Numerical Solutions of the Perturbation Equation for Linear Flow

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Abstract

Two-dimensional nondivergent perturbations of linear flow are considered. By means of a simple numerical treatment indications concerning the conditions of exponentilal instability are obtained. Numerical solutions are further used to represent the approximate development of initial perturbations up to a certain time. The correctness of this procedure is shown in the case of the Couette flow. Finally perturbations of a symmetrical harmonic velocity profile are considered in relation to the question about the instability in a barotropic atmosphere.

1. The instability problem

For a two-dimensional motion of a homogeneous incompressible and inviscid fluid the vorticity is conserved. A flow in the xy-plane with the velocity components

$$U = U(y) \qquad V = 0$$

is then stationary, and for a small perturbation of this flow expressed by a streamfunction $\Phi\ (x,\ y,\ t)$ we have the linearized equation

$$\frac{\partial \nabla^2 \Phi}{\partial t} + U \frac{\partial \nabla^2 \Phi}{\partial x} - \frac{\partial \Phi}{\partial x} \frac{d^2 U}{dy^2} = 0 \qquad (1)$$

As U only depends on y, we can develop Φ in basic solutions of the form $\varphi(y) e^{i\mu(x-ct)}$, and for $\varphi(y)$ we get the following differential equation

$$(U-c)\varphi'' - \{(U-c)\mu^2 + U''\}\varphi = 0 \qquad (2)$$

where $'$ denotes a differentiation with regard to y. Together with the necessary boundary conditions this equation determines $\varphi(y)$. Here we shall consider the simple case

$$\varphi(0) = \varphi(D) = 0 \qquad (3)$$

corresponding to rigid boundaries at $y=0$ and $y=D$.

We have now given an eigen-value problem, as we shall find for which connected values of μ and c there exist solutions of (2) and (3). If there exist solutions for complex values of c, this means that basic solutions will exist, whose amplitude will increase exponentially with time, and the considered linear flow will be unstable. The mathematical treatment of the problem is rather complicated, and generally we cannot find, even for simple functions $U(y)$, analytical expressions for the solutions $\varphi(y)$. It therefore seems natural to treat the problem by a numerical method.

In doing so we shall consider the function $\varphi(y)$ as given by its values φ_n in N points in the interval $0 \leq y \leq D$ in such a way that the interval is divided by these points in $N+1$ parts, each of the length $d = \frac{D}{N+1}$. The derivatives shall then be approximated by the corresponding quotients of finite differences. We shall here only use the simple case $N=2$ with the values φ_1 for $y_1 = \frac{D}{3}$ and φ_2 for $y = \frac{2D}{3}$. As $\varphi(O) = \varphi(D) = 0$ we then have the following expressions for φ'' at y_1 and y_2.

$$\varphi_1'' = \frac{1}{d^2}(\varphi_2 - 2\varphi_1) \qquad \varphi_2'' = \frac{1}{d^2}(\varphi_1 - 2\varphi_2)$$

Accordingly we can write the equation (2) at these two points

$$\div \{(2+\mu^2 d^2)(U_1 - c) + d^2 U_1''\}\varphi_1 + (U_1 - c)\varphi_2 = 0$$
$$(U_2 - c)\varphi_1 - \{(2+\mu^2 d^2)(U_2 - c) + d^2 U_2''\}\varphi_2 = 0 \tag{4}$$

where U_1, U_2, U_1'' and U_2'' are the values of U and U'' at y_1 and y_2 respectively. In order that these two equations shall give non-zero solutions for φ_1, φ_2 the determinant must vanish, which gives

$$c = \frac{U_1 + U_2}{2} + \frac{r}{2s} d^2(U_1'' + U_2'') \pm \frac{1}{2s} \cdot$$
$$\cdot \sqrt{\{s(U_1 - U_2) + rd^2(U_1'' - U_2'')\}^2 + 4d^4 U_1'' U_2''} \tag{5}$$

with

$$r = 2 + \mu^2 d^2, \qquad s = r^2 - 1$$

From this it is seen that a necessary condition that c can get a complex value is

$$U_1'' U_2'' < 0 \tag{6}$$

in accordance with the well-known condition of exponentiel instability for a linear flow without friction, namely that the velocity profile shall have a point of inflection. That this point of inflection shall lie between the two points y_1 and y_2 is a result of the choice $N=2$.

