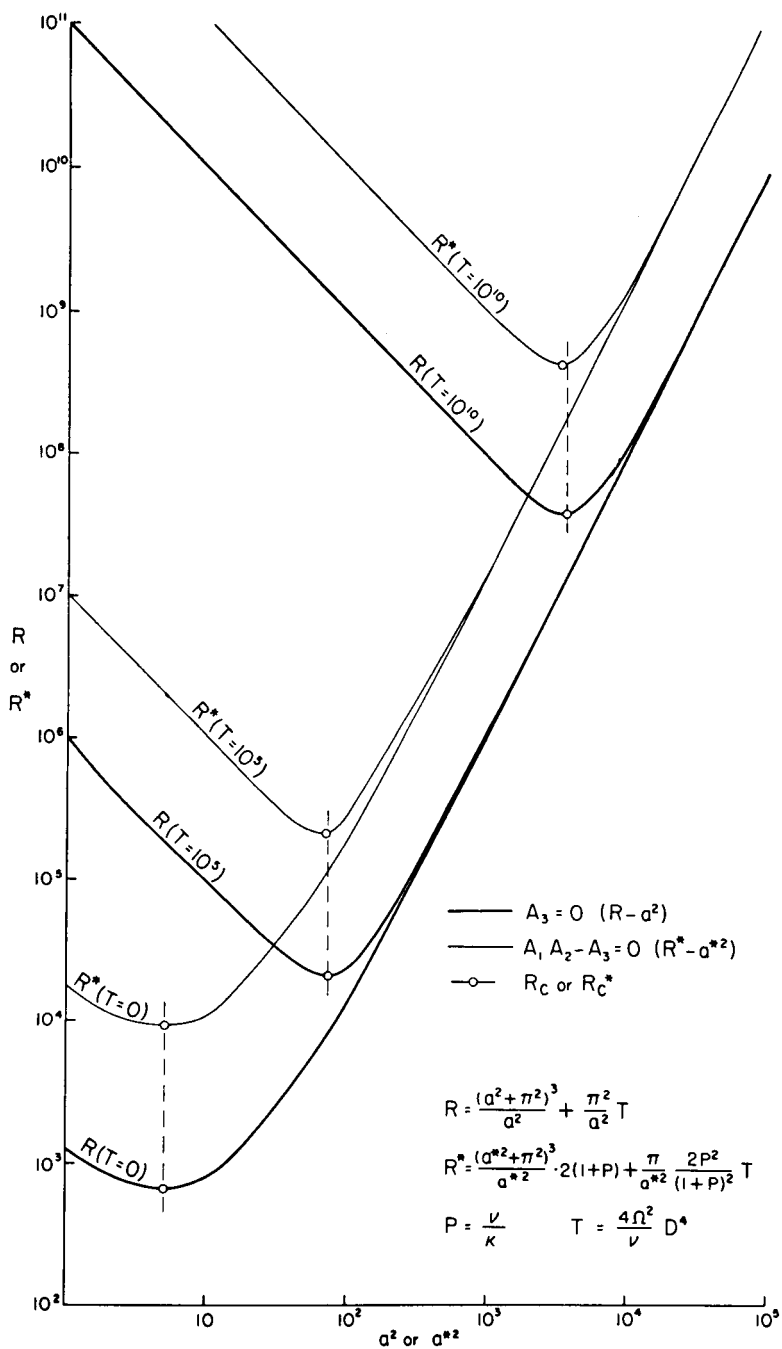


Fig. 7. Wave number versus required Rayleigh number for ordinary convective and overstable oscillating modes (starred quantities) in water ($P=6$) at several values of the Taylor number. Note minimum Rayleigh number for a specified Taylor number, designating the conditions for the first appearance of instability.

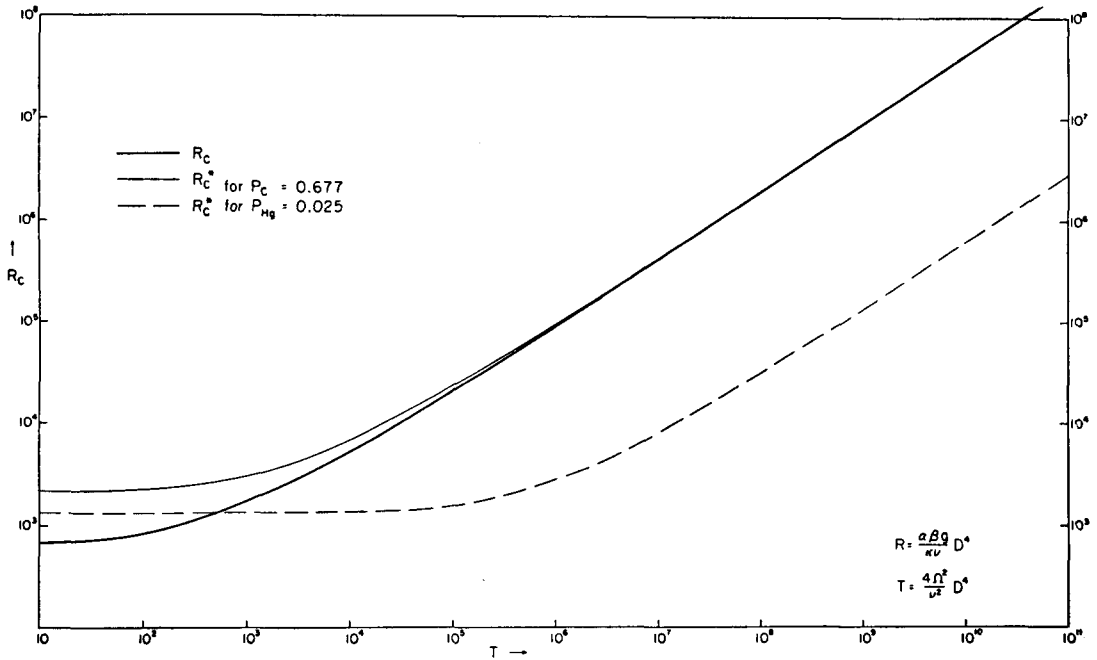


Fig. 8. Taylor number versus critical Rayleigh number for the first appearance of ordinary convection (R_c), for overstable oscillation (R_c^*) at the critical Prandtl number ($P=0.677$), and for overstable oscillation at the Prandtl number of mercury ($P=0.025$).

