A Note on the Integration of the Vorticity Equation for Quasi-Geostrophic Flow

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(Manuscript received April 18, 1954)

Abstract

By the transformation of integrals arising from the solution of the "quasi-geostrophic" equations for barotropic and simple types of baroclinic flow, it is shown that the isobaric height tendency can be expressed in terms of *first derivatives* of the heights of isobaric surfaces. This demonstrates that the procedure by which the equations are solved must tend to cancel or "average out" the large percentage errors in the geostrophic vorticity advection—i.e. errors in second and third order finite-differences. The height tendency may be computed by a simple graphical procedure, which is carried out directly on analyzed contour charts and which involves a minimum of multiplications.

1. Introduction

One of the objections frequently raised against the recently developed methods of numerical weather prediction is that they require calculation of second and third order derivatives of quantities that are measured only approximately and at widely separated points. Virtually all of those methods are based on vorticity equations for "quasi-geostrophic" flow—i.e., vorticity equations in which the true wind and vorticity (but not divergence) are approximated by the geostrophic wind and the corresponding "geostrophic vorticity". Thus, the horizontal gradient of geostrophic vorticity appears in these equations as a third derivative of the height of an isobaric surface. Coupled with the observation that the percentage errors of finite-differences generally increase as the order of the difference increases, this fact tends to show that the horizontal advection of geostrophic vorticity at individual points cannot be computed accurately from widely separated measurements. Accordingly, there has been considerable doubt that the quantities which enter into the "quasi-geostrophic" vorticity equations can be calculated exactly enough for purposes of numerical weather prediction.

One should suspect, on the other hand, that the inaccuracy of computed finite-differences *at individual points* is not as serious as it might appear from an error analysis of the *unintegrated* equations. This stems from the fact that the solution of the quasi-geostrophic vorticity equation can be expressed as an area (or volume) integral of the advection of absolute vorticity or, stated in slightly different terms, as a weighted average over a considerable area (or volume). Thus, if the errors of the computed geostrophic vorticity advection at individual points are essentially random, they will tend to be "averaged out" in the process of solving the vorticity equation. For this reason, and because most errors in reported observations are nonsystematic, the percentage errors in the *solution* of the vorticity equation are generally much less than the percentage errors in the *computed vorticity advection at individual points.*

The quasi-geostrophic vorticity equation is not, of course, usually solved by a method of direct integration. For reasons of economy and simplicity, the most widely used method of solution is the so-called "relaxation" method. It is evident, however, that the procedure of "relaxation" also tends to smooth out nonsystematic errors in the computed geostrophic vorticity advection. It can be shown, in fact, that the Richardson method of relaxation is essentially equivalent to FJØRTOFT's (1952) method of repeated averaging. This suggests that the large nonsystematic errors introduced by second and third order differences are generally "averaged out" by whatever procedure one uses to solve the vorticity equation and, accordingly, that the percentage error in the *solution* is comparable with the errors of first order differences. This conjecture would be confirmed, if the solution of the quasi-geostrophic vorticity equation could be expressed in terms of *first* derivatives of the height of an isobaric surface, thus showing that the errors in computing higher order derivatives must be cancelled out in the process of solving the vorticity equation.

The purpose of this note is to show that the solution of the vorticity equation for quasi-geostrophic flow is expressible in terms of first derivatives of the isobaric contour height. For simplicity and clarity, we shall first consider the quasi-geostrophic equation for barotropic flow, later indicating how the results in that case can be extended to certain simple types of baroclinic flow. Finally, we shall outline a graphical method for computing height and thickness tendencies from the original height and thickness charts. Although this method is not adapted to machine computations, it will be found useful for manual calculation of tendencies at a small number of strategically located points.

2. The Equations for Quasi-Geostrophic, Barotropic Flow

In accordance with the usual procedure, the quasi-geostrophic vorticity equation is derived by first eliminating the horizontal velocity divergence between the equation of continuity and the vorticity equation, and by replacing the true wind and vorticity with the geostrophic wind and geostrophic vorticity. At the same time making use of the hydrostatic equation, we may put the vorticity equation for barotropic flow in the form:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} - \nu^2\frac{\partial\psi}{\partial t} = 0 \quad (1)$$

in which t is time; x and y are rectangular coordinates directed locally toward the east and north, respectively, and measured in planes that are locally tangent to the geopotential surfaces.

