The Density Ratio at the Boundary of the Earth's Core

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Abstract

A theoretical study is made of the behaviour of a P wave incident from the mantle on the boundary of the earth's core, with special reference to phase reversals. The results are given graphically in Fig. 2, which gives the conditions for given values of the angle of emergence e and the density ratio ϱ'/ϱ or the ratio of the elastic parameters $\frac{\lambda'}{\lambda+2\mu}$ at the core boundary. The dependence of the phase changes on the density ratio is clear; for instance for $\varrho'/\varrho = 1$ (no density discontinuity) there is a change of phase both for $e < 10°$ and for $e > 55°$, whereas for $\varrho'/\varrho \geq 1.7$ there is a change of phase only for $e < 6.2°$. Observations of PcP waves around epicentral distance $= 30°$ (Fig. 3) prove conclusively that there is a first-order density discontinuity at the core boundary. They also show that $\frac{\lambda'}{\lambda+2\mu} > 0.34$. A method for determining the density ratio by more complete observations of PcP is indicated.

I. Introduction

It is generally assumed that the density has a first-order discontinuity at the surface of the earth's core at a depth of 2,900 km. See e.g. BULLEN (1947), pp. 212—219. There are no direct data concerning the density jumps at the discontinuities within the earth, of which the outer core boundary is the most important. BULLEN (1947) computes the densities within the earth on two different hypotheses, one giving a density jump from 5.66 g/cm^3 to 9.7 g/cm^3, i.e. a density ratio of 1.71, at the core boundary; the other giving densities of 5.72 g/cm^3 and 9.1 g/cm^3, i.e. a ratio of 1.59. Later these figures have only been slightly changed as consequences of other hypotheses.

In 1941 KUHN and RITTMANN published their hypothesis about the earth's interior, which entailed a density curve with no discontinuity whatsoever at the boundary of the core. A sharp change of density at the core boundary seems, however, to be most probable. GUTENBERG (1951) says that "no hypothesis can be considered a good approximation which disagrees with the result that the boundary of the outer core is very sharp." "Waves having lengths of ten km or even less are reflected at both sides of the boundary." During the last years new ideas have been proposed about the earth's interior, especially the nature of the boundaries of the two cores. I am not going into detail about these things here, as I have done that in another paper (see References). It may only be added here that Professor A. RITTMANN in a lecture in Uppsala on September 12, 1953, about the KUHN-RITTMANN hypothesis admitted the possibility of a first-order discontinuity of the density at the outer core boundary as a consequence of a phase transition in silicate under pressure.

In disentangling the problem of the density behaviour at the core boundary there are at least two ways, i.e. investigations of (a) the amplitudes of reflected waves, (b) the phase changes upon reflexion at the core boundary.

Amplitudes and travel times of seismic waves reflected and refracted at the core boundary have already been investigated to a large extent. On the other hand, phase changes do not seem to have attracted very much attention. The reason is natural, because phases (compression or dilatation) are generally difficult to determine from seismograms, except for the first pulse (P or PKP). But this does not seem to be enough reason not to investigate the matter, as it may be of some theoretical interest anyway. On the whole, quite a lot of information could be obtained from studies of the direction of initial motion of the different seismic waves.

II. Theory

The wave functions for the P waves on the mantle side of the boundary, i.e. incident and reflected P waves, for the refracted P wave, and for the reflected S wave (see Fig. 1) are respectively

$$\varphi = A e^{i\varkappa(x - \alpha z - \omega t)} + A_1 e^{i\varkappa(x + \alpha z - \omega t)}$$
$$\varphi' = A' e^{i\varkappa(x - \alpha' z - \omega t)}$$
$$\psi = B_1 e^{i\varkappa(x + \beta z - \omega t)}$$

