Prognostic Equations for the Mean Motions of Simple Fluid Systems and Their Relation to the Theory of Large Scale Atmospheric Turbulence

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Abstract

By making use of the mathematical properties of certain special types of weighted means — taken either areawise or timewise — it has been shown that a variable's deviation from its mean can be expressed in terms of the mean itself. This result has been used to derive a closed system of equations for the mean motion of an incompressible autobarotropic fluid. Comparison of the equations for area-averaged and time-averaged motion reveals that the area mean has certain advantages, inherent in the form of the equation for the unaveraged motion. The eddy transport for the case of "strong averaging" is contrasted with that derived from the classical "mixing-length" theory of eddy diffusion. The usual restrictions on the statistics of eddy fluctuations are found to be inconsistent with the unrestricted form of the eddy transport. Finally, it has been shown that these methods can be extended to the derivation of closed systems of equations for the mean motions of simple baroclinic fluids.

1. Introduction and General Remarks

In the view of most synoptic meteorologists, one of the marks of a good analysis of pressure or streamline patterns in the upper troposphere is its "smoothness" – i. e., "smooth" in the sense that it does not reflect all the minor irregularities weakly indicated by the raw pressure or wind data. To cite an example, the direction of the analyst's streamlines in the immediate neighborhood of an observing station frequently does not coincide with the reported direction of the wind at that point, but conforms to the "general" direction of the winds over the surrounding area, or lies somewhere between those two directions. Consciously or otherwise, the analyst evidently recognizes that wind reports are at least subject to instrumental and round- off errors, and may be "unrepresentative" in the respect that some of the observed irregularities in wind direction may be associated with disturbances whose scale is comparable with or less than the distance between adjacent observing stations. From this point of view, one of the most important functions of synoptic analysis is the construction of a "representative" or "smoothed" state, which is relatively insensitive to nonsystematic instrumental and round-off errors and which does not reflect irregularities that cannot be reconstructed in detail.

The analyst's subjective process of smoothing may be regarded as a rather complicated and ill-defined type of averaging. That is to say, he bases his analysis at any given reference point not only on observations in the immediate vicinity, but also on observations at a number of points in the surrounding area, attaching greatest weight to the data nearest to the reference point, less weight to data at points farther away, and no weight at all to observations at great distances from the reference point. This suggests that the analyst's "smoothed" state is roughly equivalent to a running area average, in which the raw data at a number of points in the surrounding area are weighted according to distance from the point at which the average applies. Viewed in this light, the desirability of "smoothing" is immediately clear. Owing to the nonsystematic character of round-off errors and of some types of instrumental error, the *average of a large number of reported measurements*, taken at points spaced regularly over an area surrounding a given reference point, generally differs from the corresponding *true average* by an amount less than the error in a single reported measurement at the reference point. Thus the "smoothed" state, constructed by averaging the original data, suffers less from outright error and more accurately characterizes a "true" state than does the unaveraged state revealed by the raw observations.

Any process of averaging, however, also tends to obliterate small scale fluctuations which, although their detailed structure is neither known nor observable, are nevertheless quite real. At first glance, it might appear that the resultant loss of detail might more than overbalance the advantages of reducing nonsystematic error. On the other hand, the density of reporting stations independently places a definite upper limit on the resolving power of our system of meteorological measurement. This, in fact, provides a clue as to how strongly the state should be averaged. In general, to minimize the errors of reported observations without sacrificing what little resolving power we do possess, the observed state must be averaged in such a way as to suppress fluctuations whose scale is comparable with or less than the distance between neighboring observation stations, but leaving disturbances of much larger scale essentially intact. Even so, any consistent system of averaging will result in some loss of detail; however, this is simply the price that must be paid for a more accurate and "representative" characterization of the state.

A second, and more serious objection to the characterization of a state by averaged variables is that the Navier-Stokes equations apply "in the small" – that is, they apply to the unaveraged state, in which fluctuations above the molecular scale are not smoothed out, rather than to an averaged state. For this reason, it is not strictly correct to apply the classical equations of hydrodynamics to a "smoothed" analysis, however artistically drawn, without modifying them for the effects of fluctuations that have been averaged or smoothed out. Granting, therefore, that it is desirable to carry out some kind of averaging process on the variables that characterize the state of the atmosphere, it is equally desirable that we derive from the hydrodynamical equations a new system of equations that is specially designed to apply to the averaged variables.

At this point, it is relevant to note that the meteorologist's situation is very much akin to that which motivated the development of the kinetic theory of gases and the Reynolds theory of turbulence. In the former case, owing to the impossibility of observing the position and velocity of every molecule of a gas, the physicist has been forced to characterize the state of a gas by certain statistics of its molecular state, and has derived from the Newtonian collision equations a closed system of hydrodynamical equations that apply to such quasi-continuous statistics as fluid pressure, fluid density (mean concentration of molecules),and fluid velocity (mean molecular velocity). For similar reasons, Reynolds proposed averaging the *fluid* state, in order to smooth out "turbulent" fluctuations whose scale or period is too small for observation of their structural details, but whose scale is larger than that of molecular motion. As in the kinetic theory of gases, the central problem of the Reynolds theory of turbulence is to derive a closed system of equations that apply to an *averaged* state.

According to Reynold's formal definition, what constitutes "turbulence" clearly depends on the mode of averaging – e. g., on the effective size of the domain of averaging and, accordingly, on the scale of the fluctuations that are "smoothed out". As far as this definition goes, "turbulence" is not a real and distinctive physical phenomenon, but is simply a formal and rather useful concept for dealing with fluid systems whose state is not completely known. From this same point of view, the turbulence problem, stated as an initial value problem, is essentially equivalent to the problem the meteorologist faces in trying to predict the state of the atmosphere with incomplete knowledge of its initial state.

At the same time, it must be recognized that the problem of deriving a system of equations for the synoptic analyst's "smoothed" or "representative" state does differ from the turbulence problem in at least one real and fundamental respect, namely, in that fluctuations whose scale is intermediate between the scales of the longest "planetary waves" in the mean flow and of fluctuations at the threshold of detectability are not completely smoothed out by a systematic process of averaging, and are, therefore, reflected in the mean state, though with reduced amplitude. The theory of turbulence, on the other hand, aims at deducing the interaction between the mean motion and the fluctuations that are smoothed out by averaging, from only the knowledge of a mean state that does not directly reflect any of the structural details of the fluctuations nor, for that matter, their very existence. To point up this distinction, it is useful to return to the example of the kinetic theory of gases, whose accomplished objectives are close to the unrealized ideals of the theory of turbulence. In this case, the molecular motions or fluctuations cannot be observed at all, whence the details of the motion cannot enter directly into the determination of the mean state. Nevertheless, it is possible to deduce a complete system of equations in which certain statistics of the molecular motions are the dependent variables. Fortuitously enough, those variables may be identified with quantities that can be *measured directly, without knowledge of the details of the molecular motion.* In this respect, the derivation of equations for a weighted spatial average of the *observed* motions falls short of the ideal of the theory of turbulence, and, in that sense, does not represent a fundamental approach to the turbulence problem. From the standpoint of examining the interaction between the averaged motions and the fluctuations that are smoothed out, however, it is still of some interest to derive the equations of mean motion corresponding to a given mode of averaging, whether the unaveraged motions are actually observed or not.

