The Exact Determination of the Effective Domain of Dependence of a One-dimensional Numerical Prediction Formula

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Abstract

By assuming sinusoidal initial conditions, an exact expression for determining the "cut-off" point of a one-dimensional influence function is derived. From this result one may construct tables of percent error incurred by terminating the influence function at various distances.

1. Introduction

One-dimensional numerical prediction techniques usually require the integrated product of initial conditions and an "influence function" along a latitude circle. Fortunately, the nature of the influence function is usually such that one need not know the data all the way around a given circle of latitude. That is, in order to compute a sufficiently accurate forecast, one may terminate the influence function at some large, but finite distance from the origin of computations. This distance is referred to as the radius of the "effective domain of dependence". As it turns out, the radius of the domain of dependence varies with the wavelength of the initial distribution, as well as with the phase difference between the initial distribution and the origin from which the forecast for a given point is computed. Thus, in making a forecast at a given latitude on a given day, it appears necessary to terminate the influence function at different distances for different forecast points in order to achieve the same level of accuracy everywhere.

In a recent paper, THOMPSON (1952) has shown how to obtain the Fourier integral solution of the Rossby vorticity equation in the form

$$z(\tau U t, t) = z(0, 0) + \int_{-\infty}^{0} z(\xi, 0)\, G(\xi) d\xi \quad (1)$$

where $z(x, t)$ is the height of the 500-mb surface at time t, and

$$G(\xi) = -\sqrt{\beta t}\,\frac{J_1(2\sqrt{-\beta t \xi})}{\sqrt{-\xi}}, \quad (\xi \leq 0) \quad (2)$$

is the influence function. $J_1(2\sqrt{-\beta t\xi})$ is the Bessel function of the first kind of order one, β is the N – S variation of Coriolis parameter, x is the distance, positive toward the east, τ is a constant, U is the undisturbed zonal wind speed. The essential difference between this solution and that of CHARNEY and ELIASSEN (1949) is that the latter solution is expressed as a Fourier series. Both solutions have the virtue that the "effective domain of dependence" is finite, so that one need not know the data everywhere to make a sufficiently accurate forecast.

LÖNNQVIST (1949) has applied the Charney-Eliassen formula to obtain numerical predictions for sinusoidal initial conditions. Due to the nature of the influence function in that formula, it appears that an attempt to carry out exact integrations for sinusoidal initial conditions would be an extremely difficult task. In the present study, however, it is possible, by using formula (1) above, to carry out the integration for sinusoidal initial conditions exactly, and thus compute the exact height change of the 500-mb surface. Actually, such a procedure is unnecessary for prediction of the propagation of sinusoidal initial conditions, since such waves will move with exactly the Rossby phase speed, suffering no change of amplitude or wavelength. Nevertheless, the performance of the exact integration yields information which is useful for determining quite accurate estimates of the "effective domain of dependence".

CHARNEY (1949) has derived an expression for determining the error incurred in cutting off the influence function at certain points. The integrals involved were determined empirically from an observed pressure distribution. In the present study, we evaluate the necessary integrals exactly by assuming sinusoidal initial conditions. We assume that the results obtained in such a manner will apply sufficiently well to arbitrary initial conditions.

We proceed in the following way: We first carry out the integration in (1) for sinusoidal initial conditions, and obtain an expression which is a function of the wavelength of the initial distribution and the phase difference between the sine wave and the origin from which the forecast is computed. For a given latitude and forecast period, we may then compute the height change as a function of wavelength and phase angle. We then perform the same integration, but this time terminate the integration at some finite distance from the origin. This integration yields an expression which is a function of distance out, as well as the previously mentioned parameters. Using the results of these two integrations, we are in a position to determine the finite distance out which will give a percent error equal to or less than some previously assigned maximum value.

Thus, if we let

$$z(\xi, 0) = A \sin m[-(\xi - \chi)], \quad (\xi, \chi \leq 0), \tag{3}$$

where A = amplitude, χ = phase angle, $m = \frac{2\pi}{L}$, L = wavelength, then we wish to satisfy the inequality

$$\left.\left|\frac{\int_{-\infty}^{0} A \sin m[-(\xi-\chi)]\, G(\xi) d\xi - \int_{-\xi_1}^{0} A \sin m[-(\xi-\chi)]\, G(\xi) d\xi}{\int_{-\infty}^{0} A \sin m[-(\xi-\chi)]\, G(\xi) d\xi}\right| < E, \right\} \tag{4}$$

where E is the maximum percent error allowed.