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};$$

ψ is $gz\lambda^{-1}$, the geostrophic stream function in isobaric surfaces; z is the height of an isobaric surface; g is the gravitational acceleration; λ is the Coriolis parameter;

$$J(u, v) \equiv \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}; \; \beta = \frac{\partial\lambda}{\partial y}; \; \nu = \lambda(RT_0)^{-\frac{1}{2}};$$

R is the gas constant; and T_0 is a representative value of the absolute temperature at the ground surface. All lines normal to the (x, y) plane have been treated as verticals, and all differentiations with respect to x, y, or t are carried out with pressure held fixed. Stated briefly, the problem is to integrate Eq. (1), beginning with known initial values of ψ at every point in the (x, y) plane.

The key to the usual method for solving Eq. (1) lies in regarding it as a nonhomogeneous linear equation, in which $\frac{\partial\psi}{\partial t}$ is the dependent variable, and in which the terms *not* involving time derivatives are regarded as known. Viewed in this light, Eq. (1) takes the form

$$(\nabla^2 - \nu^2)\frac{\partial\psi}{\partial t} = F(x, y, t) = \text{known function} \quad (2)$$

Now, since the function F involves space derivatives only, its initial value at every point can be computed from the given initial values of ψ. Thus, the initial values of $\frac{\partial \psi}{\partial t}$ at every point can be computed by solving Eq. (2). Having computed the time rate of change of ψ at each point, we next extrapolate linearly from the initial values of ψ, to obtain the value of ψ at every point in the (x, y) plane and at a time slightly later than the initial moment. Finally, the *predicted* values of ψ are regarded as a new set of initial data, from which F and the ψ-tendency can be calculated by repeating the procedure outlined above, and from which one can predict the values of ψ at a still later time, and so on.

The principal variations of this procedure arise from the use of different methods for solving Eq. (2). The method that will be applied here is an exact analytic method which is discussed in almost any elementary text on theoretical physics. As it turns out, equations of the same type as Eq. (2) frequently appear in the theory of forced vibrations of an elastic membrane, and their solutions are well known. If we let $\frac{\partial \psi}{\partial t}$ correspond to the displacement of a membrane, ν to the wave number of the membrane's free oscillations, and F to a periodically impressed force, the theory of elastic membranes can be applied to the solution of Eq. (2) without essential modification.

$$\frac{\partial \psi}{\partial t} = -\frac{1}{2\pi} \int \int K_0 \left[\nu \sqrt{(x-\xi)^2 + (y-\eta)^2}\right] \cdot \cdot F(\xi, \eta, t)\, d\xi d\eta \tag{3}$$

where ξ and η are dummy variables corresponding to x and y, respectively, and K_0 is the zero order Bessel Function of the second kind with imaginary argument. The indicated area integration extends over the entire (ξ, η) plane.

3. Transformation of the Vorticity Advection Integral

The difficulties discussed earlier arise in the calculation of F, a process that is necessarily carried out by the method of finite-differences. In particular, the computation of $\frac{\partial \psi}{\partial t}$ from Eq. (3) requires evaluation of a weighted integral I of the geostrophic vorticity advection, namely,

$$I = \int_A G(r) J(\psi, \nabla^2 \psi)\, dA \tag{4}$$

where $G(r) = \frac{1}{2\pi} K_0(\nu r)$; r is the distance between the fixed point (x, y) and the variable point (ξ, η); and dA is an element of area. The domain of integration A extends over the entire plane. Our present concern is to transform the integral I in such a way as to reduce the order of the derivatives of ψ. By making use of standard vector identities, we may rewrite I in the form

$$I = \int_A \nabla \cdot (G\mathbf{K} \times \zeta \nabla \psi)\, dA - \int_A \nabla G \cdot \mathbf{K} \times \zeta \nabla \psi dA$$

where $\zeta = \nabla^2 \psi$ and $\mathbf{K}$ is a unit vector directed upward. According to Gauss' Theorem, the first of the integrals in the equation above can be transformed into two line integrals, taken around a circle of infinite radius with center at (x, y), and around a circle of vanishingly small radius with center at (x, y). Both of these line integrals vanish, however; the first. because G vanishes exponentially at infinity; and the second, because ψ has continuous derivatives in the neighborhood of G's singularity at (x, y). Thus, from the preceding equation,

$$I = \int_0^\infty \int_0^{2\pi} \frac{dG}{dr} \left(\zeta \frac{\partial \psi}{\partial \Theta}\right) d\Theta dr \tag{5}$$

where Θ is the angle between the x-axis and a line connecting the points (x, y) and (ξ, η). Written in this form, I is to be interpreted as a weighted integral of the radial transport of vorticity.