To provide a concrete basis for further discussion, we shall briefly review certain points in the pattern of development of the classical theory of turbulence. For simplicity, we shall consider the motion of an incompressible autobarotropic fluid relative to a nonrotating coordinate system, with a view to predicting its *mean* motion. The *unaveraged* state of motion is governed by the well-known equation for the conservation of the vertical component of relative vorticity, taken together with the continuity equation.

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \zeta \mathbf{v} = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

in which $\mathbf{v}$ is the projection of the total vector velocity on a horizontal plane, ∇ is the horizontal vector gradient, and the vorticity ζ is the magnitude of the curl of $\mathbf{v}$. Following Reynolds' development, we next average the equations over a finite interval of time centered on time t, denoting an averaged quantity by a bar placed above it. Since the operation of averaging is commutative with differentiations with respect to time and the horizontal coordinates,

$$\frac{\partial \bar{\zeta}}{\partial t} + \nabla \cdot \overline{\zeta \mathbf{v}} = 0$$

$$\nabla \cdot \bar{\mathbf{v}} = 0$$

At this stage, it is customary to assume that a mean varies much more slowly than deviations from the mean. Thus, denoting a quantity's deviation from its mean by a "primed" quantity,

$$\overline{\zeta' \bar{\mathbf{v}}} = \overline{\bar{\zeta} \mathbf{v}'} = 0$$

Finally, by introducing the stream-function ψ, the system governing the mean motions can be put in the form

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \bar{\zeta} + \nabla \cdot \overline{\zeta' \mathbf{v}'} = 0 \qquad (1)$$

$$\bar{\mathbf{v}} = \mathbf{K} \times \nabla \bar{\psi} \qquad \bar{\zeta} = \nabla^2 \bar{\psi} \qquad (2)$$

where $\mathbf{K}$ is a unit vector directed vertically upward. The first of these equations simply states that the rate at which the *mean* vorticity of an element, travelling with the mean velocity, is created or destroyed is the net rate at which vorticity is transported in or out of a neighborhood of that element by the "eddies" that have been smoothed out.

It has been realized for some time, of course, that Eqs. (1) and (2) are not satisfactory from the standpoint of predicting the mean motions, beginning only with a given initial distribution of the mean vorticity $\bar{\zeta}$. Aside from the fact that the Reynolds' mean is not well defined (being regarded as variable for some purposes and constant for others), Eqs. (1) and (2) do not form a closed system, for the eddy transport $\overline{\zeta' \mathbf{v}'}$ cannot obviously be expressed in terms of $\bar{\zeta}$. This problem – that of deducing the form of the eddy transport – is, in fact, the key problem in the Reynolds theory of turbulence. Simultaneous with the solution of that problem would come the means for predicting the averaged state of certain simple fluid systems, whose behavior approximates that of the atmosphere. Up to the present, however, the most strenuous efforts have not succeeded in reducing Reynolds' eddy transport to the desired mathematical form.

On the other hand, it is conceivable that a closed system of equations for an averaged state might be derived by applying an equally useful, but entirely different mode of averaging. For the sake of argument, let us consider a running area mean, in which the values of the unaveraged quantity at regularly spaced points over the infinite plane are weighted less and less with increasing distance from the point at which the average applies. Now, the mean value associated with any given point is simply a linear combination of the unaveraged values at all points in the plane. Thus, corresponding to *each* point in the plane, one may write (schematically) one linear algebraic equation, in which the unknowns are the unaveraged values at *all* points in the plane, the coefficients are the values of the weighting factor at corresponding points, and the non-homogeneous term is the known mean value at the point in question. Since there are as many points as unknowns, the totality of such equations forms a closed system of linear algebraic equations which, in principle, can be solved by inverting a determinant of infinite order. The consequence of this argument is that the *unaveraged* distribution is uniquely determined by the *averaged* distribution, and can be reconstructed from it. This is evidently a sufficient, if not necessary condition for being able to express the deviation from a weighted mean in terms of the mean itself, from which it follows that the "eddy transport" can be expressed in terms of the mean motion.

The argument outlined above is, of course, a purely formal one, and is not intended to suggest that the unaveraged distribution should actually be recovered from an averaged distribution by solving a system of infinite order. As it turns out, it is possible to define an averaging process such that the unaveraged distribution can be reconstructed by repeated application of an integral operator.

Before setting out on a line of attack that differs radically from Reynolds' classical approach to the problem, it is appropriate to comment on the relative merits and disadvantages of alternative modes of averaging. To begin with, it is probably fair to say that any type of average that displays statistical stability is suitable for purposes of reducing nonsystematic error. Beyond that, the choice of a mode of averaging appears to depend primarily on experimental conditions or systems of measurement, and on the desired form of the results. It is quite probable, for example, that Reynolds chose to average time-wise simply because the most economical system of laboratory measurement is capable of greater time resolution than space resolution. The system of meteorological measurement, on the other hand, resolves about as well in space as in time. Moreover, since the problem of weather prediction is essentially an initial value problem, it is most natural for the meteorologist to average spacewise. There is one other consideration that enters into the choice of a mode of averaging, namely, the distinctive form of the equations for the unaveraged motion. From this standpoint, an area mean is preferable to the Reynolds time mean, since the vorticity equation for nondivergent or quasi-geostrophic flow is distinguished by the way in which space derivatives enter the equation.

Having dwelt at some length on background and motivation, we can now state very simply the problem that takes up the remainder of this paper. The main purpose is to develop a closed system of equations for the averaged motions of certain idealized fluids, whose behavior is, in several crucial respects, quite similar to that of the middle and upper troposphere. For clarity and simplicity, we shall first consider the motions of an incompressible autobarotropic fluid, later indicating how the methods for handling that case can be extended to the case of divergent barotropic flow and simple types of baroclinic flow. By judicious choice of a mode of area-wise averaging, it will be found possible to reconstruct the unaveraged distribution from a given averaged distribution. A further consequence of the fundamental operational property of the mean is an algorithm for inverting operators of the Laplace and Helmholtz type. Taken together, these results provide the ingredients of a method for expressing all quantities in terms of mean values that are either initially known or predictable.