Overstable Oscillation

Consider now the second condition, $A_1 A_2 - A_3 = 0$. Substituting for the coefficients from (16), (17), and (18) we have

$$2(1+P)^2(a^2 + \pi^2)^3 - Ra^2(1+P) + 2P^2 T \pi^2 = 0 \quad (28)$$

where the Prandtl number

$$P = \nu/\kappa \quad (29)$$

By operations similar to those used to find R_c from (20) we obtain

$$a_c^{*2} = \pi^2 \cosh \frac{\phi^*}{3} - \frac{\pi^2}{2} \quad (30)$$

$$R_c^* = 6(1+P)\pi^4 \left(\frac{1}{4} + \cosh \frac{\phi^*}{3} + \cosh^2 \frac{\phi^*}{3} \right) \quad (31)$$

$$\text{where } \phi^* = \cosh^{-1} \left[1 + \left(\frac{P}{1+P} \right)^2 \frac{T}{\pi^4} \right] \quad (32)$$

Since R_c^* is a function of both T and P , it is represented by a family of curves rather than

a single one as was possible with R_c . Two such curves are plotted in Fig. 8 for two values of P (thin solid and thin broken lines).

These starred quantities, deriving as they do from the criterion for the occurrence in the frequency equation of complex roots with positive real parts, are appropriate to the first appearance of the second kind of instability, oscillations of increasing amplitude. Chandrasekhar calls the phenomenon "overstability" after Eddington who developed the concept in a theory concerning the pulsations of Cepheid variable stars. Eddington's choice of the term is best explained in his own words: "In the usual kind of instability a slight displacement provokes forces tending away from equilibrium; in overstability it provokes restoring forces so strong as to overshoot the corresponding position on the other side of equilibrium and sets up an increasing oscillation" (EDDINGTON, 1926, p. 201).

In the region of larger values of T , members of the R_c^* family also parallel a $T^{1/4}$ curve. Specifically, for increasing values of T , R_c and R_c^* approach the respective asymptotes

$$R_c = \frac{3}{4^{1/3}} \pi^{1/3} T^{2/3} \cong 8.7 T^{2/3} \quad (33)$$

$$R_c^* = 6(1+P) \left(\frac{\pi^4}{4} \right)^{1/3} \left(\frac{P}{1+P} \right)^{1/3} T^{2/3} \cong \\ \cong 17.4 \cdot \frac{P^{1/3}}{(1+P)^{1/3}} T^{2/3} \quad (34)$$

At one value of P and a large value of T these asymptotes coincide. Setting (33) equal to (34) we obtain a biquadratic in P :

$$8P^4 - P - 1 = 0 \quad (35)$$

which has as its only positive root, $P_c \cong 0.677$.¹ At this critical value of the Prandtl number, the values of R_c and R_c^* practically coincide for all T greater than 10^6 (see Fig. 8) and exactly coincide when T is infinitely large.

As T approaches zero, a_c^{*2} approaches $\pi^2/2$, the same value as a_c^2 at $T=0$. Fig. 7 shows that a_c^{*2} gradually becomes smaller than a_c^2 as T increases. Thus, if both types of convection were occurring at the same non-zero value of T , the overstable cells would be somewhat larger in diameter than the ordinary type. This point leads to an interesting meteorological application in a later section.

Also as T approaches zero, R_c^* approaches $27 \frac{\pi^4}{2} (1+P) \cong 1315 (1+P)$, a value which is never smaller than twice as large as R_c for the non-rotating case. This would seem to imply that complex roots with positive real parts might exist in the case of non-rotation, provided only that the Rayleigh number be made large enough. But RAYLEIGH, JEFFREYS (1926), and PELLEW and SOUTHWELL showed that such roots could only occur with negative real parts in the non-rotating case. The point of departure of these results can be traced to the partial differential equation (13). If from this point one proceeds on the assumption that $\Omega=0$, then the third term on the left disappears and each of the remaining terms contain as a common factor the operator $(\partial/\partial t - \nu \nabla^2)$ which can be cancelled. This reduces the order of the partial differential equation by two and the degree of the frequency equation by one. But the quadratic frequency equation thus

obtained can only have complex roots with negative real parts, for the coefficient of the first degree term is always positive. On the other hand, as we have seen, the cubic frequency equation will have complex roots with positive real parts when the relation $A_1 A_2 - A_3 < 0$ among the positive coefficients is fulfilled.

The period of the overstable oscillation can be determined by considering the imaginary part of the complex roots of the frequency equation. The product of the roots of (15) taken two at a time is

$$n(m+ib) + n(m-ib) + m^2 + b^2 = A_2 \quad (36)$$

But at the critical value R_c^* , $m=0$; so (36) becomes $b^2 = A_2$. Rewriting (17) we have

$$b^2 = \frac{\kappa^2}{D^4} \frac{P}{(a_c^{*2} + \pi^2)} \cdot \left. \begin{aligned} & \cdot [(2+P)(a_c^{*2} + \pi^2)^3 - R_c^* a_c^{*2} + P T \pi^2] \end{aligned} \right\} \quad (37)$$

Since b is the imaginary part of a complex root of the frequency equation, the period of oscillation is

$$\tau = \frac{2\pi}{b} \quad (38)$$

Summary of Instability Criteria

So the question of which type of instability appears before the other is decided by an intrinsic characteristic of the fluid medium, the Prandtl number, and the rotational parameter, the Taylor number. For fluids with $P > 0.677$ (e.g., water, air), R_c is smaller than R_c^* for all T , and therefore ordinary convective stability will always occur first. For fluids with $P < 0.677$ (e.g., mercury), the critical Rayleigh curves for the two types of instability cross at some point, R_c^* becoming the smaller in regions of large T (see Fig. 8). Thus, ordinary convection would be the first to occur at low values of T , and the overstable oscillating mode would take priority at large values of T . In the case of fluids with $P \geq P_c$, the R_c curve alone describes a general criterion for the breakdown of stability; in fluids with $P < P_c$, the appropriate critical curve would be the envelope containing the R_c curve from $T=0$ to the crossover point (a function of P) and the R_c^* curve from the crossover to infinitely large T .

The curves plainly show the stabilizing effect of rotation first pointed out by Jeffreys, R_c

¹ An equivalent value was obtained by Chandrasekhar, though he writes the reciprocal, $\kappa/\nu = 1.478$.

increasing rapidly beyond $T=10^2$ for ordinary convection, and beyond some $T(P)$ for overstable oscillation. Jeffreys' observation that high viscosities could actually aid instability can also be seen. Since $T \propto \nu^{-1/2}$ and $R \propto T^{1/2}$ for large T , it readily follows that $\beta \propto \nu^{-1/2}$.

It has been estimated that high conductivity might also assist instability by "preventing the (stabilizing) effects of rotation from being dominant", but this does not follow since α appears only in the denominator of R_c . There is, as it were, no early need for a convective regime to set in if the conductivity is high, for large amounts of heat can be transferred by conduction alone.