As given by Eq. (5), I still involves second order derivatives of ψ. To reduce the order further, we shall next express ζ (the Laplacian of ψ) in terms of derivatives with respect to the polar coordinates (r, Θ). Bearing in mind that G depends only on r, we may then rewrite Eq. (5) as

$$I = \int_0^\infty \frac{dG}{dr} \int_0^{2\pi} \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \Theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) \right] \frac{\partial \psi}{\partial \Theta} d\Theta dr$$

$$I = \int_0^\infty \frac{1}{2r^2} \frac{dG}{dr} \int_0^{2\pi} \frac{\partial}{\partial \Theta} \left(\frac{\partial \psi}{\partial \Theta} \right)^2 d\Theta dr +$$

$$+ \int_0^\infty \frac{1}{r} \frac{dG}{dr} \int_0^{2\pi} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) \frac{\partial \psi}{\partial \Theta} \partial \Theta dr \qquad (6)$$

In the first integral on the right side of Eq. (6), the integration with respect to Θ can be carried out directly. Since $\frac{\partial \psi}{\partial \Theta}$ is continuous, the first integral vanishes. The second may be integrated by parts, as follows,

$$I = \int_0^\infty \frac{1}{r} \frac{dG}{dr} \int_0^{2\pi} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \Theta} \right) d\Theta dr -$$

$$- \int_0^\infty \frac{dG}{dr} \int_0^{2\pi} \frac{\partial \psi}{\partial r} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial \Theta} \right) d\Theta dr$$

or
$$I = \int_0^{2\pi} \int_0^\infty \frac{1}{r} \frac{dG}{dr} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \Theta} \right) dr d\Theta -$$

$$- \int_0^\infty \frac{1}{2} \frac{dG}{dr} \int_0^{2\pi} \frac{\partial}{\partial \Theta} \left(\frac{\partial \psi}{\partial r} \right)^2 d\Theta dr \qquad (7)$$

In the second integral on the right side of Eq. (7), the integration with respect to Θ can be performed directly. Since $\frac{\partial \psi}{\partial r}$ is continuous, the second integral vanishes. The first integral on the right side of Eq. (7) is again integrated by parts, this time by integrating first with respect to r.

$$I = \int_0^{2\pi} \int_0^\infty \frac{\partial}{\partial r} \left(\frac{dG}{dr} \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \Theta} \right) dr d\Theta -$$

$$- \int_0^{2\pi} \int_0^\infty r \frac{d}{dr} \left(\frac{1}{r} \frac{dG}{dr} \right) \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \Theta} dr d\Theta \qquad (8)$$

In the first integral on the right side of Eq. (8), the integration with respect to r can be carried out directly. That integral vanishes, for $\frac{dG}{dr}$ vanishes exponentially at infinity, and the derivatives of ψ are continuous in the neighborhood of the singularity of $\frac{dG}{dr}$ at $r = 0$. Finally, the integral I may be put in the form

$$I = - \int_0^\infty H(r) \int_0^{2\pi} \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \Theta} d\Theta dr \qquad (9)$$

where $H(r) = r \frac{d}{dr} \left(\frac{1}{r} \frac{dG}{dr} \right)$. It is noteworthy that Eq. (9) expresses I as a weighted integral of the radial transport of tangential (angular) momentum, illustrating the connection between the principles of conservation of angular momentum and vorticity.

Substituting the result of Eq. (9) into Eq. (3), we may now write the solution of the vorticity equation for quasi-geostrophic, barotropic flow as

$$\frac{\partial \psi}{\partial t}(x, y) = - \int_0^\infty H(r) \int_0^{2\pi} \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \Theta} d\Theta dr +$$

$$+ \int_0^\infty r G(r) \int_0^{2\pi} \beta \frac{\partial \psi}{\partial \xi} d\Theta dr \qquad (10)$$

The points to be emphasized about this formula are that $G(r)$ and $H(r)$ are known analytic functions which can be computed with any desired degree of accuracy, and that $\frac{\partial \psi}{\partial t}$ is expressed in terms of *first order derivatives* of ψ. This demonstrates that the large percentage errors incurred in computing higher order derivatives of ψ must be cancelled out in the process of solving the vorticity equation for the ψ-tendency. As conjectured earlier, the percentage errors in computing the ψ-tendency from Eq. (10) are much less than the errors in the computed geostrophic vorticity advection at individual points.