Finally, for purposes of orientation, the results of this study are compared with one of the principal conclusions of the classical mixing-length theory of eddy diffusion. By introducing the mixing-length hypothesis without the usual restrictions on the statistical properties of turbulent fluctuations, it is shown that the eddy transport is directed normal to the gradient of mean vorticity, rather than down the gradient, and that the coefficient of eddy transfer depends on the correlation between eddy displacements in one direction and eddy velocities in a direction normal to it. This clearly indicates that the process by which mean vorticity is generated or destroyed is not the classical process of eddy diffusion, and that assumptions about the statistics of turbulent fluctuations should not be introduced without regard to the nature of the quantity whose unaveraged value is conserved.

2. The Weighted Area Mean as a "Smoothing" Operator

As a preliminary to introducing mean values into the hydro-dynamical equations, we shall first discuss the general mathematical properties of a weighted area average which, as will be seen later, is specially adapted to the form of those equations.

The unaveraged variables under consideration are, in general, continuous and continously differentiable functions of the rectangular coordinates x and y in an infinite "horizontal" plane, and time t. Corresponding to the spatial distribution of a variable at a fixed time, we shall define a mean value of that variable, denoted by placing a bar over the unaveraged quantity.

$$\overline{\varphi}(x, y) \equiv \frac{\alpha^2}{2\pi}\int\int K_0(\alpha r)\,\varphi(\xi, \eta)\,d\xi d\eta \quad (3)$$

where φ is a typical function of x and y, α is a real constant, K_0 is the zero order Bessel function of the second kind with imaginary argument, ξ and η are the coordinates of the variable point (dummy variables of integration corresponding to x and y), and r is the distance between the fixed point (x, y) and the variable point (ξ, η) – i. e., $r^2 = (x-\xi)^2 + (y-\eta)^2$. The indicated area integration extends over the entire plane. It must be kept in mind that x and y, the coordinates of the point at which the average applies, are constant with respect to the integration.

With regard to the properties of this average, we first note that it is normalized, in the sense that the mean value of "one" is still "one". It follows that, if φ is finite over the entire plane, the average is always convergent, for the absolute magnitude of the average cannot exceed the maximum unaveraged value without regard to sign. The weighting function $K_0(\alpha r)$ – or more simply, K – decreases monotonically with increasing distance from the point (x, y) and is never negative. It has a logarithmic singularity at (x, y) and approaches zero like $(\alpha r)^{-\frac{1}{2}} e^{-\alpha r}$ with increasingly great distance from (x, y). Thus K satisfies the essential conditions on a proper weighting function, namely, that it never gives negative weight and places less and less weight with increasing distance from the point at which the average applies.

The constant α, which is unspecified in value and which we are free to fix, deserves special comment. From the foregoing discussion, it can be seen that, as α is increased, relatively less weight is attached to points at a large fixed distance from the point (x, y), relative to the weight attached to points at a somewhat smaller distance. Thus the value of α determines the "effective" size of the domain of averaging, and thereby fixes the scale of transition between the small-scale fluctuations that are smoothed out by averaging and the large scale disturbances that are left essentially intact. In that sense, one might speak of this smoothing process as "averaging of modulus α", and define "turbulence of modulus α" as the fluctuations that are smoothed out by that mode of averaging. In any case, "turbulence" in the formal sense must be defined with reference to a particular type of average.

To illustrate the role played by the "scale" parameter α, we shall apply the average to a function that is independent of y and varies sinusoidally in the x-direction. By a straightforward integration, it can be shown that, if $\varphi = \cos \beta x$,

$$\bar{\varphi} = \left(\frac{\alpha^2}{\alpha^2 + \beta^2}\right) \varphi \qquad (4)$$

Two extreme cases may be distinguished, according as $\alpha < \beta$ or $\alpha > \beta$.

$$\lim_{\frac{\alpha}{\beta} \to \infty} \bar{\varphi} = \varphi \qquad \lim_{\frac{\alpha}{\beta} \to 0} \bar{\varphi} = 0$$

This result may be interpreted from two different points of view. Relative to a fixed value of α, disturbances of very small wave number (large scale) are virtually unaffected by the process of averaging, while disturbances of very large wave number (small scale) are completely obliterated. Similarly, if the unaveraged distribution is a continuous spectrum of wavelike components, the average distribution will reflect only those components whose wave numbers are comparable with or less than the scale parameter α. Relative to *disturbances of a fixed scale*, on the other hand, averaging for a very small value of α tends to smooth out the disturbances, whereas averaging for a very large value of α leaves them almost intact. Accordingly, we shall later speak of the case where α is much greater than some characteristic wave number as "weak averaging" and, where it is much less, as "strong averaging".

3. The Fundamental Operational Property of the Mean

Although the "bar"-operator has been defined as a weighted average, it also has many of the features of a transform or integral operator, and it will frequently be convenient to think of it in that light. Our next concern is to investigate some of the operational properties of the mean, regarding it as a special type of double transform.

The fundamental operational property of the mean, from which all others are deducible, stems from the peculiar character of the weighting function K. It may be verified by direct differentiation that K satisfies the Helmholtz equation

$$\nabla^2 K = \alpha^2 K \qquad (5)$$

The usefulness of this relation is most easily demonstrated by applying the bar-operator to the Laplacian of a typical variable φ. According to the definition of Eq. (3),

$$\overline{\nabla^2 \varphi} = \frac{\alpha^2}{2\pi} \int\int K \nabla^2 \varphi d\xi d\eta$$

By making use of well-known vector identities, the equation above can be rewritten in the form

$$\overline{\nabla^2 \varphi} = \frac{\alpha^2}{2\pi} \int\int \nabla \cdot (K \nabla \varphi - \varphi \nabla K)\, d\xi d\eta + $$
$$+ \frac{\alpha^2}{2\pi} \int\int \varphi \nabla^2 K d\xi d\eta$$

We next substitute into the equation above the value of $\nabla^2 K$ given by Eq. (5), and again introduce the definition of the bar-operator, with the result that

$$\overline{\nabla^2 \varphi} = \alpha^2 \bar{\varphi} +$$
$$+ \lim_{\left\{ \substack{R \to \infty \\ \varrho \to 0} \right.} \frac{\alpha^2}{2\pi} \int\int_S \nabla \cdot (K \nabla \varphi - \varphi \nabla K) \cdot d\xi d\eta.$$

where the domain of integration S is an annular region bounded by a circle Γ of radius R and a circle γ of somewhat smaller radius ϱ, both with center at (x, y). Thus, as R becomes infinite and ϱ approaches zero, the domain S will cover the entire plane. This limiting process is necessitated by the singularity of K at (x, y).