There is still no proof that an S wave exists in the outer core. α is connected with the angle of emergence e by the relation

$$\alpha = \operatorname{tg} e$$

The notations are the same as used by JEFFREYS (1926). Quantities referring to the transmitted P wave are denoted by accents, reflected waves by the suffix 1. The elastic equations of motion which must be satisfied by the wave functions are

$$\varrho \frac{\partial^2 \varphi}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \varphi$$

$$\varrho \frac{\partial^2 \psi}{\partial t^2} = \mu \nabla^2 \psi$$

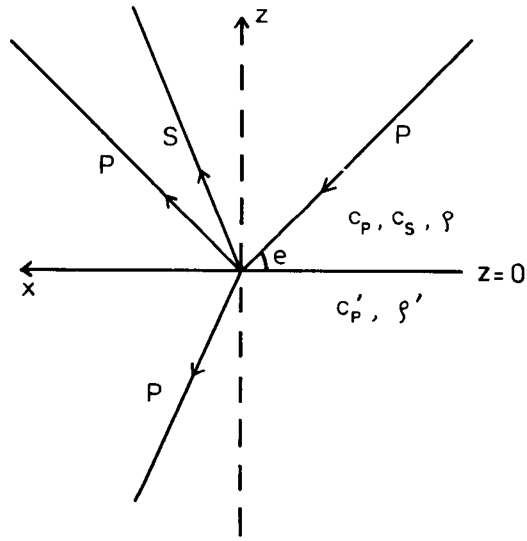

Fig. 1. P wave incident on the core boundary from the mantle.

They give the following relations

$$1 + \alpha'^2 = \left(\frac{c_P}{c'_P}\right)^2 (1 + \alpha^2)$$

$$1 + \beta^2 = \left(\frac{c_P}{c_S}\right)^2 (1 + \alpha^2),$$

enabling the calculation of α' and β for given values of α. c_P and c_S are the velocities of P and S waves just outside the core, and c'_P the velocity of P waves just inside the core. ϱ and ϱ' are the densities. The boundary conditions (here between a solid and a fluid) are that the vertical displacement w and the elastic stress components p_{zz} and p_{zx} are continuous across the surface $z = 0$. The general expressions for these quantities are

$$w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial x}$$

$$p_{zz} = \lambda \nabla^2 \varphi + 2\mu \left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial z \partial x}\right)$$

$$p_{zx} = \mu \left(2 \frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right)$$

This gives us three equations with the ratios of the amplitudes A_1/A, B_1/A, A'/A as the unknowns. These equations are respectively

$$\frac{1}{\alpha'}\cdot\frac{A_1}{A}+\frac{1}{\alpha'}\cdot\frac{B_1}{A}+\frac{A'}{A}=\frac{\alpha}{\alpha'}$$

$$\frac{A_1}{A}+\frac{2\beta}{\beta^2-1}\cdot\frac{B_1}{A}-\frac{\varrho'(\beta^2+1)}{\varrho(\beta^2-1)}\cdot\frac{A'}{A}=-1$$

$$\frac{2\alpha}{1-\beta^2}\cdot\frac{A_1}{A}+\frac{B_1}{A}=\frac{2\alpha}{1-\beta^2}$$

Solving for A_1/A we obtain $A_1/A = 1-\frac{2}{X}$ where

$$X = 1+\frac{4\alpha\beta}{(\beta^2-1)^2}+\frac{\varrho'(\beta+1)^2\alpha}{\varrho(\beta^2-1)^2\alpha'}$$

This means that we have $A_1/A \gtreqless 0$ for $X \gtreqless 2$ respectively or for $\frac{4\alpha\beta\varrho\alpha'+\varrho'(\beta^2+1)^2\alpha}{\varrho(\beta^2-1)^2\alpha'} \gtreqless 1$ resp. For given values of the wave velocities c_P, c_P', and c_S this relation may be written as

$$f(\varrho'/\varrho, e) \gtreqless 1.$$

This means that the angle of emergence e for which $f = 1$ and $A_1/A = 0$ depends upon the density ratio at the core boundary.