4. The Quasi-Geostrophic Equations for Simple Types of Baroclinic Flow

Our next aim is to show that the solutions of the quasi-geostrophic equations for certain simple types of baroclinic flow can also be expressed in terms of first derivatives of the heights of isobaric surfaces. If this is possible, then whatever conclusions one may draw regarding the errors in the solution of the barotropic vorticity equation can be extended to baroclinic flow as well.

Either by direct specification or through finite-difference approximations, the "two-parameter" baroclinic models developed by EADY (1952), ELIASSEN (1952), THOMPSON (1953), SAWYER and BUSHBY (1953), and CHARNEY and PHILLIPS (1953) all contain the implicit assumption that the direction of the vertical wind shear (or the direction of the horizontal temperature gradient) is independent of height, and are all governed by equations of the same general type. For the present purposes, it will be convenient to deal with a system of equations whose mathematical form is similar to that of the barotropic vorticity equation. In an earlier report (THOMPSON, 1953), the author has shown that the equations for quasi-geostrophic flow in which the isotherm patterns in different isobaric surfaces are geometrically similar can be put in the form

$$\frac{\partial}{\partial t}\nabla^2\bar{\psi} + J(\bar{\psi}, \nabla^2\bar{\psi}) + aJ\bar{\varphi}, \nabla^2\bar{\varphi}) + \beta\frac{\partial\bar{\psi}}{\partial x} = 0 \quad (11)$$

$$\frac{\partial}{\partial t}\nabla^2\bar{\varphi} + J(\bar{\psi}, \nabla^2\bar{\varphi}) + J(\bar{\varphi}, \nabla^2\bar{\psi}) + bJ(\bar{\varphi}, \nabla^2\bar{\varphi}) + + \beta\frac{\partial\bar{\varphi}}{\partial x} - \mu^2 J(\bar{\psi}, \bar{\varphi}) - \mu^2\frac{\partial\bar{\varphi}}{\partial t} = 0 \quad (12)$$

where the "barred" quantities are vertical averages, integrated with respect to pressure through the entire depth of the atmosphere; $\varphi = RT\lambda^{-1}$; T is the absolute temperature; a and b are nondimensional constants that are determined from a representative vertical wind profile; and μ^2 is a constant that depends on a representative measure of the static stability through a very deep layer. The quantity $\bar{\psi}$ is the geostrophic stream function for the vertically averaged wind. Similarly, $\bar{\varphi}$ is the geostrophic stream function for the *difference* between the vertically averaged wind and the wind at the ground surface. That is, owing to the hydrostatic relationship between ψ and φ,

$$\bar{\varphi} = \bar{\psi} - \psi_0 \quad (13)$$

where $\psi_0 = gz_0\lambda^{-1}$ and z_0 is the height of an isobaric surface next to the ground.

It will be noted that the only terms of Eqs. (11) and (12) which involve derivatives of second and third order, which do not involve time derivatives, and which do not already have the mathematical form $J(\alpha, \nabla^2\alpha)$ occur in the combination

$$J(\bar{\psi}, \nabla^2\bar{\varphi}) + J(\bar{\varphi}, \nabla^2\bar{\psi})$$

With the aid of Eq. (13), however, the expression above may be rewritten as

$$J(\bar{\psi}, \nabla^2\bar{\psi} - \nabla^2\psi_0) + J(\bar{\varphi}, \nabla^2\bar{\varphi} + \nabla^2\psi_0)$$

$$= J(\bar{\psi}, \nabla^2\bar{\psi}) + J(\bar{\varphi}, \nabla^2\bar{\varphi}) - J(\psi_0, \nabla^2\psi_0) \quad (14)$$

The fact to be stressed is that the terms of this expression do have the mathematical form $J(\alpha, \nabla^2\alpha)$. Since the transformations by which one passes from Eq. (4) to Eq. (9) are valid for *any* scalar variable ψ, area integrals of the terms of Eq. (14) can be transformed to integrals of the type shown in Eq. (9). Finally, because Eqs. (11) and (12) are of the same form as Eq. (2), it follows that $\frac{\partial\bar{\psi}}{\partial t}$ and $\frac{\partial\bar{\varphi}}{\partial t}$ can be expressed in terms of *first derivatives* of $\bar{\psi}$ and $\bar{\varphi}$. This argument shows that the solutions of the quasi-geostrophic equations for simple types of baroclinic flow are not subject to the large percentage errors of second and third order finite-differences.