The condition (6) is, however, not a sufficient one. This can be demonstrated by considering an anti-symmetrical profile

$$U(y) = \div U(D \div y)$$

for which (6) is satisfied. Equation (5) is then reduced to

$$c = \pm \frac{1}{s}\sqrt{(sU_1 + rd^2 U_1'')^2 - d^4 U_1''^2} \tag{7}$$

Replacing U_1'' with the quotient of finite differences $\frac{1}{d^2}(U_0 \div 3U_1)$, where $U_0 = U(0)$, we get that c becomes complex if

$$\left.\begin{aligned} &|U_1 \mu^4 d^4 + (U_0 + U_1)\mu^2 d^2 + 2U_0 - 3U_1| < \\ &\qquad < |U_0 - 3U_1| \end{aligned}\right\} \tag{8}$$

With $N=2$ we can only describe a very smooth profile, and we assume that U_0 and U_1 are both positive. First we see that (8) can only be fulfilled if

$$3U_1 > U_0 \tag{9}$$

This is in accordance with the further condition of instability, which states that the numerical value of the vorticity $|U'|$ shall have a maximum at the point of inflection, cf. FJØRTOFT (1950). But even this is not a sufficient criterion. With $3U_1 > U_0$, (8) gives that c is complex when

$$\mu^2 d^2 < 2 - \frac{U_0}{U_1} \tag{10}$$

but this can only be the case if

$$2U_1 > U_0 \tag{11}$$

As a particular example we consider the anti-symmetrical, harmonic profile

$$U = A \sin\left\{\frac{2\pi}{H}\left(\frac{D}{2} - y\right)\right\}, \quad H \geqq D \tag{12}$$

This profile has a point of inflection at $y = \frac{D}{2}$, where the numerical value of the vorticity has a maximum. (6) and (9) are then satisfied, but the third condition (11) is only fulfilled when

$$D > \frac{H}{2} \tag{13}$$

i. e. when the distance between the rigid boundaries is larger than half the profile wavelength. This result is proved by HØILAND (1952) for perturbations of infinite wavelength. The above simple considerations indicate that a linear flow with a velocity profile given by (12) and (13) is unstable for perturbations with wavelength L larger than a certain value, in our crude approach given by (10), i. e.

$$L > \frac{2\pi}{3} \frac{D}{\sqrt{2 - \frac{U_0}{U_1}}}$$

Fig. 1 shows the imaginary part c_{im} of c as function of L for two such profiles with respectively $H = \frac{3}{2} D$ and $H = D$.

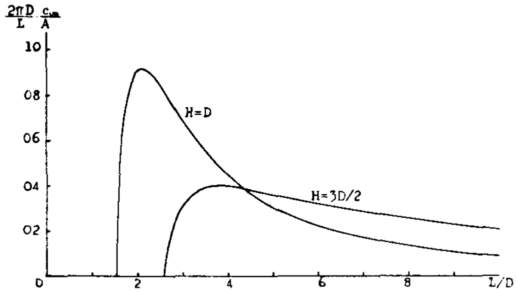

Fig. 1. The curves show the imaginary part of μc obtained by the approximation $N = 2$ for the anti-symmetrical harmonic profile with respectively $H = \frac{3}{2} D$ and $H = D$.

2. The initiail-value problem

From a total set of solutions of the eigenvalue problem we can compose the general solution $\Phi(x, y, t)$ of the perturbation equation (1), and this general solution can be adapted to a given initial perturbation.

By the numerical method used in the foregoing approximate values of $\varphi(y)$ can be obtained at some points and then by interpolation for all values of y in the considered interval. In connexion with the initial-value problem, however, it seems more appropriate to find the approximate solutions $\varphi^\star(y)$ by a somewhat different method, based on a development of $\varphi^\star(y)$ in a finite sine series

$$\varphi^\star(y) = \sum_{n=1}^{N} a_n \sin \frac{n\pi y}{D}, \quad n = 1, 2, 3, \text{---}, N \tag{14}$$