It remains to evaluate the limit indicated in the equation above. Since φ has no singularities at all and K has no singularities in S, the area integral over S may be transformed by Gauss' theorem into a line integral taken around a path C which encloses S. The path of integration is traversed by proceeding around Γ in a counter-clockwise direction, along the left hand edge of a "cut" adjoining Γ and γ, around γ in a clockwise direction, back along the right-hand edge of the "cut", and so on around Γ to the starting point. Thus

$$\overline{\nabla^2\varphi} = \alpha^2\bar{\varphi} + \lim_{\substack{R\to\infty \\ \varrho\to 0}} \frac{\alpha^2}{2\pi}\int_c \left(K\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial K}{\partial n}\right)dC$$

in which $\frac{\partial\varphi}{\partial n}$ and $\frac{\partial K}{\partial n}$ are the projections of $\nabla\varphi$ and ∇K on a unit vector directed outward and normal to the curve C, and dC is an increment of length along C. It is evident, however, that the contributions along one edge of the "cut" and along the other edge exactly cancel, so that

$$\overline{\nabla^2\varphi} = \alpha^2\bar{\varphi} + \lim_{R\to\infty} \frac{\alpha^2 R}{2\pi}\int_0^{2\pi} \left(K\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial K}{\partial r}\right)_R d\Theta$$

$$- \lim_{\varrho\to 0} \frac{\alpha^2\varrho}{2\pi}\int_0^{2\pi} \left(K\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial K}{\partial r}\right)_\varrho d\Theta$$

where Θ is the angle between the x-axis and a radius from (x, y). The second term on the righthand side of the equation above vanishes as R becomes infinite, since $RK_0(\alpha R)$ and $RK_1(\alpha R)$ approach zero exponentially as R becomes infinite. The third term on the right-hand side may be evaluated by the techniques developed to prove Cauchy's residue theorem. Owing to the singularity of K at (x, y), it yields a finite contribution proportional to the value of φ at the point (x, y) where the average applies. The final result, in fact, is simply that

$$\overline{\nabla^2\varphi} = \alpha^2(\bar{\varphi} - \varphi) \tag{6}$$

This is the fundamental operational property of the mean. Taken as it stands, it carries the rather remarkable implication that the deviation of a variable from its corresponding local mean is proportional to the mean of its Laplacian.

4. The Commutative Properties of the Bar-Operator

There are a number of other convenient operational properties of the bar-operator that follow directly from the fundamental property given in Eq. (6). For example, applying the bar-operator to both sides of Eq. (6),

$$\overline{\overline{\nabla^2\varphi}} = \alpha^2(\bar{\bar{\varphi}} - \bar{\varphi})$$

We also apply the bar-operator to $\nabla^2\bar{\varphi}$, which is given by Eq. (6) as

$$\overline{\nabla^2\bar{\varphi}} = \alpha^2(\bar{\bar{\varphi}} - \bar{\varphi})$$

Subtracting the second of the two equations above from the first,

$$\overline{\overline{\nabla^2\varphi} - \nabla^2\bar{\varphi}} = 0$$

Now, it can be shown that the weighted average of a function can be everywhere zero if and only if the function itself is identically zero. In particular,

$$\overline{\nabla^2\varphi} = \nabla^2\bar{\varphi}$$

Thus, the bar-operator may be commuted with the Laplacian operator.

With this result, it is a relatively easy matter to show that the bar-operator can be commuted with all vector differentiations. That is to say,

$$\overline{\nabla\varphi} = \nabla\bar{\varphi}$$

$$\overline{\nabla\cdot\mathbf{A}} = \nabla\cdot\bar{\mathbf{A}}$$

$$\overline{\nabla\times\mathbf{A}} = \nabla\times\bar{\mathbf{A}}$$

for all scalar variables φ and all vectors $\mathbf{A}$. Moreover, since the order of integration with respect to the space coordinates and differentiation with respect to time can be reversed, the bar-operator is commutative with time differentiation.

$$\overline{\frac{\partial\varphi}{\partial t}} = \frac{\partial\bar{\varphi}}{\partial t}$$

As will be found later, all these properties are useful in manipulating the equations of mean motion.

5. Two Inversion Formulae

By a simple rearrangement of Eq. (6), the fundamental operational property of the mean can be put in the form

$$\varphi - \bar{\varphi} + \frac{1}{\alpha^2}\overline{\nabla^2\varphi} = 0$$

Denoting the bar-operator $\overline{(\ )}$ by $L(\ )$, and successively applying the operator $L(\ )$ to the equation above $(n-1)$ times, we generate the following array of equations.

$$\varphi - L(\varphi) + \frac{1}{\alpha^2} L(\nabla^2\varphi) = 0$$

$$L(\varphi) - L^2(\varphi) + \frac{1}{\alpha^2} L^2(\nabla^2\varphi) = 0$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$L^{n-2}(\varphi) - L^{n-1}(\varphi) + \frac{1}{\alpha^2} L^{n-1}(\nabla^2\varphi) = 0$$

$$L^{n-1}(\varphi) - L^n(\varphi) + \frac{1}{\alpha^2} L^n(\nabla^2\varphi) = 0$$

We next add all these equations, noting that the second term of the ith equation just cancels the first term of the $(i+1)$st. Thus,

$$\varphi = L^n(\varphi) - \frac{1}{\alpha^2} \sum_{i=1}^{n} L^i(\nabla^2\varphi)$$

Finally, we shall consider the limit of the expression above as n becomes infinite. At the same time commuting the Laplacian and L ( ) operators,

$$\varphi = \varphi^\star - \frac{1}{\alpha^2} \sum_{i=1}^{\infty} \nabla^2 L^i(\varphi) \tag{7}$$

where $\varphi^\star$ is the limit of $L_n(\varphi)$ as n becomes infinite.

Our next concern is to examine the behavior of $L^n(\varphi)$ for increasingly large values of n. Now, it is intuitively obvious that the result of applying a *normalized* averaging process to a function infinitely many times will tend to a definite limit. That is to say, $\varphi^\star$ exists. A necessary condition for the existence of $\varphi^\star$ is that one more application of the bar-operator leaves the limit unchanged – i. e.,

$$\overline{\varphi^\star} = \varphi^\star$$

But the fundamental operational property of the mean, coupled with its commutative properties, implies that

$$\nabla^2\overline{\varphi^\star} = \alpha^2(\overline{\varphi^\star} - \varphi^\star)$$

from which it follows that

$$\nabla^2\varphi^\star = 0$$

This is simply Laplace's equation, whose solutions are well known. By hypothesis, $\varphi^\star$ is nonsingular and becomes infinite at infinity more slowly than an exponential function. The only function $\varphi^\star$ that satisfies both Laplace's equation and the condition above is of the form

$$\varphi^\star = \alpha x + by + c$$

where a is the *arithmetic* average of $\frac{\partial\varphi}{\partial x}$, taken over the entire plane, b is the corresponding average of $\frac{\partial\varphi}{\partial y}$, and c is an arbitrary constant depending on the location of the origin. With reference to meteorological variables, the arithmetic mean over a very large area is highly persistent, so that $\varphi^\star$ may be regarded as known for purposes of short range prediction.