The validity of Stommel's limit, rotational influence negligible in fluid layers of depth less than $(\nu/\Omega)^{1/2}$, becomes evident when this value is substituted for D in the Taylor number. The result is $T=4$, a point clearly in the flat portion of the curve, R_c having increased only about one per cent over its value at $T=0$. For a ten per cent increase in the critical Rayleigh number, the limit becomes $D \leq 1.7(\nu/\Omega)^{1/2}$; for a one hundred per cent increase, the constant multiple is 3.3.

Cell Shapes

Experimental observations have shown that cells of hexagonal shape most often appear (centers of adjacent cells are usually located near the vortices of an equilateral triangle, cf. Figs 1 and 10). Beginning with an expression for w representing a hexagonal distribution (Christopherson, 1940),

$$w = \frac{W_0}{2} \left[\cos \frac{2\pi}{3L} (3^{1/2}x + y) + \cos \frac{2\pi}{3L} (3^{1/2}x - y) + \cos \frac{4\pi}{3L} y \right] \cos \frac{\pi}{D} z \quad (39)$$

where W_0 is an arbitrary constant and L is the length of one side, an expression for the streamlines at the top surface can be derived; for large T ,

$$\Psi = \left(\cos \frac{2\pi}{3L} 3^{1/2}x \right) \left(\cos \frac{2\pi}{3L} y \right) + \frac{1}{2} \cos \frac{4\pi}{3L} y \quad (40)$$

Because of the restriction to large T (considering negligibly small those terms which do not contain T), equation (40) represents concentric circles instead of spirals. These streamlines and

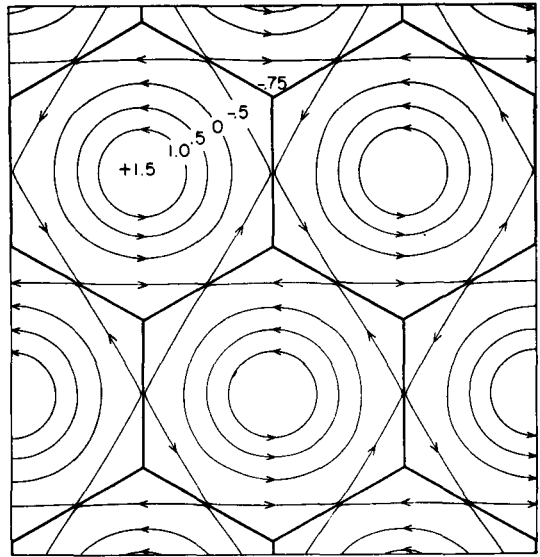


Fig. 9. Theoretical streamlines for the top surface of hexagonal convection cells at a high Taylor number. Numerical data represent arbitrary units of relative vorticity.

a set of relative vorticity values are plotted in Fig. 9.

The effect of rotation upon the overall shapes of cells, i.e., the increase of the cells' depth-diameter ratio, can be demonstrated by a further consideration of the two dimensional wave number. From equation (39)

$$a^2 = D^2 \left(\frac{4\pi}{3L} \right)^2 \quad (41)$$

Combining this with the distance between the centers of adjacent hexagons as an average cell diameter, we obtain for the depth-diameter ratio

$$\frac{D}{d} = \frac{3^{1/2}}{4\pi} a_c \quad (42)$$

for which a_c may be determined from (25). The increase in depth-diameter ratios with increasing rates of rotation can be seen in Table 2.

Table 2

$T =$	0	10^3	10^4	10^9	10^{12}
$D/d =$	0.3	0.5	1.8	5.7	18.0

The increase in cellular depth-diameter ratios with increasing Taylor number in water at 25° C.

Table 3

	D/d	
	Predicted	Measured
Fig. 1	2.1	2.1
Fig. 2	14.3	15.5
Fig. 10	5.8	5.6

Cellular depth-diameter ratios predicted theoretically compared to those measured in several of the illustrated experiments.

Table 3 lists the ratios predicted by (42) for several of the experiments illustrated and compares them with actual measurements.

The Experimental Apparatus

The program of experiments intended to test the theoretical results was designed with the non-dimensional Rayleigh and Taylor parameters in mind. Water was used as both a readily available medium and one whose physical properties are accurately known and tabulated (DORSEY, 1940). It remained for the experimenter to measure the angular velocity of the rotating model and the depth, mean temperature, and vertical lapse rate of the fluid. By changing the depth and/or the angular velocity a range of experiments at different values of the Taylor number was carried out, measuring the lapse rate at which convection began in each one as a dependent variable. In this way, an array of points along the R_c , T curve was determined.

For steady rotation of the fluid models a leveled, eighteen inch diameter, three inch wide steel ring was mounted on a vertical shaft. The ring provided a load bearing surface with provision for leveling, centering, and clamping various experimental models in place. The shaft was driven through V-belts by a constant speed induction motor with a continuously variable transmission providing a range of speeds at the mounting ring of from zero to fifty revolutions per minute. An electric counter-timer circuit measured the time required for any whole number of revolutions to one hundredth second accuracy.

Containers for the experiments were made from open end pyrex glass cylinders of twelve inch nominal outside diameter. Pyrex bottoms were cemented to one end of each cylinder,

electrically conducting pyrex heating plates for experiments heating from below and ordinary pyrex plate for those with cooling from above.

Power for the heating plates was supplied by a variable transformer and was carried into the rotating system through mercury slip-rings. A wattmeter measured the input; but due to extraneous losses, only a fraction of the indicated input was used to drive the cells, so the meter readings could only be used comparatively. Cooling from above was usually accomplished by evaporation at the free surface, but a few trials were made with an external heat sink. A second, thin walled cylinder of slightly smaller diameter was mounted in the upper part of the main cylinder, its bottom in contact with the water surface. Cool water from a thermostatically controlled mixing valve was then continuously circulated through the upper cylinder, making it an approximately constant heat sink of adjustable capacity.

Visual observations were made with the aid of a rotoscope, a reversing prism mounted with its optical axis along the axis of rotation of the turntable and rotated at exactly half the table's speed by a selsyn drive geared to the main shaft. With the aid of this device, all motions in the model are seen relative to an apparently stationary turntable.

Top surface circulations were made visible by small tracer particles of aluminum powder, their movements either viewed directly with the aid of the rotoscope or photographed with the time exposure, "streak" technique. An electronic flash triggered just before the end of an exposure marked the direction in which the particles were travelling. Fig. 10 is a positive reversal of one such "streak-flash" photo. Evaporation cooling was driving cells with descending cores, and although there is some confusion where streaks overlap and where spurious flashes appear because of reflections from the particles, the majority of flashes indicate that the circulation was cyclonic (southern hemisphere sense, since the basic rotation was clockwise).