The boundary conditions (3) are then satisfied and (2) can be written

$$\left.\begin{aligned} &\sum_{n=1}^{N} \left\{(c - U)\left(\mu + n^2 \frac{\pi^2}{D^2}\right) - U''\right\} \cdot \\ &\quad \cdot a_n \sin \frac{n\pi y}{D} = 0 \end{aligned}\right\} \tag{15}$$

We now make the approximations

$$\left.\begin{aligned} U \sin \frac{n\pi y}{D} &= \sum_{m=1}^{N} F_{nm} \sin \frac{m\pi y}{D} \\ U'' \sin \frac{n\pi y}{D} &= \sum_{m=1}^{N} G_{nm} \sin \frac{m\pi y}{D} \end{aligned}\right\} \tag{16}$$

where

$$F_{nm} = \frac{2}{D} \int_0^D U \sin \frac{n\pi y}{D} \sin \frac{m\pi y}{D} dy$$

$$G_{nm} = \frac{2}{D} \int_0^D U'' \sin \frac{n\pi y}{D} \sin \frac{m\pi y}{D} dy$$

Equation (15) then gives N linear, homogeneous equations in the N coefficients a_n.

$$\left.\begin{aligned} &c\left(\mu^2 + n^2 \frac{\pi^2}{D^2}\right) a_n - \sum_{m=1}^{N} \cdot \\ &\cdot \left\{\left(\mu^2 + m^2 \frac{\pi^2}{D^2}\right) F_{mn} + G_{mn}\right\} a_m = 0 \end{aligned}\right\} \tag{17}$$

The condition that this system of equations shall possess solutions different from the zero-solution, is that its determinant is zero, which yields a relation between c and μ, in form of an algebraic equation. For a fixed value of μ we get generally N values $c^{(n)}$, real or complex. For each $c^{(n)}$ we can find, apart from an arbitrary factor, a set $a_m^{(n)}$ and the corresponding numerical solution $\varphi^{\star(n)}$. The N functions $\varphi^{\star(n)}$ (y) can be composed to a general expression for the perturbation function given by

$$\Phi^\star (x, y, t) = \sum_{n=1}^{N} \varphi^{\star(n)} e^{i\mu(x - c^{(n)}t)} \tag{18}$$

The N arbitrary constants entering this expression can be fixed by adapting it to a given initial perturbation of the form

$$e^{i\mu x} \sum_{n=1}^{N} B_n \sin \frac{n\pi y}{D}.$$

Since the perturbation function must be real, we must take either the real or the imaginary part of (18).

It can now be expected, that the expression (18) will represent with some approximation the development of the initial perturbation up to a certain time. Denoting the error by $\psi(x, y, t)$ we have by definition

$$\Phi (x, y, t) = \Phi^\star (x, y, t) + \Psi (x, y, t) \tag{19}$$

Writing formally

$$\left.\begin{aligned}\frac{\partial\nabla^2\Phi^\star}{\partial t} &= \sum_{n=1}^{N} q_n(x,t)\sin\frac{n\pi y}{D}\\ U\frac{\partial\nabla^2\Phi^\star}{\partial x}-\frac{\partial\Phi^\star}{\partial x}U'' &= \sum_{n=1}^{\infty} r_n(x,t)\sin\frac{n\pi y}{D}\end{aligned}\right\}\quad(20)$$

$\Phi^\star$ has been determined in such a way that

$$r_n(x,t)=q_n(x,t),\quad n\leq N$$

By substitution from (19) and (20) in (1) we obtain

$$\frac{\partial\nabla^2\Psi}{\partial t}+U\frac{\partial\nabla^2\Psi}{\partial x}-\frac{\partial\Psi}{\partial x}U''+R(x,y,t)=0\quad(21)$$

where

$$R(x,y,t)=\sum_{n=N+1}^{\infty} r_n(x,t)\sin\frac{n\pi y}{D}$$

As $\Psi=0$ at $t=0$, Ψ will at first grow only in consequence of the term $R(x, y, t)$. The smaller R is, the longer time it lasts before the error becomes essential.