5. A Graphical Method for Computing the Vorticity Advection Integral

The most laborious and time-consuming step in the computation of the ψ-tendency from Eq. (10) is the evaluation of the integral I, given in Eq. (9). We shall next outline a simple graphical procedure whereby the integral I can be evaluated directly from analyzed isobaric contour charts, with a minimum of numerical calculation. The first step of this procedure is to draw in the contours of the isobaric surface (e.g., 500 millibar surface) for *unit* intervals of ψ. The second step is to superimpose on the contour chart a transparent overlay, on which are drawn a set of concentric "coordinate" circles spaced dr apart, as illustrated in Figure I. The common center of these circles is placed on the point at which the ψ-tendency is desired. Another set of "reference" circles is drawn midway between the "coordinate" circles.

Thus far, the area of integration has been divided into a set of nested rings of width dr. The next step is to subdivide each ring into smaller elements of area, bounded by radial segments passing through the intersections of the contours with the appropriate "reference" circle and subtending angular increments $d\Theta$. A typical element of area is the shaded area S in Figure I. It remains to find the contribution from each element of a given ring, the net contribution from a given ring, and finally, the total contribution from all rings.

A fact that simplifies the procedure considerably is that Eq. (9) can be written in the form

$$I = -\int_0^\infty H(r) \int_0^{2\pi} d_r\psi d_\Theta\psi$$

where $d_r\psi$ is the variation of ψ over a radial displacement dr and $d_\Theta\psi$ is the variation of ψ over an angular displacement $d\Theta$. Fixing attention on the typical area element shown in Figure I, we see that the increments $d\Theta$ have been chosen in such a way that $d_\Theta\psi$ is always unity, since the contours are drawn for *unit* intervals of ψ. Thus, the element of the Θ-integration has the numerical magnitude of $d_r\psi$, the difference between the values of ψ at the midpoints A and B of the circular segments which bound S. The sign to be affixed to that difference is positive or negative, according as the angle γ at which the contours cross the circles is acute or obtuse. The element of the r-integration is obtained simply by adding up all the differences $d_r\psi$ for each ring (summing with regard to sign) and by multiplying each sum by the appropriate value of the weighting function $-H(r)$. Finally, the integral I is evaluated by summing up the contributions from all rings. Since $H(r)$ decreases rapidly with increasing r, the r-integration need not be extended to infinity in order to insure that the error is less than any preassigned limit.

It is worth noting that this procedure involves only as many multiplications as there are rings and, accordingly, that it can be carried out very quickly with only an adding machine. This method is not, of course, suited to automatic machine computation, owing to the irregular size of the area elements. It is, however, ideally adapted to manual computation of ψ-tendencies at a limited number of isolated points—e.g., points on trough- and ridge-lines and inflection points.

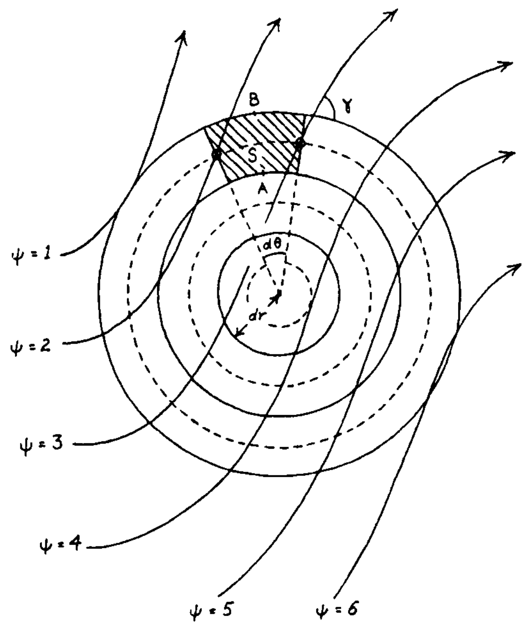

Fig. 1.

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