Internal cell motions were traced with nigrosine ink and photographed by conventional instantaneous flash technique (Figs 3 and 4). To illuminate the smoke tracer (ammonium chloride) circulating in cells in air, the light from the electronic flash was collimated by a

horizontal slit and projected through the experimental cylinder at a selected level while the camera viewed the cross section from above (Figs 5 and 6).

The vertical temperature gradient was the crucial variable of the experiment. It was evident from the theoretical results that the lapse rates would be so very small that a sensitive measuring system would be required. The critical Rayleigh number for ten centimeters of water rotating at ten revolutions per minute calls for a lapse rate of only 0.025°C per cm. Further, the temperature sensitive elements would have to be small enough so as not to interfere with the delicate cellular circulations, and they would have to be able to transmit their information to a measuring device outside the rotating system.

These requirements were met with copper-constantan wire thermocouples. Ten junctions connected in a series thermopile gave an output of a little better than two hundred microvolts per degree Centigrade and also provided a five point sample of the temperature at each level, thus serving to average out the horizontal temperature gradients within the cells. The mean temperature of the fluid was taken at the end of each run with a mercury thermometer and used to determine the tabulated values of α , κ , and ν .

The temperature signal was carried from the model to the rotating member of a set of mercury slip-rings, six concentric copper cylinders mounted in a lucite plate that turned with the main shaft. The stator consisted of a thicker lucite plate in which six concentric circular troughs had been cut, each then filled to about two-thirds of its depth with mercury. The copper cylinders rotated freely in the stator troughs with their lower edges well under the mercury surface. The device maintained three virtually constant resistance circuits without contact "noise" between the rotating and non-rotating parts of the apparatus. The mercury slip-rings introduced little or no detectable errors into the thermopile circuit, for there was no opportunity for a temperature difference to build up across the thermocouple consisting of the copper rotor, the continuously mixed mercury, and the copper terminals of the stator.

From the mercury troughs, the signal was carried to an automatic potentiometer that balanced the small voltage input and provided

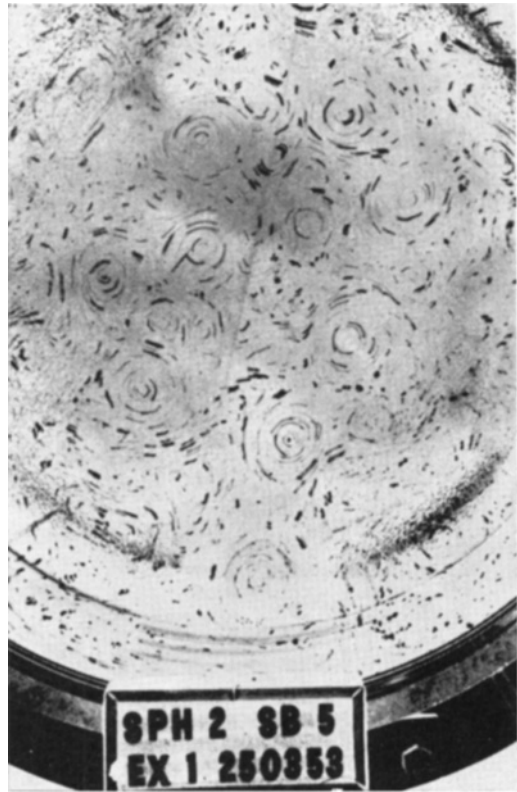


Fig. 10. A positive reversal of a streak photograph of the top surface of cells driven by evaporation cooling in water. Water 22.6°C ; air 23.3°C ; depth 18 cm; rotating 5.0 rpm; Taylor number 1.2×10^9 ; exposure time four seconds. (Approx. one half size.)

a proportional current which was fed in turn to a recording milliammeter. The entire system, used in conjunction with a ten junction thermopile, was capable of detecting temperature differences reproducible within $\pm 0.01^{\circ}\text{C}$.

Experimental Verification of the R_c Criterion

It was evident that lapse rate measurements should be made with as steady conditions as possible in order best to approximate the assumptions of the theoretical analysis. It seemed that this could be done by supplying only the minimum amount of heating from below necessary to drive the convective circulation. This was investigated by studying the character of vertical temperature difference records of the type illustrated in Fig. 11. In the curve at the top, the heating rate is too low to produce convective instability; the vertical temperature

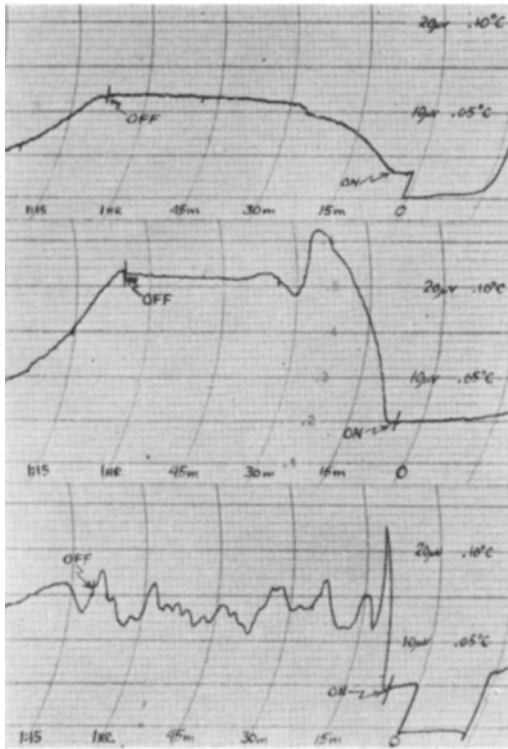


Fig. 11. Vertical temperature difference records made at three different rates of heating from below. Depth 3 cm; rotating 10 rpm; thermopiles at 1.0 and 1.9 cm levels; one, three, and ten watts heating plate input, top to bottom.

difference trace rises gradually and asymptotically approaches some constant value where the vertical transfer of heat by conduction alone is balanced by the losses at the top surface. In the bottom curve, the heating rate is too large for steady convective heat transfer; the temperature trace rises sharply, breaks down suddenly as unsteady cells are established, and continues to oscillate with the changing vertical circulations. The middle curve is characteristic of a heating rate somewhat greater than the minimum, in this case, three watts, whereas the minimum was something nearer two watts. The trace rises smoothly to a supercritical peak value, breaks down as convection begins, and goes into a "damped oscillation" that levels out at the critical lapse rate value after the initial overturning is completed and approximately steady state convection has been fully established. In the experimental series, greater than minimum heating rates were used, since

with bare minimums it became apparent it would take hours to establish cells. In general, rates that could establish cells in from fifteen to twenty minutes were found convenient; visual observations of the cellular circulations under these conditions showed them to be as steady as could be desired.