A further consequence of the argument outlined above is that

$$\lim_{i\to\infty} \nabla^2 L^i(\varphi) = 0$$

For this reason, and because each application of the L ( ) operator involves multiplication by a factor of α^2, the series on the righthand side of Eq. (7) converges for sufficiently small α. It is demonstrable, therefore, that the limits indicated on the righthand side of Eq. (7) exist. Writing Eq. (7) in slightly different form,

$$\varphi = \varphi^\star - \frac{1}{\alpha^2} \sum_{i=1}^{\infty} L^i(\nabla^2\varphi) \tag{8}$$

This equation may be regarded as an inversion formula for solving equations of the Poisson type. That is to say, given the distribution of $\nabla^2\varphi$ over the entire plane, φ can be computed from it by repeated application of the integral operator L ( ). Stated in this way, the inversion formula is reminiscent of a result obtained by FJØRTOFT (1952), through a formal expansion of the finite-difference form of the Laplacian operator.

A more striking consequence of the operational properties of the mean can be derived by applying the Laplacian operator to both sides of Eq. (8). Noting that $\nabla^2\varphi^\star = 0$, and letting $\nabla^2\varphi = \Phi$,

$$\Phi = -\frac{1}{\alpha^2} \nabla^2 \sum_{i=0}^{\infty} L^i(\overline{\Phi}) \tag{9}$$

It must be borne in mind that the equation above is an *identity*, which holds for all functions Φ. One may, therefore, define an "unaveraging" operator $L^{-1}(\ )$ which, when applied to the averaged distribution $\overline{\Phi}$, yields the unaveraged distribution Φ. That is, according to Eq. (9),

$$L^{-1}(\ ) = -\frac{1}{\alpha^2}\nabla^2\sum_{i=0}^{\infty}L^i(\ )$$

In this respect, Eq. (9) is an inversion formula for computing the *unaveraged* distribution corresponding to a given *averaged* distribution. By following a line of reasoning similar to that sketched out in the preceding paragraph, it can be shown that the summation on the righthand side of Eq. (9) converges to a definite limit.

In order to illustrate how the unaveraged distribution is to be computed from the mean, and to examine the conditions under which the computational procedure converges rapidly, we shall return to the example of Eq. (4). In that case, Eq. (9) reduces to

$$\Phi = \frac{\beta^2}{\alpha^2}\sum_{i=0}^{\infty}\left(\frac{\alpha^2}{\alpha^2+\beta^2}\right)^i\left(\frac{\alpha^2}{\alpha^2+\beta^2}\,\Phi\right)$$

By summing the geometric series exactly, it is easily verified that the equation above is an identity. It is also evident that the series converges for all values of α and β, and that it converges rapidly for values of α much less than the wave number. The latter case, which corresponds to "strong averaging", is the one of greatest interest in connection with the development of equations for the mean motion.

6. The Equation of Mean Motion for an Incompressible, Auto-barotropic Fluid

Up to this point, our discussions have dealt only with the mathematical properties of a particular type of average – i. e., with mathematical definitions and identities. Our next concern is to apply those mathematical methods and results to the problem of deriving a closed system of *physical* equations for the averaged motion of a fluid.

In order to present general features of method in the simplest possible form, and to facilitate comparisons with the classical theory of atmospheric turbulence, we shall consider the motions of an incompressible auto-barotropic fluid relative to a non-rotating coordinate system. This, of course, is the model discussed earlier in the introduction, in connection with Reynolds' formulation of the turbulence problem. The *unaveraged* motions of the fluid are governed by the following closed system of equations.

$$\frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\nabla\zeta = 0$$

$$\mathbf{v} = \mathbf{K}\times\nabla\psi \qquad \zeta = \nabla^2\psi$$

where $\mathbf{v}$ is the horizontal component of velocity, ζ is the vertical component of relative vorticity, φ is the streamfunction, and $\mathbf{K}$ is a unit vector directed vertically upward. To obtain the equations for the averaged motion, we simply apply the bar-operator to the equations above.

$$\frac{\partial\overline{\zeta}}{\partial t} + \overline{\mathbf{v}\cdot\nabla\zeta} = 0 \tag{10}$$

$$\overline{\mathbf{v}} = \mathbf{K}\times\nabla\overline{\psi} \tag{11}$$

$$\overline{\zeta} = \nabla^2\overline{\psi} \tag{12}$$

As was also pointed out earlier, the key problem is to express the mean vorticity advection $\overline{\mathbf{v}\cdot\nabla\zeta}$ in terms of the mean vorticity $\overline{\zeta}$. If it is possible to do so, then Eq. (10) provides a method for predicting the future distribution of $\overline{\zeta}$ from its known initial distribution.

For later purposes, it will be convenient to decompose the mean vorticity advection by introducing the deviation from the mean. Denoting the deviation of a variable from its mean by a "prime", we may state the following general result, which is a direct consequence of the fundamental operational property of the mean. [Eq. (6)].

$$\varphi' = -\frac{1}{\alpha^2}\nabla^2\overline{\psi}$$

Applying this result to the stream-function,

$$\psi' = -\frac{1}{\alpha^2}\nabla^2\overline{\psi} = -\frac{1}{\alpha^2}\overline{\zeta}$$

$$\mathbf{v}' = -\frac{1}{\alpha^2}\mathbf{K}\times\nabla\overline{\zeta}$$

$$\zeta' = -\frac{1}{\alpha^2}\nabla^2\overline{\zeta}$$

The expressions above are now introduced into the vorticity advection.

$$\mathbf{v}\cdot\nabla\zeta = \bar{\mathbf{v}}\cdot\nabla\bar{\zeta} + \mathbf{v}'\cdot\nabla\bar{\zeta}' + \bar{\mathbf{v}}\cdot\nabla\zeta' + \mathbf{v}'\cdot\nabla\zeta' =$$
$$= \bar{\mathbf{v}}\cdot\nabla\bar{\zeta} - \frac{1}{\alpha^2}\bar{\mathbf{v}}\cdot\nabla(\nabla^2\bar{\zeta}) + \frac{1}{\alpha^4}J(\bar{\zeta}, \nabla^2\bar{\zeta})$$

in which J is the Jacobian, or "functional" determinant. Finally, averaging the equation above, again making use of Eq. (6), and substituting into Eq. (10),

$$\frac{\partial\bar{\zeta}}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\bar{\zeta} + \frac{1}{\alpha^4}\overline{J(\bar{\zeta}, \nabla^2\bar{\zeta})} +$$
$$+ \frac{1}{\alpha^2}\nabla\cdot\overline{(\nabla\nabla\cdot\bar{\zeta}\bar{\mathbf{v}} - \bar{\mathbf{v}}\nabla^2\bar{\zeta})} = 0 \qquad (13)$$

Formally, at least, this equation represents the solution of the problem that was originally set. Starting with the known initial distribution of $\bar{\zeta}$, we may compute $\bar{\psi}$ by making use of Eq. (12) and the inversion formula for solving equations of the Poisson type [Eq. (8)]. Having computed $\bar{\psi}$, it is then possible to compute $\bar{\mathbf{v}}$ from Eq. (11). In this way, since they involve no time derivatives, all terms of Eq. (13) *except the first* can be computed from the initial distribution of $\bar{\zeta}$. Thus Eq. (13) gives the instantaneous rate of change of $\bar{\zeta}$ at some arbitrarily chosen initial moment, in terms of the spatial distribution of $\bar{\zeta}$ at that time. This property enables us to predict the distribution of $\bar{\zeta}$ at a time slightly later than the initial moment, whence the process can be repeated indefinitely. This is sufficient to demonstrate that Eq. (13) is a true prognostic equation, and that Eqs (10), (11), and (12) constitute a closed system of equations for the mean motion.