The curve represented by the equation

$$f(\varrho'/\varrho, e) - 1 = 0$$

is given in Fig. 2 (curve aa), assuming $c_P =$ 13.7 km/sec, $c_P' = 8.0$ km/sec, and $c_S = 7.25$ km/sec. For $e = 0°$ this equation gives an infinite value of ϱ'/ϱ. The limit for $e = 90°$ of the density ratio ϱ'/ϱ is equal to c_P/c_P', i.e. 1.71. The curve has a minimum with $\varrho'/\varrho \simeq 0.4$ at $e \simeq 25°$. As e is generally not measured, a second curve (bb) is also drawn in Fig. 2, giving the relation between e and the angular distance Δ between epicentre and station for a surface focus. The relation $e - \Delta$ is determined by the equation

$$\cos e = \frac{c_P}{R}\frac{dT}{d\Delta},$$

where R = the radius of the core (3,470 km) and $\frac{dT}{d\Delta}$ has been computed from the travel times of PcP. For a deep-focus earthquake this curve $e(\Delta)$ is not directly applicable but needs a correction as a given e then always corresponds to a somewhat smaller Δ.

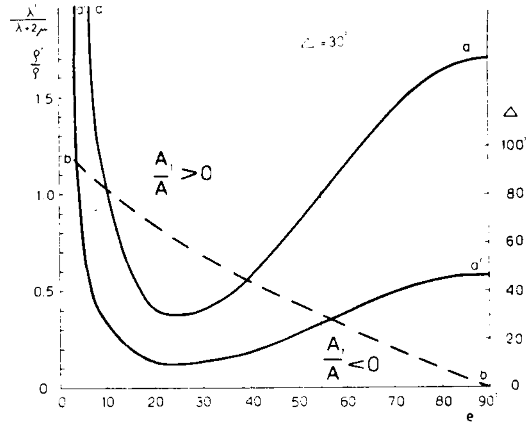

Fig. 2. Curve aa represents the equation $f(\varrho'/\varrho, e) = 1$, i.e. $A_1/A = 0$.
Curve a′a′ represents the equation $f_1\left(\frac{\lambda'}{\lambda+2\mu}, e\right) = 1$.
The ordinate scale is the same for ϱ'/ϱ as for $\frac{\lambda'}{\lambda+2\mu}$.
Curve bb gives e (Δ) for a surface focus.

Substituting $\frac{\lambda'}{\lambda+2\mu}$ for ϱ'/ϱ by means of the relation

$$(c_P/c_P')^2 = \frac{\lambda+2\mu}{\lambda'}\cdot\varrho'/\varrho$$

in the equation

$$f(\varrho'/\varrho, e) = 1$$

this becomes

$$\frac{4\alpha\beta}{(\beta^2-1)^2}+\frac{\lambda'}{\lambda+2\mu}\cdot\frac{1+\alpha'^2}{1+\alpha^2}\cdot\frac{(\beta^2+1)^2\alpha}{(\beta^2-1)^2\alpha'}=1$$

which may be written as

$$f_1\left(\frac{\lambda'}{\lambda+2\mu}, e\right) = 1.$$

It is given as curve a′a′ in Fig. 2.

The curves aa and a′a′ in Fig. 2 correspond to $A_1/A = 0$, the areas above the curves to $A_1/A > 0$, and the areas below the curves to $A_1/A < 0$. Naturally only the parts of the diagram where $\varrho'/\varrho \geq 1$ are of practical importance.

The behaviour of the P wave on reflexion at the outside of the core with regard to phase changes is immediately clear from Fig. 2 for any value of ϱ'/ϱ and e (or Δ). If for instance there is no first-order density discontinuity, i.e. $\varrho'/\varrho = 1$, we get a reversal of phase for $e < 10°$ ($\Delta > 82°$) and for $e > 55°$ ($\Delta < 30°$; an accurate calculation gives $54.65°$ as the limiting value of e) and there is no reversal of phase between these limits. On the other hand, assuming a density ratio of $\varrho'/\varrho \geqslant 1.7$, there is a phase change only for small e-values ($e < 6.2°$), but no phase change for other e-values. This is the marked difference between this case and cases of smaller density ratio, and particularly the case of continuity.