A series of experiments were carried out in which the mean temperature at five different levels was measured within thirty seconds by switching the recording system across five different thermopiles. These showed that the lapse rate became essentially linear in the middle layers of the fluid ten or fifteen minutes after convection had begun, but that it was steeper than average within both the lowest centimeter where heat was being added and in the top two or three centimeters near the evaporation cooled free surface. Excess cooling was reduced by partially restricting the ventilation of the top surface with a porous lid. In all subsequent experiments, average lapse rates were measured in the middle layers of the fluid. The thermopile grids were set as far apart vertically as was possible in the particular fluid depth, but they were kept clear of the excessively warm lower and cold upper layers.

In Fig. 12 the values of critical Rayleigh numbers obtained in these experiments are plotted along a section of the theoretical R_c , T curve. The close correspondence between the experimental points and the theoretical curve agrees with Chandrasekhar's result that the two free and the one rigid, one free boundary cases are closely equivalent except when the influence of rotation is low, for the curve was derived under the assumption of two free boundaries whereas the points were observed in a model in which one boundary was rigid.

Experimental Verification of the R_c^* Criterion

At the time of the present preliminary report, experiments designed to investigate the overstable mode have begun. Mercury, with a Prandtl number of 0.025, well under the critical 0.677, is the fluid used in these trials. The dashed curve of Fig. 8 is the theoretical relationship for the occurrence of the second kind of instability in mercury, again under the assumption of two free bounding surfaces. For all values of the Taylor number in excess of 10^3 , R_c^* is less than R_c and therefore the overstable mode will be the first to occur.

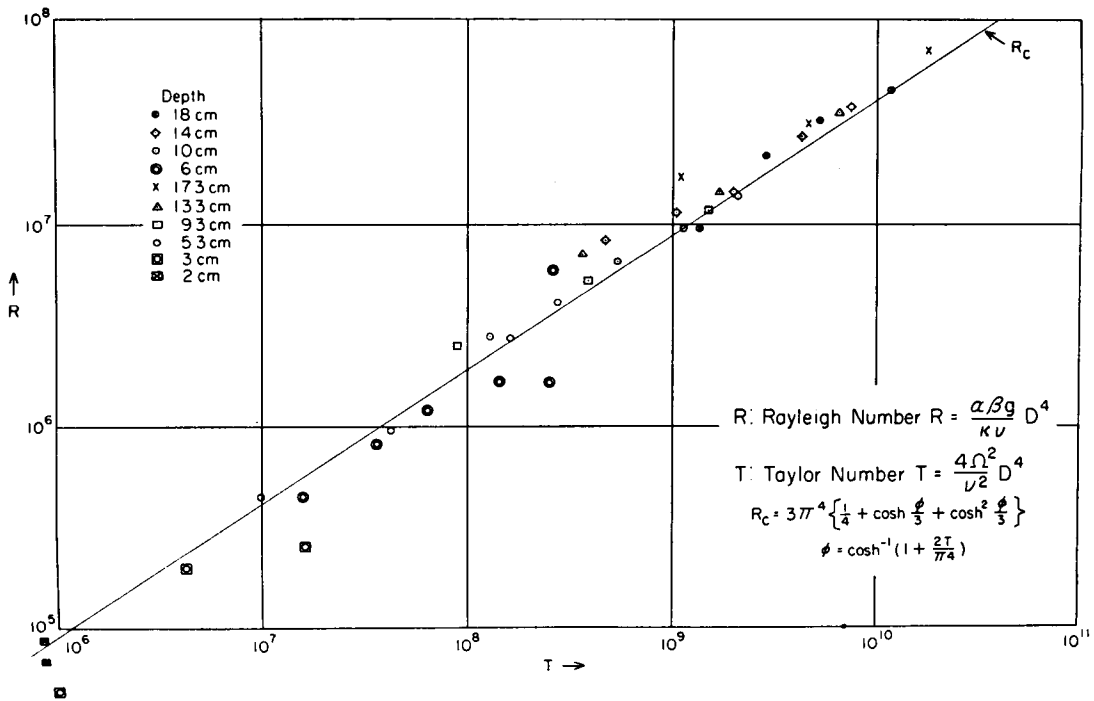


Fig. 12. Experimental results versus theoretical curve for ordinary cellular convection in water.

The first trials in this series have shown that the phenomenon of overstability does exist (FULTZ, NAKAGAWA & FRENZEN, 1954). A five centimeter deep layer of mercury was heated from below while rotating approximately ten revolutions per minute. Several cell-like circulations, each about two centimeters in diameter, appeared at the top surface, their motion made visible by small particles of impurities. Each cell periodically reversed the direction of its relative rotation, implying alternating divergence and convergence at the top surface and hence alternating ascent and descent in the central core. The period of the oscillation was about fifteen seconds, very close to the 13.9 seconds predicted by equation (38).

Recalling Eddington's definition of overstability, "... a slight displacement . . . provokes restoring forces so strong as to overshoot the corresponding position on the other side of equilibrium", one can describe a mechanism for these oscillating cells. If in a system heated from below we have a fluid of sufficiently small Prandtl number, i.e., a fluid with a high enough thermometric conductivity and low enough kinematic viscosity, then the lower

layers readily absorb heat and, expanding, soon become sufficiently buoyant to overcome the restraint imposed by the viscosity. Convective cells begin to form in the usual manner, the cores carrying the hot fluid upward and the cell walls bringing colder fluid downward to replenish the lower layers. Because of its high conductivity and a consequent ability to absorb an excess amount of heat before vertical motion can begin, the fluid attains more buoyancy than that required merely to overcome the weak viscous stresses. Therefore the overturning takes place rather more violently than is the case with steady, ordinary convection cells. The rapid transfer of hot fluid upward and cold fluid downward destroys the unstable density stratification, and the fluid begins to decelerate. But since the fluid began its motion with an excess buoyancy, it initially "overshoots the corresponding position on the other side of equilibrium". Excess cooling in the upper layers, again made possible by the high conductivity, drives the other half of the cycle in a similar manner, but in the opposite direction. In practice, the fluid will not continue to overshoot further and further at the end of

each swing, since there are practical limits to the fluid velocities and hence to the amplitude of oscillation. In short, during the oscillation the fluid alternates between two states of unstable stratification, one the equivalent of supercritical heating from below and the other, of supercritical cooling from above.