7. The Case of "Strong Averaging"

Although Eq. (13) comprises a satisfactory solution to the problem that was originally set, it is difficult to interpret this result as it stands. For simplicity and ease of comparison, we shall next consider the case of "strong averaging" – i. e., averaging for which the scale parameter α is much less than some characteristic wave number. In this case, the weighted average of the vorticity advection displays the same properties as an unweighted average, and takes on a more familiar form.

By introducing the relationship between the mean and deviations from the mean [Eq. (6)], the product of any two typical variables u and v may be written as

$$\overline{uv} = \overline{\bar{u}\bar{v}} + (\overline{\bar{u}v'} + \overline{\bar{v}u'}) + \overline{u'v'}$$
$$= \bar{u}\bar{v} - \frac{1}{\alpha^2}(\bar{u}\nabla^2\bar{v} + \bar{v}\nabla^2\bar{u}) + \frac{1}{\alpha^4}\nabla^2\bar{u}\nabla^2\bar{v}$$
$$= \bar{u}\bar{v} - \frac{1}{\alpha^2}\nabla^2(\bar{u}\bar{v}) + \frac{2}{\alpha^2}\nabla\bar{u}\cdot\nabla\bar{v} + \frac{1}{\alpha^4}\nabla^2\bar{u}\nabla^2\bar{v}$$

where, as before, a prime denotes the deviation from the mean. Applying the bar-operator to the equation above, and again making use of the fundamental operational property of the mean,

$$\overline{uv} = \bar{u}\bar{v} + \frac{2}{\alpha^2}\overline{\nabla\bar{u}\cdot\nabla\bar{v}} + \frac{1}{\alpha^4}\overline{\nabla^2\bar{u}\nabla^2\bar{v}} \qquad (14)$$

For purposes of estimate, we now ascribe to the disturbances in the *mean* flow a characteristic wave number ν^1. Thus the *relative* orders of magnitude of the second and third terms on the righthand side of Eq. (14) are

$$\frac{1}{\alpha^2}\overline{\nabla\bar{u}\cdot\nabla\bar{v}} \rightarrow \frac{\nu^2}{\alpha^2}$$
$$\frac{1}{\alpha^4}\overline{\nabla^2\bar{u}\nabla^2\bar{v}} \rightarrow \frac{\nu^4}{\alpha^4}$$

In general, therefore, if α is much less than the characteristic wave number of the disturbances in the mean flow (the longest wave components in the unaveraged spectrum), the third term on the righthand side of Eq. (14) is much greater than the second. In other words, in the case of "strong averaging", the weighted average of a product reduces to

$$\overline{uv} = \bar{u}\bar{v} + \overline{u'v'} \qquad (15)$$

This property, it will be recognized, is peculiar to the usual unweighted mean or, more generally, to a "weakly" weighted mean.

Specializing the general result of Eq. (15), we may now put Eq. (13) in the alternative forms

$$\left.\begin{aligned} \frac{\partial\bar{\zeta}}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\bar{\zeta} &= -\nabla\cdot\overline{\zeta'\mathbf{v}'} && \text{a} \\ &= -\frac{1}{\alpha^4}\overline{J(\bar{\zeta}, \nabla^2\bar{\zeta})} && \text{b} \\ &= -\frac{1}{\alpha^2}\overline{J(\zeta, \bar{\zeta})} && \text{c} \end{aligned}\right\} \qquad (16)$$

As pointed out earlier, the conditions under which the equations above apply are also those for which the series in the inversion formulae (8) and (9) converge most rapidly. From this point onward, the case of "strong averaging" is the one with which we shall be primarily concerned.

The righthand side of Eq. (16) represents the rate at which the *mean* vorticity of an imaginary element, traveling with the *mean* velocity, is created or destroyed or, from another point of view, reflects the interaction between the mean flow and the fluctuations that have been smoothed out by averaging. According to Eq. (16a), mean vorticity is generated by a net convergence of the "eddy" transport of vorticity – i. e., the transport of vorticity effected by the "eddies" or fluctuations that have been smoothed out. From the distinctive form of Eq. (16 c), it may be seen that no mean vorticity is created or destroyed if the unaveraged vorticity distribution is lined symmetrical, a fact that has been independently and adequately established by earlier writers.

8. The Relation of the Equation of Mean Motion to the Theory of Turbulent Diffusion

According to 16 a, the equation of "strongly averaged" motion can be written as

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \bar{\zeta} + \nabla \cdot \overline{\zeta' \mathbf{v}'} = 0$$

This equation, it will be recognized, is identical in form to the Reynolds-averaged vorticity equation. Our present purpose is to compare the form of the eddy transport $\overline{\zeta' \mathbf{v}'}$ – which, in this case, can be deduced exactly – with the form derived in the classical theory of eddy diffusion. Following the usual method of development, we introduce the "mixing-length" hypothesis, imagining that turbulent fluctuations of vorticity are brought about by small displacements of fluid elements from their mean positions, each conserving its original mean vorticity.

$$\overline{\zeta' \mathbf{v}'} = -\overline{(\mathbf{r}' \cdot \nabla \bar{\zeta}) \mathbf{v}'}$$

where $\mathbf{r}'$ is to be interpreted as an eddy displacement. Subject to several restrictions, including the assumption that eddy displacements in one direction and eddy velocities in the direction normal to it are uncorrelated, the expression above can be transformed into

$$\overline{\zeta' \mathbf{v}'} = -(\overline{\mathbf{r}' \cdot \mathbf{v}'}) \nabla \bar{\zeta}$$

This result simply states that the eddy transport is directed down the gradient of mean vorticity, and that the coefficient of eddy diffusion $\overline{\mathbf{r}' \cdot \mathbf{v}'}$ depends on the correlation between eddy displacements and eddy velocities in the same direction. If, as is usually supposed in the classical theory of eddy diffusion, the latter correlation is always positive, then extreme values of mean vorticity would tend to disappear as time went on.