We shall illustrate this method by considering the simple Couette flow

$$U=\frac{A}{D}y\qquad 0\leq y\leq D\qquad(22)$$

for which the exact solution of the perturbation equation (1) can be obtained directly, as discussed by FJØRTOFT (1951). Equation (2) can be written in this case

$$(U-c)(\varphi''-\mu^2\varphi)=0\qquad(23)$$

and the numerical treatment based on (14) yields

$$\left.\begin{aligned}c\left(\mu^2+n^2\frac{\pi^2}{D^2}\right)a_n-\sum_{m=1}^{N}\left(\mu^2+m^2\frac{\pi^2}{D^2}\right)\cdot\\ \cdot F_{mn}a_m=0\end{aligned}\right\}\quad(24)$$

with

$$F_{nn}=\frac{A}{2}$$

$$F_{mn}=\begin{cases}0 & n+m \text{ even } n\neq m\\ \div A\dfrac{8}{\pi^2}\dfrac{nm}{(n^2-m^2)^2} & n+m \text{ odd}\end{cases}$$

Choosing $N=4$ we obtain

$$\left.\begin{aligned}c^{(1)}=0.19\,A,\quad c^{(2)}=0.40\,A,\\ c^{(3)}=0.60\,A,\quad c^{(4)}=0.81\,A\end{aligned}\right\}\quad(25)$$

and the corresponding solutions

$$\begin{aligned}\varphi^{\star(1)}&=a^{(1)}\left(\sin\frac{\pi y}{D}+1.58\,k_{1,2}\sin\frac{2\pi y}{D}+\right.\\&\left.+1.50\,k_{1,3}\sin\frac{3\pi y}{D}+1.04\,k_{1,4}\sin\frac{4\pi y}{D}\right)\\ \varphi^{\star(2)}&=a^{(2)}\left(\sin\frac{\pi y}{D}+0.64\,k_{1,2}\sin\frac{2\pi y}{D}-\right.\\&\left.-0.62\,k_{1,3}\sin\frac{3\pi y}{D}-1.08\,k_{1,4}\sin\frac{4\pi y}{D}\right)\\ \varphi^{\star(3)}&=a^{(3)}\left(\sin\frac{\pi y}{D}-0.64\,k_{1,2}\sin\frac{2\pi y}{D}-\right.\\&\left.-0{,}62\,k_{1,3}\sin\frac{3\pi y}{D}+1.08\,k_{1,4}\sin\frac{4\pi y}{D}\right)\\ \varphi^{\star(4)}&=a^{(4)}\left(\sin\frac{\pi y}{D}-1.58\,k_{1,2}\sin\frac{2\pi y}{D}+\right.\\&\left.+1.50\,k_{1,3}\sin\frac{3\pi y}{D}-1.04\,k_{1,4}\sin\frac{4\pi y}{D}\right)\end{aligned}\qquad(26)$$

where

$$k_{1,n}=\frac{\mu^2+\dfrac{\pi^2}{D^2}}{\mu^2+n^2\dfrac{\pi^2}{D^2}}$$

Taking $\mu=\frac{\mu}{D}$ we obtain functions $\varphi^{\star(1)}(y)$ and $\varphi^{\star(2)}(y)$ as illustrated in fig. 2. They approximate the corresponding exact solutions of (23)

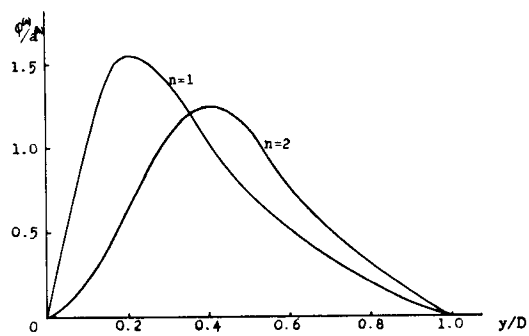

Fig. 2. The numerical solutions $\varphi^{\star(n)}(y)$ with $N=4$ for the Couette flow corresponding to $c^{(1)}=0.19A$ and $c^{(2)}=0.40A$.