In a rotating system the tendency for fluid rings to resist horizontal divergence supplies an additional restoring force when the rings are forced to move away from their equilibrium position by the appearance and disappearance of fluid in their centers. It may be that thermal expansion effects alone are insufficient to main-

interpreted from the standpoint of the several modes of heat transfer involved. The trials illustrated were carried out in a three centimeter deep layer of water, rotating at ten revolutions per minute. The temperature values recorded are the differences between the 1.0 and 1.9 centimeter levels. The only change in the experiment between the separate traces is in the heating rate.

When the power supplied is one watt, heat is transferred vertically by conduction alone. Approximately steady state convective heat transfer occurs when the system is heated with three watts input at the plate, while ten watts

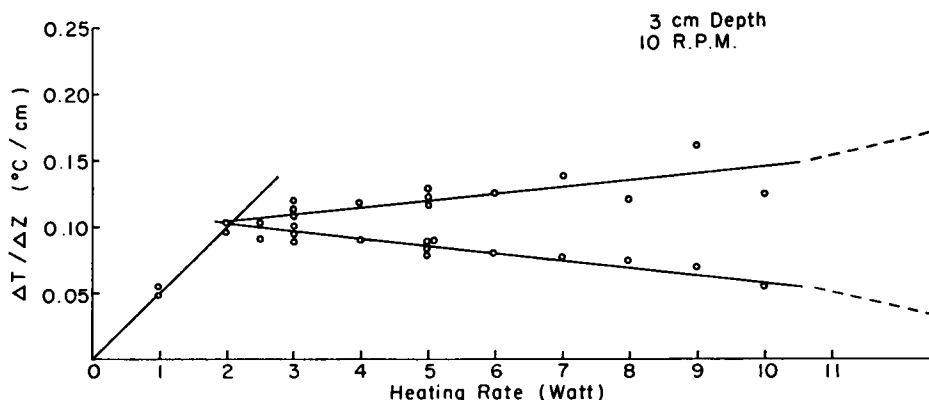


Fig. 13. Heating rate versus characteristic lapse rate values. Between 0 and 2 watts, the curve represents the maximum steady lapse rate attained; beyond 2 watts the upper branch represents the maximum supercritical lapse rate attained, and the lower branch represents the value at which the lapse rate steadies or around which it oscillates after convection has begun.

tain overstable oscillations, and that for this reason the occurrence of this mode is not physically possible in non-rotating systems. Since this additional restoring force is a result of rotation, this is at least consistent with the theoretical result obtained by the inclusion of a non zero rotation in the frequency equation, i.e., the increase of the equation's degree which makes possible the appearance of complex roots with positive real parts and their associated overstable mode.

Regimes of Vertical Heat Transfer

The experiments of the type illustrated in Fig. 11, those involving the changing character of the temperature trace with changing rates of heating, were first carried out merely with the idea of determining the minimum heat required for convection. But they can also be

input leads to an unsteady convective transfer regime. An interesting relation becomes apparent when characteristic temperatures from traces obtained in experiments of this type are plotted against their heating rates (Fig. 13). In the conductive regime (0 to 2+ watts), the heating rate is plotted against the steady, maximum temperature difference attained. At heating rates greater than the minimum required for convection, we plot two characteristic temperature differences, the amplitude of the first peak which breaks down when convection begins, and the average value at which the trace later steadies or around which it oscillates.

The upper branch of Fig. 13, the plot of the supercritical peak attained before convection, uniformly increases with increasing heating rates up to regions in which the breakdown occurs sharply, almost immediately after the

heat is turned on. The lower branch, representing a value proportional to the average vertical temperature difference that can be maintained with the prevailing rates of heat input and vertical transfer, decreases continuously in an approximately linear fashion over the range of relative small heating rates covered by Fig. 13. The triple point located by the intersection of the three curves designates the critical lapse rate and the minimum heating rate required for convection. The discontinuous slope consisting of the conductive and the steady convective vertical temperature difference is similar to curves first obtained in non-rotating models by SCHMIDT and MILVERTON (1935).

The lower branch, initially almost linear, curves and approaches some very small temperature difference (zero as far as these rough determinations can show) as an asymptote. At 30 watts input, this difference was about 0.2°C ; at 50 watts input, it was less than 0.01°C . As far as these admittedly crude measurements can be interpreted, the curve shows no obvious discontinuities. This would seem to imply a continuous spectrum of eddies for thermal exchange extending from steady convection through increasingly unsteady convection to a regime of random, turbulent eddies.

APPLICATIONS TO METEOROLOGY

Cloud phenomena often considered to be examples of Benard cellular convection are commonly observed in the atmosphere (BRUNT, 1937). Altocumulus (especially the floccus variety) and cirrocumulus have been viewed as fields of Benard cells with ascending currents in their centers; cumulus mammatus and the rarer, net-like cirrus reticulatus are examples of cells with descending cores, the first a case in which the descending air remains saturated, and the second, descent in which dessication occurs. That one could say that it is the stabilizing influence of rotation which helps make these two atmospheric circulation types possible is quite unlikely since the changes in the required critical lapse rates in the atmosphere with rotation are extremely small (see below). The question of an explanation for these cloud phenomena remains open, but their analogues with respect to circulation direction have been shown in the laboratory and are available for further study. There are, however, several areas

in which these ideas can be at least qualitatively applied to meteorological problems. As early as 1928 Jeffreys pointed out that rotational stability theory should be of importance to meteorology.

In connection with this discussion, the reader is reminded that the visible cloud usually constitutes only a small part of a convective cell, generally only a part of the cell's core. The clear spaces between individual clouds are the descending portions of cells, so that the average distance between clouds is probably a fair measure of the cell diameter. The height of the cell is the full depth of the convective layer, not merely the height of the visible cloud. For low level clouds of the cumulus variety, the convective layer usually extends from near the ground to some level above the height of the cloud tops, since the height to which visible clouds grow is more often limited by their entraining of drier air from the environment rather than by their reaching the upper limit of the unstable layer. In the examples treated below, we shall, for simplicity, assume that the cell core remains saturated and that the visible cloud does reach the top of the unstable layer. The cloud heights considered, then, are the maximum that could be attained provided that sufficient moisture was available.