2. Evaluation of the Integrals

Let

$$\left.\begin{aligned} B = & \int_{-\infty}^{0} A \sin m[-(\xi-\chi)](-\sqrt{\beta t}) \cdot \\ & \cdot \frac{J_1(2\sqrt{-\beta t \xi})}{\sqrt{-\xi}} d\xi, \; (\xi, \chi \leq 0), \end{aligned}\right\} \tag{5}$$

which, because $(\xi, \chi \leq 0)$, may be written

$$B = -A\sqrt{\beta t} \int_{-\infty}^{0} \sin m(|\xi| - |\chi|) \frac{J_1(2\sqrt{+\beta t|\xi|})}{\sqrt{|\xi|}} d\xi \tag{6}$$

If we write

$$y = 2\sqrt{\beta t|\xi|} \tag{7}$$

we obtain

$$B = A \int_{0}^{\infty} - \sin(\alpha^2 y^2 - K) J_1(y) dy, \tag{8}$$

where

$$\left.\begin{aligned} \alpha^2 &= \frac{m}{4\beta t} \\ K &= m|\chi| \end{aligned}\right\} \tag{9}$$

Therefore

$$B = Im[A \int_{0}^{\infty} J_1(y) e^{iK} e^{-p^2 y^2} dy], \tag{10}$$

where

$$p^2 = i\alpha^2, \; (i = \sqrt{-1}) \tag{11}$$

From WATSON (1944), p. 394 (5):

$$\int_{0}^{\infty} J_{2\nu}(ay) e^{-p^2 y^2} dy = \frac{\sqrt{\pi}}{2p} e^{-\frac{a^2}{8p^2}} I_\nu\left(\frac{a^2}{8p^2}\right) \tag{12}$$

With $\nu = \frac{1}{2}$, $a = 1$, we find:

$$B = Im\left[\frac{A\sqrt{\pi}}{4\alpha i^{1/2}} e^{iK} e^{\frac{i}{8\alpha^2}} I_{1/2}\left(-\frac{i}{8\alpha^2}\right)\right], \quad (13)$$

where $I_{1/2}$ is the Bessel function of the first kind for imaginary argument.

Since

$$-\pi < \arg\left(-\frac{i}{8\alpha^2}\right) < \frac{\pi}{2}, \quad (14)$$

we may use the formula (Watson, p. 77)

$$I_{1/2}\left(-\frac{i}{8\alpha^2}\right) = e^{-\frac{\pi i}{4}} J_{1/2}\left(-\frac{i}{8\alpha^2} e^{\frac{\pi i}{2}}\right) \quad (15)$$

Making use of (15) and the expression

$$J_{1/2}(\chi) = \sqrt{\frac{2}{\pi\chi}} \sin\chi, \quad (16)$$

and noting that

$$m = \frac{2\pi}{L} \quad (17)$$

where L is the wavelength of the initial distribution, (13) finally reduces to

$$B = -2A \sin\left(\frac{\beta L t}{4\pi}\right) \cos\left(\frac{\beta L t}{4\pi} + \frac{2\pi|\chi|}{L}\right) \quad (18)$$

The expression (18) may be used to compute the exact height change of a given sinusoidal wave for various phase angles ($|\chi|$) and various wavelengths L at given latitudes for a given forecast period.

The next step is to evaluate the same integral as (5), but with the lower limit replaced by $(-\xi_1)$. We therefore consider the integral

$$\left.\begin{aligned} B_1 = \int_{-\xi_1}^{0} & A \sin m[-(\xi - \chi)](\sqrt{-\beta t}) \cdot \\ & \cdot \frac{J_1(2\sqrt{-\beta t \xi})}{\sqrt{-\xi}} d\xi, \ (\xi, \chi \leq 0), \end{aligned}\right\} \quad (19)$$

or

$$B_1 = A\sqrt{\beta t}\int_0^{|\xi_1|} \sin m(|\xi| - |\chi|) \cdot \frac{J_1(2\sqrt{\beta t|\xi|})}{\sqrt{|\xi|}} d\xi \quad (20)$$