Two special cases of the general equation

$$A_1/A = 1 - 2/X$$

are of interest.

(1) $e = 0°$, i.e. grazing incidence. In this case $\alpha = 0$, $X = 1$, and $A_1/A = -1$, independent of ϱ'/ϱ. This means a total reflexion with reversal of phase.

(2) $e = 90°$, i.e. normal incidence. In this case $\alpha = \infty$;

$$\lim_{\alpha=\infty} X = 1 + \frac{\varrho'}{\varrho}\frac{\alpha}{\alpha'} = 1 + \frac{\varrho'}{\varrho}\frac{c'_P}{c_P},$$

and

$$\frac{A_1}{A} = 1 - \frac{2}{1 + \frac{\varrho'}{\varrho}\frac{c'_P}{c_P}}$$

$c_P/c'_P = 13.7/8.0 = 1.7$ and $\varrho'/\varrho = 1.7$ according to one of BULLEN's (1947) hypotheses. For these values of the ratios, $A_1/A = 0$ for normal incidence, i.e. no PcP wave is obtained. For another value of ϱ'/ϱ we get $A_1/A \neq 0$, but most probably rather small.

III. Observations and Discussion

It is often difficult to determine from seismic records whether a wave begins as compression or as dilatation, especially for the later phases. Furthermore the direction of motion of P and PcP cannot be directly compared (except for grazing incidence), as they start in different directions from the hypocentre, and the initial motion may be different along different directions. This difficulty may be overcome at least for deep-focus earthquakes by comparing PcP and PcS which start in nearly the same direction.

It may prove helpful to investigate the direction of motion of PcP (compression or dilatation) for a given shock at a number of stations in about the same azimuth but at different epicentral distances. If at some distance a change compression to dilatation or vice versa is found for PcP, this may either be due to a change of the initial motion due to different direction from the hypocentre, or it may be due to a change on the reflexion at the core. If for a larger number of shocks, investigated in this way in different azimuths, there is a change compression to dilatation or vice versa for PcP always at the same epicentral distance the reason is certainly the reflexion at the core, but if the changes occur at different distances they must be ascribed to the earthquake mechanism.

In Fig. 3 are reproduced a number of records, also given in Table 1, with epicentral distances around 30° and with especially strong PcP.

They are partly Central American earthquakes at intermediate depth recorded at

Table 1. Earthquakes with strong PcP. Reproductions in Fig. 3

O = origin time, φ, λ = latitude and longitude of epicentre, H = focal depth (n normal), M = earthquake magnitude, Δ = geographical epicentral distance.

No.	Date	O GMT	φ	λ	H km	M	Seismograph station	Δ
1	19 May 1934	104737	14 ¾ N	91 ¼ W	120	6¼	Tinemaha	32.8°
2	19 May 1934						Pasadena	31.1°
3	20 Jan. 1939	204027	13 ½ N	91 ½ W	70	6½	Pasadena	31.8°
4	23 Sept. 1943	150044	15 N	91 ½ W	110	6¾	Pasadena	30.7°
5	16 July 1948	071230	14 ½ N	92 W	100	6¼	Pasadena	30.7°
6	25 Mar. 1952	033511	34.3 N	23.4 E	n	6	Kiruna	33.5°
7	23 June 1953	015312	36 N	25 E	100	5½	Kiruna	31.9°
8	23 June 1953						Uppsala	24.4°
9	21 Oct. 1953	183951	38.3 N	20.8 E	n	6½	Kiruna	29.5°