It should also be noted that the rotation that plays the stabilizing role in the cumulus problem is the absolute rotation of the fluid in which the convection occurs. In a coordinate system on the face of the rotating earth, a rotation around the z axis analogous to the rotation of our cylindrical model is $\Omega_e \sin \phi$, the vertical component of the earth's rotation. As is often done, we specify that the motions occur in higher latitudes in order that the horizontal component be small. Thus we could write for $4\Omega^2$ in the Taylor number (22),

$$4(\Omega_e \sin \phi)^2 = f^2 \quad (43)$$

where f is the Coriolis parameter. But more generally, the atmospheric fluid possesses a rotation of its own, relative to the earth; so a better approximation of the $4\Omega^2$ factor for atmospheric convection problems would be

$$(f + \zeta)^2 = Z^2 \quad (44)$$

where ζ and Z are the relative and absolute vorticities, respectively. Since it does not imme-

diately follow that horizontal shear would lead to the same rotational effects, it is assumed that the major portion of the relative vorticity of the general current results from the curvature component. Thus we can write for the equations of motion

$$\frac{\partial u}{\partial t} - Zv = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + \nu \nabla^2 u \quad (45)$$

plus similar forms for the other two equations. In this way we substitute the local absolute vorticity for 2Ω in the Taylor number in order to apply the theory to atmospheric convection.

The Critical Lapse Rate

The concept of a critical Rayleigh number for convection has been extended to compressible fluids by JEFFREYS (1930). Assuming that the atmosphere is a perfect gas, we have the energy equation (Goldstein).

$$\rho J c_p \frac{D\theta}{Dt} - \frac{Dp}{Dt} = \rho J c_p \kappa \nabla^2 \theta + \Phi \quad (46)$$

After perturbation analysis, the pressure term can be reduced to

$$-\frac{w}{\rho J c_p} \frac{\partial p}{\partial z} = w \gamma_d \quad (47)$$

where γ_d , the dry adiabatic lapse rate, arises by substitution of the hydrostatic relation. Writing $\partial\theta/\partial z = -\gamma$, we obtain the simplified perturbation energy equation

$$\frac{\partial \theta}{\partial t} - w(\gamma - \gamma_d) = \kappa \nabla^2 \theta \quad (48)$$

where the primes have been omitted as before. Comparison with equation (12) shows $(\gamma - \gamma_d) = \beta$, a small amount which must be added to the dry adiabatic lapse before convection can begin. It follows that a static atmospheric layer with a lapse rate equal to the dry adiabatic is analogous to the isothermal liquid layer assumed in the theoretical analysis.

However, discussion of this slight increase over the dry adiabatic lapse rate that rotation makes necessary for the initiation of convection in the atmosphere becomes academic when the small size of β is realized. For a one kilometer deep layer of air with a mean tempera-

Table 4

Absolute vorticity sec ⁻¹	Eddy viscosity cm ² sec ⁻¹		
	10 ³	10 ⁴	10 ⁵
10 ⁻⁴	77 m	245 m	775 m
3 × 10 ⁻⁴	45 m	141 m	447 m
10 × 10 ⁻⁴	24 m	77 m	245 m

Minimum atmospheric depths required to increase the critical Rayleigh number by ten per cent for several values of absolute vorticity and eddy viscosity.

ture of zero degrees Centigrade, an absolute vorticity of 10⁻⁴ sec⁻¹, and eddy coefficients of viscosity and conductivity of the order of 10³ cm² sec⁻¹, a calculation of R_c gives a required β of only 2.5×10^{-5} degrees per kilometer. If we ask for a set of conditions which will increase the lapse rate by just one per cent of the dry adiabatic, we obtain the improbable picture of a ten kilometer deep convective layer in which, if the eddy coefficients are of the order of 10⁶, an absolute vorticity of 1.5×10^{-2} sec⁻¹, one hundred fifty times as large as the middle latitude Coriolis parameter, would be required.

But appreciable percentage changes in the critical Rayleigh number can be caused by rotational influences of the order of those found in the atmosphere. Table 4 lists the minimum required depths for a ten per cent increase in R_c as indicated by the modified Stommel limit, $1.7(\nu/\Omega)^{1/2}$ considering several combinations of absolute vorticity and eddy viscosity. For minimum depths necessary to double R_c under these same conditions, multiply the tabular entries by 1.9.

Thus in atmospheric convection layers, the influence of rotation might be of some importance, if not in causing significant increases in lapse rates, then perhaps in bringing about secondary effects such as the increase of cells' depth-diameter ratios or the occurrence of overstable oscillations.

The Depth of Atmospheric Convection

The association of deep convection in the atmosphere with cyclonic relative vorticity has been noted by, among others, PETTERSEN et al. (1946); they gave as a forecasting rule, "Deep and very deep convection (to heights above 600 and 400 millibars, respectively)

should not be forecast unless the upper air charts show indication of cyclonic (relative) vorticity". Of 73 cases of "deep convection", 62 were found to have occurred with cyclonic vorticity and only one with anticyclonic vorticity as measured at the 750 millibar level. In ten cases, the relative vorticity was either zero or indeterminate. Of 30 cases of "very deep convection", 28 occurred with cyclonic vorticity and none with definitely anticyclonic vorticity. Similar association was found by CRUTCHER et al. (1950) who, on a monthly, year round basis, found positive linear correlations from .73 to .89 between the cloud top height and the relative vorticity measured at 700 millibars in 454 observations on the Washington to Bermuda air route. This group pointed out that there was no adequate dynamic explanation for this association, especially in view of existing ideas which call for maximum positive vertical motion ahead of rather than on the trough line.

The expression for the depth-diameter ratio of convection cells as a function of the basic rotation (42) accurately predicts the measurements of ordinary cells in water. It can also be used for numerical examples which show a quantitative relation between deep atmospheric convection and areas of cyclonic relative vorticity.

For a shallow convection case, consider a D/d ratio of unity, applying for example, to fair weather cumulus in cells one kilometer deep. We shall use a reasonable and convenient value for the eddy viscosity, $10^{3\frac{1}{2}}$. From (42) we find $a_c^2 = 52.5$; equation (25) then determines $T = 4 \times 10^4$, and from (22) we determine the absolute vorticity of the general current = $0.6 \times 10^{-4} \text{ sec}^{-1}$. In middle latitudes, the relative vorticity would be anticyclonic, $-0.4 \times 10^{-4} \text{ sec}^{-1}$.

For an illustration of deep convection, consider a D/d ratio of two, applying to cumulus congestus in cells five kilometers deep. A reasonable coefficient of eddy viscosity for the larger volume is $10^{4\frac{1}{2}}$. For these conditions, $a_c^2 = 212$, $T = 2 \times 10^6$, and the corresponding absolute vorticity of the general current is $1.8 \times 10^{-4} \text{ sec}^{-1}$. The relative vorticity is cyclonic, $0.8 \times 10^{-4} \text{ sec}^{-1}$ in middle latitudes.