If we let

$$|\xi| = \frac{y^2}{4\beta t} \quad (7)$$

(20) becomes

$$B_1 = -A \int_0^{2\sqrt{\beta t|\xi_1|}} \sin\left(\frac{my^2}{4\beta t} - m|\chi|\right) J_1(y)\, dy \quad (21)$$

With the following substitutions:

$$\left.\begin{aligned} b &= 2\sqrt{\beta t|\xi_1|} \\ y &= bz \\ m|\xi_1| &= \frac{w}{2} \end{aligned}\right\} \quad (22)$$

and $K = m|\chi|$ as in (9) above, (21) finally becomes

$$\left.\begin{aligned} B_1 = A\Big[& \sin K \int_0^1 b \cos\left(\frac{wz^2}{2}\right) J_1(bz)\, dz - \\ & - \cos K \int_0^1 b \sin\left(\frac{wz^2}{2}\right) J_1(bz)\, dz \end{aligned}\right\} \quad (23)$$

The integrals in (23) are expressible in terms of the Lommel functions of two variables, $U_n\left(\frac{b^2}{w}, b\right)$.

From Watson, p. 543 (1 & 2):

$$\int_0^1 bJ_1(bz) \cos\left(\frac{wz^2}{2}\right) dz = \sin\left(\frac{w}{2}\right) U_1\left(\frac{b^2}{w}, b\right) - \\ - \cos\left(\frac{w}{2}\right) U_2\left(\frac{b^2}{w}, b\right) + U_2\left(\frac{b^2}{w}, 0\right)$$

$$\int_0^1 bJ_1(bz) \sin\left(\frac{wz^2}{2}\right) dz = -\cos\left(\frac{w}{2}\right) U_1\left(\frac{b^2}{w}, b\right) - \\ - \sin\left(\frac{w}{2}\right) U_2\left(\frac{b^2}{w}, b\right) + U_1\left(\frac{b^2}{w}, 0\right) \quad (24)$$

Where

$$\left.\begin{aligned} U_n\left(\frac{b^2}{w}, b\right) &= \sum_{r=0}^{\infty} (-1)^r \left(\frac{b}{w}\right)^{n+2r} J_{n+2r}(b \\ U_n\left(\frac{b^2}{w}, 0\right) &= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{b^2}{2w}\right)^{n+2r}}{\Gamma(n+2r+1)} \end{aligned}\right\} \quad (25)$$

Thus

$$B_1 = A\Big[\cos\left(K - \frac{w}{2}\right) U_1\left(\frac{b^2}{w}, b\right) - \\ - \sin\left(K - \frac{w}{2}\right) U_2\left(\frac{b^2}{w}, b\right) + \sin K U_2\left(\frac{b^2}{w}, 0\right) - \\ - \cos K U_1\left(\frac{b^2}{w}, 0\right)\Big] \quad (26)$$

If we now substitute (22), (7), (17), (9) into (26), we obtain

$$B_1 = A\Big[\cos\frac{2\pi}{L}(|\xi_1| - |\chi|)U_1\Big(\frac{\beta Lt}{\pi}, 2\sqrt{\beta t|\xi_1|}\Big) + + \sin\frac{2\pi}{L}(|\xi_1| - |\chi|)U_2\Big(\frac{\beta Lt}{\pi}, 2\sqrt{\beta t|\xi_1|}\Big) + + \sin\frac{2\pi|\chi|}{L}U_2\Big(\frac{\beta Lt}{\pi}, 0\Big) - - \cos\frac{2\pi|\chi|}{L}U_1\Big(\frac{\beta Lt}{\pi}, 0\Big)\Big] \qquad (27)$$

If we consider what happens to B_1 for large $|\xi_1|$, we find

$$\lim_{|\xi_1|\to\infty} B_1 = A\Big[\sin\frac{2\pi|\chi|}{L}U_2\Big(\frac{\beta Lt}{\pi}, 0\Big) - - \cos\frac{2\pi|\chi|}{L}U_1\Big(\frac{\beta Lt}{\pi}, 0\Big)\Big], \qquad (28)$$

since the U_1 and U_2 functions go to zero for large $|\xi_1|$.

It may now readily be shown that the right side of (28) is indeed equal to the right side of (18).