Table 1

t	$Q_1^\star$	$Q_2^\star$	$Q_3^\star$	$Q_4^\star$	$Q_1^{(1)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	$Q_4^{(2)}$	$Q_5^{(1)}$	$Q_6^{(2)}$
0	1.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
T	0.84	0.52	−0.16	0.02	0.85	0.50	−0.17	0.00	−0.02	0.00
2T	0.47	0.71	−0.50	−0.14	0.50	0.68	−0.50	−0.19	0.00	−0.03
3T	0.13	0.50	−0.69	−0.52	0.17	0.50	−0.65	−0.50	0.20	0.00
4T	−0.03	0.14	−0.50	−0.91	0.00	0.19	−0.50	−0.65	0.50	0.21

with singularities at respectively $y_1 = \frac{D}{5}$ and $y_2 = \frac{2D}{5}$, where $U(y_1) = c^{(1)}$ and $U(y_2) = c^{(2)}$.

By means of the functions (26) we can compose the general perturbation function (18). If we take as the initial perturbation

$$\Phi(x, y, 0) = B \sin \mu x \cdot \sin \frac{\pi y}{D} \qquad (27)$$

we get

$$\begin{aligned}\Phi^\star(x, y, t) = B\Big\{ & Q_1^\star \sin \mu \left(x - \frac{A}{2}t\right) \sin \frac{\pi y}{D} + \\ & + k_{1,2} Q_2^\star \cos \mu \left(x - \frac{A}{2}t\right) \sin \frac{2\pi y}{D} + \\ & + k_{1,3} Q_3^\star \sin \mu \left(x - \frac{A}{2}t\right) \sin \frac{3\pi y}{D} + \\ & + k_{1,4} Q_4^\star \cos \mu \left(x - \frac{A}{2}t\right) \sin \frac{4\pi y}{D}\Big\}\end{aligned} \qquad (28)$$

where

$$Q_1^\star = 0.30 \cos(0.3\,\mu A t) - 0.70 \cos(0.1\,\mu A t),$$
$$Q_2^\star = 0.48 \sin(0.3\,\mu A t) + 0.44 \sin(0.1\mu\,At)$$
$$Q_3^\star = 0.45\{\cos(0.3\,\mu A t) - \cos(0.1\,\mu A t)\},$$
$$Q_4^\star = 0.32 \sin(0.3\,\mu A t) - 0.76 \sin(0.1\,\mu A t)$$

The values of $Q^\star$ at different times are given in table 1, where $\mu A T = \pi$ or $T = \frac{L}{2A}$.

For t up to about $2T$ $Q_4^\star$ is still small compared with the other ones, which indicates that the approximation $N=4$ is sufficient for describing the development up to that time. That this is really the case can be seen by considering the exact solution. For the Couette flow equation (1) reduces to

$$\frac{\partial \nabla^2 \Phi}{\partial t} + U \frac{\partial \nabla^2 \Phi}{\partial x} = 0$$

Considering again the initial perturbation (27) we have for $\nabla^2\Phi(x, y, t)$ the exact solution

$$\nabla^2 \Phi = -\left(\mu^2 + \frac{\pi^2}{D^2}\right) B \sin\{\mu(x - Ut)\} \sin \frac{\pi y}{D} \qquad (29)$$

This expression can be written as the following series

$$\nabla^2\Phi = -\left(\mu^2 + \frac{\pi^2}{D^2}\right) B \sum_{n=1}^{\infty} \left[\sin\left\{\mu\left(x - \frac{A}{2}t\right)\right\} \cdot Q_n^{(1)}(t) + \cos\left\{\mu\left(x - \frac{A}{2}t\right)\right\} Q_n^{(2)}(t)\right] \sin \frac{n\pi y}{D}$$

which gives

$$\Phi = B \sum_{n=1}^{\infty} k_{1,n} \left[Q_n^{(1)}(t) \sin\left\{\mu\left(x - \frac{A}{2}t\right)\right\} + Q_n^{(2)}(t) \cos\left\{\mu\left(x - \frac{A}{2}t\right)\right\}\right] \sin \frac{n\pi y}{D} \qquad (30)$$

Here $Q_n^{(1)} = 0$ for n even and $Q_n^{(2)} = 0$ for n odd, exactly as in the numerical solution (28), while

$$\left.\begin{aligned} Q_n^{(1)} &= 2\,\mu A t \left\{\frac{1}{(\mu A t)^2 - (n-1)^2\pi^2} - \frac{1}{(\mu A t)^2 - (n+1)^2\pi^2}\right\} \sin \frac{\mu A t}{2}, \quad n \text{ odd} \\ Q_n^{(2)} &= -2\,\mu A t \left\{\frac{1}{(\mu A t)^2 - (n-1)^2\pi^2} - \frac{1}{(\mu A t)^2 - (n+1)^2\pi^2}\right\} \cos \frac{\mu A t}{2}, \quad n \text{ even}\end{aligned}\right\} \qquad (31)$$

The values of $Q_n^{(1)}$ and $Q_n^{(2)}$ at different times are given in table 1. Up to $2T$ there is a good agreement with the numerical solution.