These are only fictional examples, designed for illustrative purposes; but it should be noted that reasonable cell dimensions and not un-

reasonable values of the eddy viscosity lead to vorticity values of reasonable order. It would be interesting to obtain actual D/d ratios for clouds and to test this relation quantitatively. The measurements required would be the average distance between clouds for the cell diameter, the height of the cloud tops above the ground as an approximation of the cell depth, and, of course, the absolute vorticity of the general current.

Clear and Cloudy Areas

An approximation of the Prandtl number for air in the case of atmospheric convection is the ratio of the eddy, rather than the molecular, coefficients of viscosity and conductivity. Even though the eddy values separately vary with the scale of the circulations involved, i.e., with the depth of the convective layer, their ratio is generally considered to retain an approximately constant value near unity. At any rate, it is of the same order as the critical Prandtl number, 0.677, but probably somewhat larger. This, combined with the fact that the required lapse rates (in excess of the dry adiabatic) for the two types of convection and the difference between them are very small gives rise to the possibility that both the ordinary convective and the overstable oscillating cellular modes might occur simultaneously in the atmosphere. With a large enough rate of heating from below, one sufficient to maintain the lapse rate required by R_c^* as well as that required by R_c , this would be possible.

Along with this possibility, we have the earlier observation that as the Taylor number increases, the critical wave number for the overstable mode gradually becomes smaller than that for ordinary convection. This effect is visibly evident in the relatively shifting minimums of the wave number curves in Fig. 7. So if the two convective types are both occurring at some non-zero value of the Taylor number, the wave number of the ordinary type will be somewhat greater than that of the oscillatory type and a "beat frequency" pattern of resonance and interference can develop.

Consider a field of simultaneously occurring convective types in a layer three kilometers deep under the influence of an absolute vorticity of 10^{-4} sec^{-1} , assuming an eddy viscosity of $10^4 \text{ cm}^2 \text{ sec}^{-1}$. For these values we find that $T = 8 \times 10^5$, $a_c^2 = 153.$, and $a_c^{*2} = 94.4$. Assuming

that the two types occur in the form of superimposed fields of hexagons, we find for their respective depth-diameter ratios, $D/d=1.71$, $(D/d^*)=1.34$. Since they are both in a three kilometer layer, their average diameters are $d=1.75$ km and $d^*=2.24$ km. From these it can be determined that the central cores of the two types of cells closely coincide in widely separated areas located over the field at the vertices of equilateral triangles, 8.1 kilometers on a side. In the central areas of the triangles and along the middle sections of their sides, the two convective patterns interfere in varying degrees, the descending walls of either type damping the rising cores of the other. This model is consistent with the frequently observed distributions of convective clouds, e.g. trade cumulus, in relatively small cloudy areas with large clear areas in between. If the centers of the cloud groups of the eight kilometer triangular distribution are traveling ten meters per second, they will pass over or near a ground observer at eleven to thirteen minute intervals. This is an entirely reasonable period, according to observers who have timed the passage of groups of trade cumulus.

In areas of partial interference, the occurrence or non-occurrence of visible clouds would be determined by the resultant strength of the partially reinforced cores and the amount of moisture available for condensation. If conditions were such that only strongly reinforced convective cores could produce visible clouds, the pattern would be small cloudy areas separated by large clear spaces. Similarly, if only a small amount of convective lifting were needed, there would be large cloudy areas separated by small clear spaces. The tallest clouds would be found where complete resonance occurs, in the centers of the cloudy areas. Again, this model could be checked by aerial observation, measuring the distance between cloudy areas in addition to the distance between individual clouds, the depth of the convective layer, and the absolute vorticity of the general current.

The period of the overstable component of the interference pattern of the numerical example above, obtained by substituting in equations (37) and (38), is of the order of one hundred hours. Its large magnitude is due to the close proximity of the R_c and R_c^* curves in a fluid with a Prandtl number so close to the

critical value. In other words, the nominally oscillating component would be practically steady throughout the life of the convection field. But the interference pattern would still be a slowly changing one since the other component, the ordinary cells, would be in an unsteady condition; for since the lapse rate must be large enough for R_c^* , and since the Prandtl number is greater than the critical value, it follows that the lapse rate would exceed the minimum requirements of R_c . One would expect an interference pattern undergoing random changes in time and space, much the way patches of altocumulus are seen to change.

Conclusion

In all of the foregoing illustrative examples, the only effects considered were those of the basic rotation of the fluid environment and the thermal instability produced by uniform heating from below. Obviously, the symmetric models that result could, at best, only rarely occur in the earth's notoriously non-uniform, non-stationary atmosphere. Horizontal variations in the rate of heating from below and the presence of horizontal and vertical wind shears are a few disturbing factors that are seldom absent.

In the model experiments, a deliberate effort was made to eliminate all influences other than those of the basic rotation and the vertical temperature gradient. The fluid was allowed to reach relative rest (solid rotation) before the convective circulations were begun, and uniform heat and cold sources at the bottom and top and careful insulation of the cylinder's sides kept the average horizontal temperature gradient near zero. With these precautions, the conditions of each of the rotating Benard cell experiments determined a point in the Rayleigh, Taylor number plane of Fig. 12.

In the "dishpan" experiments of Fultz, the Taylor number is still an appropriate rotational parameter, but the principal thermal gradient involved is horizontal. There is a horizontal analogue of the Rayleigh number for use with this model containing the horizontal instead of the vertical temperature gradient, a horizontal distance instead of the vertical depth, and a horizontal (centrifugal) acceleration instead of the vertical acceleration of gravity. The dishpan experimental conditions, then, can

be located as points on a "horizontal Rayleigh number", Taylor number plane.

These two planes bound a space in which the conditions for any rotating system with horizontal and vertical temperature gradients can be located. Fultz has pointed out that in such a space an appropriate coordinate combining the vertical and horizontal Rayleigh numbers could have the form of a Richardson number, the horizontal temperature gradients determining a vertical shear term through the thermal wind relations. Throughout the space, in analogy with the correspondence of specific areas of the vertical Rayleigh, Taylor number plane with regimes of heat transfer and modes of circulation, certain volumes would delimit the conditions for the occurrence of similarly specific phenomena. It was noted above that dishpan type wave motions occurred in the rotating Benard-cell experiments if the horizontal temperature gradients were not minimized. This is an example of an experiment whose characteristic point had moved out of the vertical Rayleigh-Taylor plane into a region of the three dimensional, vertical Ray-

leigh, horizontal Rayleigh, Taylor number space, a region that corresponded to wave motions. Similarly, dishpan experiment points appear to reach a volume of the space corresponding to Benard convection cells when the Taylor number is large enough, the horizontal temperature gradients are moderate, and the vertical temperature gradient is not zero.

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