Thus

$$\lim_{|\xi_1|\to\infty} B_1 = B \qquad (29)$$

which, of course, is to be expected. Therefore, from (27), we have,

$$B - B_1 = -A\Big[\cos\frac{2\pi}{L}(|\xi_1| - |\chi|) U_1\Big(\frac{\beta Lt}{\pi}, 2\sqrt{\beta t|\xi_1|}\Big) + \sin\frac{2\pi}{L}(|\xi_1| - |\chi|) U_2\Big(\frac{\beta Lt}{\pi}, 2\sqrt{\beta t|\xi_1|}\Big)\Big] \qquad (30)$$

Finally,

$$E = \left|\frac{B - B_1}{B}\right| = \left|\frac{1}{2\sin\Big(\frac{\beta Lt}{4\pi}\Big)\cos\Big(\frac{\beta Lt}{4\pi} + \frac{2\pi|\chi|}{L}\Big)} \cdot \left[\cos\frac{2\pi}{L}(|\xi_1| - |\chi|) U_1\Big(\frac{\beta Lt}{\pi}, 2\sqrt{\beta t|\xi_1|}\Big) + + \sin\frac{2\pi}{L}(|\xi_1| - |\chi|) U_2\Big(\frac{\beta Lt}{\pi}, 2\sqrt{\beta t|\xi_1|}\Big)\right]\right| \qquad (31)$$

3. Results

The result (31) is exact. It is unfortunate that available tables of Lommel functions (See E. LOMMEL, 1886, pp. 229–239) make it impossible to carry the calculations out to very

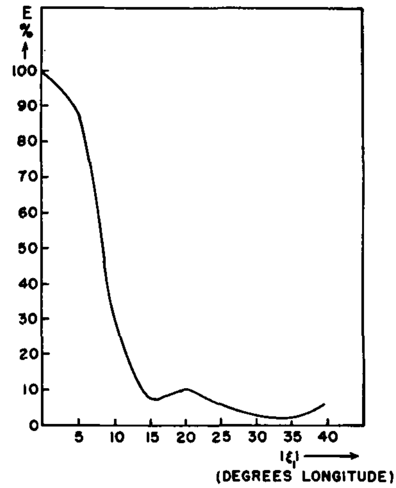

Fig. 1. Absolute value of error (E) in percent, vs. distanc out ($|\xi_1|$) from origin. $t = 1$ day, latitude $= 45°$ N, $L = 90°$ longitude, $|\chi| = 0$.

large distances from the origin. Fig. 1 shows the graph of E *vs* $|\xi_1|$ for $t = 1$ day, latitude 45° N, $L = 90°$ longitude, $|\chi| = 0$. From the graph it is apparent that one may be assured of an error < 10 % if the numerical integration is carried out beyond 20° longitude from the origin. In fact, at 35° away, the error is only 2.6 %. The rest of the curve will oscillate with peaks of decreasing amplitude, much like the Bessel functions, and finally go to zero.

It is of interest to note that, in previous applications of equation (1), we have used $|\xi_1| = 35°$ longitude as a rough estimate of the cut-off point. It now turns out that this was a fortunate choice, particularly when the forecast is being made for a point of zero phase angle of initial conditions.

In practice, since it is clear from (31) that E varies with $|\chi|$ as well as with β, t, and L, one would construct a table of E for various $|\chi|$, β, t, and L for quick reference in preparation of forecasts.

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REFERENCES

CHARNEY, J. G., 1949: On a physical basis for numerical prediction of large-scale motions in the atmosphere. *Journal of Meteorology*, **6**, 6, pp. 371 – 385.

CHARNEY, J. G., and ELIASSEN, A., 1949: A numerical method for predicting the pertubations of the middle latitude westerlies. *Tellus*, **1**, 2, pp. 38 – 54.

LOMMEL, E., 1886: Die Beugungserscheinungen einer kreisrunden Schirmchens, Theoretisch und experimentell. *Abhandlungen Der Mathematisch-Physikalischen Classe der Königlich Bayerischen Akademie der Wissenschaften*. München, Band 15, pp. 229 – 329.

LÖNNQVIST, O., 1949: The numerical prediction method for upper air profiles tried on some regular type profiles. *Tellus*, **1**, 4, pp. 53 – 58.

THOMPSON, P. D., 1952: Notes on the theory of large-scale disturbances in atmospheric flow with applications to numerical weather prediction. *Geophysical Research Paper*, No. 16, 107 pp.

WATSON, G. N. 1944: *A Treatise on the Theory of Bessel-Functions*. Cambridge University Press, 804 pp.