In the present case, on the other hand, there is no real compulsion to introduce restrictions on the statistics of the eddy motions, for the eddy velocity $\mathbf{v}'$ is expressible in terms of the mean vorticity gradient and vice versa. Thus, according to the results of section 6,

$$\begin{aligned}\overline{\zeta' \mathbf{v}'} &= -\overline{(\mathbf{r}' \cdot \nabla \bar{\zeta}) \mathbf{v}'} \\ &= -\overline{(\mathbf{K} \cdot \mathbf{r}' \times \mathbf{v}')(\mathbf{K} \times \nabla \bar{\zeta})}\end{aligned}$$

or, approximately,

$$\overline{\zeta' \mathbf{v}'} = -(\overline{\mathbf{K} \cdot \mathbf{r}' \times \mathbf{v}'})(\mathbf{K} \times \nabla \bar{\zeta})$$

In contradistinction to the classical theory, this result states that the eddy transport is directed *normal* to the mean vorticity gradient, and that the coefficient of eddy transfer depends on the correlation between eddy displacements in one direction and eddy velocities in the direction normal to it.

It should be emphasized that it is not even necessary to introduce the mixing-length hypothesis in the present case, for the exact form of the eddy transport of vorticity is implicit in Eq. (13). The comparison above is, however, useful in demonstrating that the process by which mean vorticity is generated or destroyed is not one of pure eddy diffusion, and in pointing out the dangers of imposing arbitrary restrictions on the statistics of the eddy fluctuations, however plausible they might otherwise appear.

9. Extensions to Simple Types of Baroclinie Flow

The foregoing results, it will be realized, apply to a very special type of flow, and are intended mainly to sketch out the pattern that a more general development might follow. Some generalizations – e. g., the inclusion of effects engendered by referring the motions to a rotating coordinate system – require only slight modifications of the preceding theory and need no elaboration. The extension of this theory to systems whose behavior is controlled by basically different physical mechanisms is not, however, so obvious. Our present concern is to show how the methods developed in the foregoing sections can be used to derive equations for the mean quasi-geostrophic motions of a simple baroclinic fluid. Divergent barotropic flow is included under the latter as a special case.

The equations for the simple types of baroclinic flow recently studied by ELIASSON (1952), EADY (1952), THOMPSON (1953), BUSHBY and SAWYER (1953), and CHARNEY and PHILLIPS (1953) can, in general, be put into the form of a simultaneous system of two equations.

$$\frac{\partial}{\partial t}(\nabla^2\Psi - \mu_1^2\Psi) = -J(\Psi, \nabla^2\Psi) - AJ(\Phi, \nabla^2\Phi) - \beta\frac{\partial\Psi}{\partial x} \qquad (17\text{a})$$

$$\frac{\partial}{\partial t}(\nabla^2\Phi - \mu_2^2\Phi) = -J(\Psi, \nabla^2\Phi) - J(\Phi, \nabla^2\Psi) - BJ(\Phi, \nabla^2\Phi) + \mu_2^2 J(\Psi, \Phi) - \beta\frac{\partial\Phi}{\partial x} \qquad (17\text{b})$$

Here the dependent variables Ψ and Φ are the stream-functions for the vertically-integrated mean wind and thermal wind, respectively. The quantities A, B, μ_1, and μ_2 are regarded as known constants, β is the northward derivative of the Coriolis parameter, and the coordinate x is directed toward the east. Since the right hand sides of Eqs (17a) and (17b) do not involve time derivatives, it is evident that they can be expressed in terms of $\overline{\Psi}$ and $\overline{\Phi}$, by making use of Eq. (6). Accordingly, if the bar-operator is applied to the equations above, they may be regarded as a means for predicting $(\nabla^2 - \mu_1^2)\overline{\Psi}$ and $(\nabla^2 - \mu_2^2)\overline{\Phi}$. The remaining question is whether or not $\overline{\Psi}$ and $\overline{\Phi}$ can be computed from the predicted values of $(\nabla^2 - \mu_1^2)\overline{\Psi}$ and $(\nabla^2 - \mu_2^2)\overline{\Phi}$, to regenerate the initial conditions at a slightly later time.

Stated in general terms, the crux of the matter is whether or not it is possible to compute any typical variable φ from a known distribution of $(\nabla^2 - \mu^2)\varphi$. Although there are already well known methods for doing so, we shall present an integral operator method similar to the inversion procedure developed in section 5. As before, we generate an array of equations, obtained by successive applications of the L ( ) operator to Eq. (6), and by multiplying the ith equation in the array by

$$\frac{1}{\alpha^2}\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)^{i-1}.$$

$$\varphi - \left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)L(\varphi) = -\frac{1}{\alpha^2}L(\nabla^2\varphi - \mu^2\varphi)$$

$$\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)\left[L(\varphi) - \left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)L^2(\varphi)\right] = -\left[\frac{1}{\alpha^2}L^2(\nabla^2\varphi - \mu^2\varphi)\right]\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)^{n-1}\left[L^{n-1}(\varphi) - \left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)L^n(\varphi)\right] = -\left[\frac{1}{\alpha^2}L^n(\nabla^2\varphi - \mu^2\varphi)\right]\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)^{n-1}$$

We next add all these equations, noting that the second term on the left hand side of the ith equation exactly cancels the first term of the $(i+1)$st.

$$\varphi = \left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)^n L^n(\varphi) - \frac{1}{\alpha^2}\sum_{i=1}^{n}\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)^{i-1}\cdot L^i(\nabla^2\varphi - \mu^2\varphi)$$

It will now be supposed that μ is less than α – a restriction that is not very severe, since μ is generally much less than the wave number of the large scale disturbances. In that case, the limiting form of the equation above is

$$\varphi = -\frac{1}{\alpha^2}\sum_{i=1}^{\infty}\left(\frac{\alpha^2-\mu^2}{\alpha^2}\right)^{i-1}L^i(\nabla^2\varphi - \mu^2\varphi) \qquad (18)$$

This result shows that φ can be computed from a given distribution of $(\nabla^2 - \mu^2)\varphi$. Since μ is less than α, the righthand side of the equation above converges more rapidly than the series in Eq. (8), which has already been found convergent. It can be concluded, therefore, that the methods developed in preceding sections will lead to a closed system of equations for the mean quasi-geostrophic motions of simple baroclinic fluids.

10. A Weighted Time Mean as a Smoothing Operator

In order to point up the advantages of the space mean, and to indicate how the foregoing methods can be extended to time means, we shall derive the fundamental operational properties of a weighted time-average. In this connection, it should be noted that a time mean – if it is to be of any use for purposes of prediction – should not depend on unaveraged values at times later than a specified "moment of latest information." Also, it is intuitively evident that the mean should place greatest weight on the state at the "moment of latest information", with decreasing weight on the state in the increasingly remote past. This suggests an average of the form

$$\bar{\varphi}(t) = k \int_{-\infty}^{t} e^{k(\tau - t)} \varphi(\tau) d\tau \qquad (19)$$

where τ is a dummy variable of integration corresponding to t, and k is an unspecified constant that we are free to fix. The average is normalized, in that the mean value of "one" is "one". The exponential factor acts as a weighting function which attaches no weight on data beyond time t, greatest weight on data at time t, and decreasing weight to data at times in the increasingly distant past. It also satisfies the formal conditions on a proper weighting function, namely, that it is monotonic and never negative.