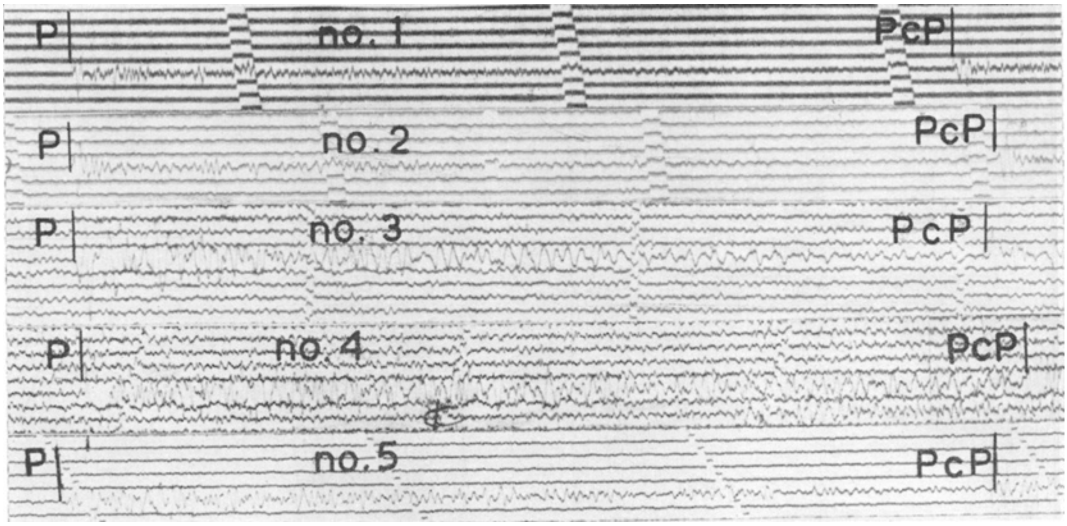

Fig. 3a. Records of PcP around $\Delta = 30°$. The numbers on the records refer to Table 1.

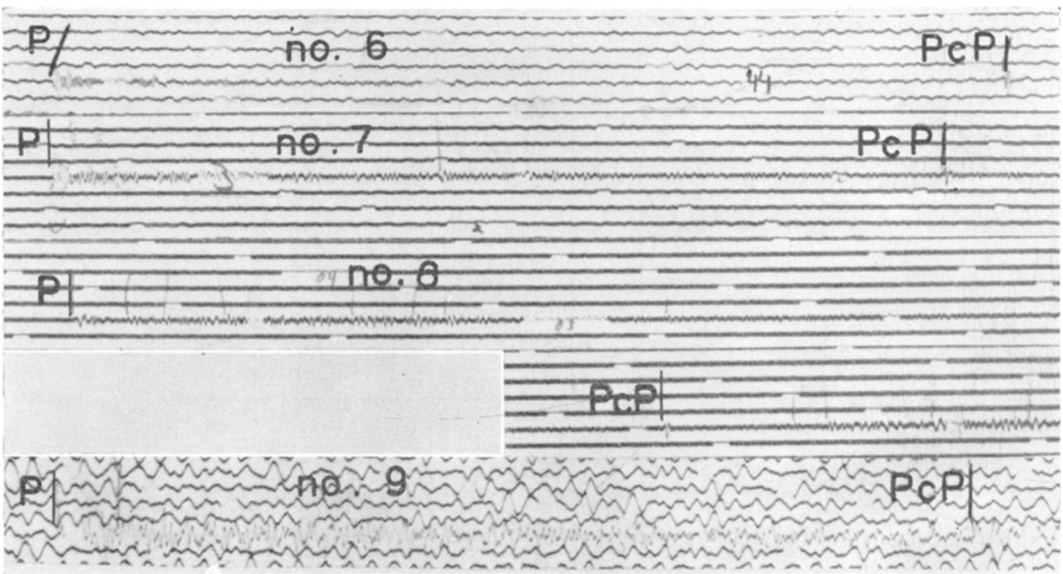

Fig. 3b. Records of PcP around $\Delta = 30°$. The numbers on the records refer to Table 1.