3. A symmetrical, harmonic velocity profile

In this section we shall consider a velocity given by

$$U = -A\cos\frac{2\pi y}{D} \qquad 0 \leq y \leq D \quad (32)$$

The instability of such a linear flow is possibly of some interest for the study of the large-scale atmospheric motion.

For this profile the developments (16) are very simple and the equations (17) become

$$\left.\begin{aligned}
&\left\{\left(\mu^2+\frac{\pi^2}{D^2}\right)c-\frac{1}{2}\left(\mu^2-\frac{3\pi^2}{D^2}\right)A\right\}a_1+ \\
&\qquad +\frac{1}{2}\left(\mu^2+5\frac{\pi^2}{D^2}\right)Aa_3=0 \\
&\left(\mu^2+4\frac{\pi^2}{D^2}\right)ca_2+\frac{1}{2}\left(\mu^2+12\frac{\pi^2}{D^2}\right)Aa_4=0 \\
&- - - - - - - - - - \\
&\frac{1}{2}\left\{\mu^2+[(n-2)^2-4]\frac{\pi^2}{D^2}\right\}Aa_{n-2}+ \\
&\qquad +\left(\mu^2+n^2\frac{\pi^2}{D^2}\right)ca_n+ \\
&\qquad +\frac{1}{2}\left\{\mu^2+[(n+2)^2-4]\frac{\pi^2}{D^2}\right\}Aa_{n+2}=0 \\
&- - - - - - - - - - \\
&\frac{1}{2}\left\{\mu^2+[(N-3)^2-4]\frac{\pi^2}{D^2}\right\}Aa_{N-3}+ \\
&\qquad +\left\{\mu^2+(N-1)^2\frac{\pi^2}{D^2}\right\}ca_{N-1}=0 \\
&\frac{1}{2}\left\{\mu^2+[(N-2)^2-4]\frac{\pi^2}{D^2}\right\}Aa_{N-2}+ \\
&\qquad +\left(\mu^2+N^2\frac{\pi^2}{D^2}\right)ca_N=0
\end{aligned}\right\} \quad (33)$$

This system of equations consists of two independent parts, the one containing a_n with n odd, the other a_n with n even. Accordingly, we can consider those perturbations separately which are either symmetrical or anti-symmetrical in y.

Taking $N=4$ the following two equations in a_1 and a_3 are obtained

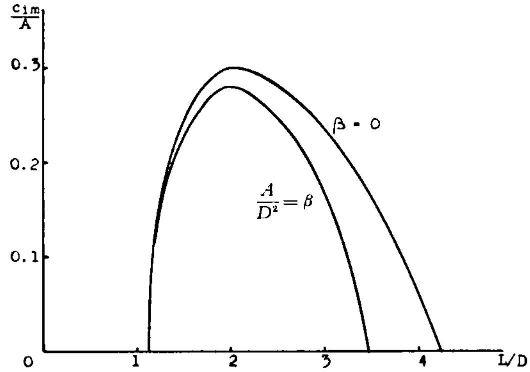

Fig. 3. The figure illustrates the stabilizing effect of the varying Coriolis parameter showing the imaginary part of c given by (35), ($\beta = 0$), and of c given by (39) with $\frac{A}{D^2}=\beta$.

$$\left.\begin{aligned}
&\left\{\left(\mu^2+\frac{\pi^2}{D^2}\right)c-\frac{1}{2}\left(\mu^2-3\frac{\pi^2}{D^2}\right)A\right\}a_1+ \\
&\qquad +\frac{1}{2}\left(\mu^2+5\frac{\pi^2}{D^2}\right)Aa_3=0 \\
&\frac{1}{2}\left(\mu^2-3\frac{\pi^2}{D^2}\right)Aa_1+\left(\mu^2+9\frac{\pi^2}{D^2}\right)ca_3=0
\end{aligned}\right\} \quad (34)$$