Applying this averaging operator to a sinusoidal time series, for example, we find that the average value of $\varphi = \cos \lambda\tau$ is

$$\bar{\varphi} = \sin \Theta \sin (\lambda t + \Theta)$$

where

$$\Theta = \sin^{-1} \frac{k}{\sqrt{k^2 + \lambda^2}}$$

so that, in this case,

$$\lim_{\frac{k}{\lambda} \to \infty} \bar{\varphi} = \varphi \qquad \lim_{\frac{k}{\lambda} \to 0} \bar{\varphi} = 0$$

Thus the role of k is very similar to that of the "scale parameter" α, and what has previously been said of comparisons between α and the characteristic wave number is equally true of comparisons between k and the frequency.

With regard to operational properties of the mean, it will be recognized that the "smoothing operator" defined by Eq. (19) is related to the Laplace transform, and that it has similar properties. Applying the bar-operator to $\frac{\partial \varphi}{\partial t}$, for example, and integrating by parts,

$$\overline{\frac{\partial \varphi}{\partial t}} = k \int_{-\infty}^{t} e^{k(\tau - t)} \frac{\partial \varphi}{\partial \tau} d\tau =$$

$$= k [\varphi e^{k(\tau - t)}]_{-\infty}^{t} - k^2 \int_{-\infty}^{t} e^{k(\tau - t)} \varphi d\tau$$

Whence, by definition,

$$\overline{\frac{\partial \varphi}{\partial t}} = k(\varphi - \bar{\varphi}) \qquad (20)$$

This is the fundamental operational property of the time-mean. By direct differentiation of Eq. (19), it may be verified that

$$\overline{\frac{\partial \varphi}{\partial t}} = \frac{\partial \bar{\varphi}}{\partial t}$$

Moreover, since the order of integration with respect to time and differentiations with respect to the space coordinates is reversible, the bar-operator may be commuted with all vector differentiations.

Eq. (20) enables us to derive a closed system of equations for the time-averaged motions, for deviations from the mean can be expressed in terms of the mean itself. That is, denoting deviations by primed quantities,

$$\varphi' = \frac{1}{k} \frac{\partial \bar{\psi}}{\partial t}$$

Applying this result to the stream-function,

$$\psi' = \frac{1}{k} \frac{\partial \bar{\psi}}{\partial t}$$

$$\mathbf{v}' = \frac{1}{k} \mathbf{K} \times \frac{\partial}{\partial t} \nabla \bar{\psi}$$

$$\zeta' = \frac{1}{k} \frac{\partial \bar{\zeta}}{\partial t}$$

We now introduce the expressions above into the time-averaged vorticity equation for incompressible auto-barotropic flow.

$$\frac{\partial \bar{\zeta}}{\partial t} + \overline{\mathbf{v} \cdot \nabla \zeta} = 0$$

After considerable manipulation, the equation above can be put in a form equivalent to that of the Reynolds-averaged equation.

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \bar{\zeta} + \frac{1}{k^2} \overline{J\left(\frac{\partial \bar{\psi}}{\partial t}, \frac{\partial \bar{\zeta}}{\partial t}\right)} = 0 \qquad (21)$$

Since $\bar{\mathbf{v}} = \mathbf{K} \times \nabla \bar{\psi}$ and $\bar{\zeta} = \nabla^2 \bar{\psi}$, this equation involves only one unknown, – i. e., the mean stream-function $\bar{\psi}$ – and thus constitutes a closed system of equations for the time-averaged motions.

Although Eq. (21) states exactly and concisely how *mean* vorticity is generated by the fluctuations that are smoothed out in the averaging process, it must be noted that the time derivative of the mean stream-function enters nonlinearly. In general, therefore, the mean stream-function tendency cannot be obtained from Eq. (21) in closed form. On further reflection, it will be seen that this defect would be present in any system of time-averaging of this type. On the other hand, as was pointed out earlier, space-averaging leads to equations which are linear in the time derivatives and which can, accordingly, be solved by known methods.

11. Summary and Conclusions

The problem of deriving a closed system of equations for the mean motion of a fluid has been approached from the standpoint of reconstructing the unaveraged distribution of a variable from the distribution of its average. For purposes of mathematical simplicity, it has been found convenient to consider weighted averages, in which the weighting functions are chosen with regard to the peculiar form of the equations for the unaveraged motion. The averaging or "smoothing" operations associated with these weighted means are essentially transforms or integral operators, and have similar operational properties.

By making use of the operational properties of a weighted area mean, we have derived an inversion formula [Eq. (9)] for computing the *unaveraged* distribution of a variable from its *averaged* distribution.

An integral operator method [Eqs (8) and (18) for inverting operators of the Laplace and Helmholtz types has been developed as a natural by-product. Finally, it has been shown that a certain type of weighted time mean has operational properties analogous to those of the area mean.

To illustrate general features of method, the mathematical results of the first part have been used to derive a closed system of equations for the mean motion of an incompressible autobarotropic fluid, relative to a nonrotating coordinate system [Eqs (13) and (21)]. This derivation has been carried out for both area and time averages, either of which is satisfactory from the standpoint of reducing nonsystematic instrumental and round-off errors in reported observations. The equation for the area-averaged motion, however, is *linear* in the time derivatives and can be solved by known methods, whereas the equation for the time-averaged motion is *nonlinear* in the time derivatives and cannot be solved in closed form. This result, which stems from the distinctive form of the equation for the unaveraged motion, indicates that the area mean is preferable for purposes of applying the theory of mean motions.

The case of "strong averaging", corresponding to a suitably chosen value of an adjustable scale parameter, is of greatest intrinsic interest, and has been examined in some detail. In that case, the area-averaged vorticity equation is identical in form to the Reynolds-averaged equation. By introducing the "mixing-length" hypothesis without the usual restrictions on the statistics of eddy fluctuations, it is shown that the eddy transport of vorticity is directed normal to the mean vorticity gradient, and that the coefficient of eddy transfer depends on the correlation between eddy displacements in one direction and eddy velocities in the direction normal to it. This result, of course, is at direct variance with the classical theory of eddy diffusion.

Finally, we have outlined a general pattern of development by which these methods can be used to derive a closed system of equations for the area-averaged motions of a compressible barotropic fluid and of certain simple types of baroclinic fluids. It appears probable that more general methods for inverting the process of averaging will be capable of extension to a wide variety of physical problems.

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