Pasadena and Tinemaha (Benioff electromagnetic vertical seismometers), and partly Mediterranean earthquakes recorded at Kiruna and Uppsala (Grenet-Coulomb electromagnetic short-period vertical seismometers). A focal depth of 100 km means only a decrease of 0.2° in epicentral distance (from 30.0° to 29.8°) for the same angle of emergence (e) at the core as for a surface focus. The depth is therefore of no consequence here. The variations of A_1/A with Δ are continuous (see the general theory), and the slight deviations from $\Delta = 30°$ in our cases are therefore also of no importance. A comparison of direction of first motion of P and PcP would require more data as already indicated. But if ϱ'/ϱ were $= 1$, then $A_1/A = 0$ for $\Delta = 30°$, and this would be valid for any P wave incident on the core, independent of the incident amplitude A and independent of its phase. The fact, evident from our observations, that PcP may be very strong also for $\Delta = 30°$ is conclusive proof that $\varrho'/\varrho > 1$ at the core boundary. We also note that both P and PcP arrive as compressions both at Uppsala ($\Delta = 24.4°$) and at Kiruna ($\Delta = 31.9°$) for the earthquake on 23 June 1953 (No. 7 and 8). There is no change of phase of PcP when the distance $\Delta = 30°$ is passed.

It may possibly be objected that PcP is not always so strong at $\Delta = 30°$ as in the cases given here. This is true, as the cases presented here have exceptionally strong PcP. But a weak or possibly absent PcP at $\Delta = 30°$ in another case does not invalidate our result, as such a circumstance cannot be due to the core but must be due to the earthquake mechanism.

From the relation

$$\left(\frac{c_P}{c'_P}\right)^2 = \frac{\lambda + 2\mu}{\lambda'} \cdot \frac{\varrho'}{\varrho}$$

it is at least theoretically possible to explain the velocity discontinuity $c_P - c'_P$ in three different ways.

(1) $\frac{\varrho'}{\varrho} = 1$ and $\frac{\lambda'}{\lambda + 2\mu} \neq 1$. This case has already been excluded.

(2) $\frac{\varrho'}{\varrho} > 1$ and $\frac{\lambda'}{\lambda + 2\mu} = 1$.

(3) $\frac{\varrho'}{\varrho} > 1$ and $\frac{\lambda'}{\lambda + 2\mu} \neq 1$.

A knowledge of the numerical value of ϱ'/ϱ is required in order to decide between (2) and (3). From the study of records for $\Delta \simeq 30°$ (Fig. 3) and the graphs in Fig. 2 the results may be summarized as follows. $\varrho'/\varrho \leq 1$ is excluded, leaving $\varrho'/\varrho > 1$ as the only possibility. This means that $A_1/A > 0$ for $\Delta = 30°$. And this in turn entails that $\frac{\lambda'}{\lambda + 2\mu} > 0.34$. According to BULLEN's hypotheses $\varrho'/\varrho = 1.59$—1.71 and $\frac{\lambda + 2\mu}{\lambda'} = 1.70$, i.e. $\frac{\lambda'}{\lambda + 2\mu} = 0.59$.

It is true that certain simplifying assumptions underlie the theory but it is believed that corrections for them would only be of second order.

The study of PcP could be used also for a determination of the ratio ϱ'/ϱ. We write the general equation for A_1/A in the following way

$$\frac{A_1}{A} = 1 - \frac{2}{X} = F\left(\frac{\varrho'}{\varrho}, \Delta\right)$$

e is a given function of Δ for a given depth of focus. The velocities c_P, c_S, and c'_P are supposed known. The relation obviously means that a determination of A_1/A should make it possible to compute ϱ'/ϱ. But A_1/A is strictly not the same as the amplitude ratio PcP/P at a given station, as this would presuppose equal radiation of energy in all directions from the focus. Instead it would be necessary first to determine the earthquake mechanism, i.e. the orientation of its fault plane and the direction of slip along this plane, by means of compression and dilatation of P waves or by other means at a large number of stations. A_1 is the observed PcP amplitude and A has to be computed from observations of P. Another difficulty has been observed by MARTNER (1950) and ERGIN (1953), namely that PcP may be complicated by minor phases; see also ERGIN (1952).

Acknowledgements

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