They possess non-zero solutions when

$$c=\frac{1}{4}\frac{\mu^2D^2-3\pi^2}{\mu^2D^2+\pi^2}A\pm\frac{1}{4}\frac{A}{\mu^2D^2+\pi^2}\cdot \sqrt{\frac{\mu^2D^2-3\pi^2}{\mu^2D^2+9\pi^2}(5\mu^4D^4+30\pi^2\mu^2D^2-7\pi^4)} \quad (35)$$

c becomes complex for wavelength

$$1.2\,D < L < 4.2\,D$$

The imaginary part c_{im} of c as function of L is shown in fig. 3. It has its maximum value about $L=2D$ $(\mu=\frac{\pi}{D})$. The two equations in a_2 and a_4 give

$$c=\pm\frac{A}{2}\mu D\sqrt{\frac{\mu^2D^2+12\pi^2}{(\mu^2D^2+4\pi^2)(\mu^2D^2+16\pi^2)}}$$

which are real for all μ.

In table 2 are given the values of $c^{(n)}$ corresponding to $\mu=\frac{\pi}{D}$ for the different approximations $N=1, 2, - - -, 8$. It is seen that

Table 2

N	$\frac{c^{(1)}}{A}$	$\frac{c^{(2)}}{A}$	$\frac{c^{(3)}}{A}$	$\frac{c^{(4)}}{A}$	$\frac{c^{(5)}}{A}$	$\frac{c^{(6)}}{A}$	$\frac{c^{(7)}}{A}$	$\frac{c^{(8)}}{A}$
1	−0.50							
2	−0.50	0						
3	−0.25 +0.30i	0	−0.25 −0.30i					
4	−0.25 +0.30i	0.20	−0.25 −0.30i	−0.20				
5	−0.38 +0.30i	0.20	−0.38 −0.30i	−0.20	0.27			
6	−0.38 +0.30i	0	−0.38 −0.30i	0.46	0.27	−0.46		
7	−0.34 +0.18i	0	−0.34 −0.18i	0.46	−0.36	−0.46	0.54	
8	−0.34 +0.18i	0.14	−0.34 −0.18i	−0.14	−0.36	0.63	0.54	−0.63

the flow is unstable only for the symmetrical part of the perturbations.

Considering the initial symmetrical perturbation

$$\Phi(x, y, 0) = B \sin \frac{\pi x}{D} \sin \frac{\pi y}{D}$$

we get with the approximation $N=8$ the solution

$$\Phi^\star = B\left\{S_1 \sin \frac{\pi y}{D} + S_3 \sin \frac{3\pi y}{D} + S_5 \sin \frac{5\pi y}{D} + S_7 \sin \frac{7\pi y}{D}\right\}$$

$$S_1 = 1.65\, M \sin \frac{\pi}{D}(x + 0.34\, At) + 0.55\, P \cos \frac{\pi}{D}(x + 0.34\, At) - 2.28 \sin \frac{\pi}{D} \cdot (x + 0.36\, At) - 0.02 \sin \frac{\pi}{D}(x - 0.54\, At)$$

$$S_3 = -0.11\, M \sin \frac{\pi}{D}(x + 0.34\, At) - 0.26\, P \cos \frac{\pi}{D}(x + 0.34\, At) + 0.21 \sin \frac{\pi}{D} \cdot (x + 0.36\, At) + 0.01 \sin \frac{\pi}{D}(x - 0.54\, At)$$

$$S_5 = 0.07\, M \sin \frac{\pi}{D}(x + 0.34\, At) - + 0.01\, P \cos \frac{\pi}{D}(x + 0.34\, At) - 0.14 \sin \frac{\pi}{D} \cdot (x + 0.36\, At) - 0.01 \sin \frac{\pi}{D}(x - 0.54\, At)$$

$$S_7 = 0.04\, M \sin \frac{\pi}{D}(x + 0.34\, At) + 0.01\, P \cos \frac{\pi}{D}(x + 0.34\, At) - 0.08 \sin \frac{\pi}{D} \cdot (x + 0.36\, At) + 0.04 \sin \frac{\pi}{D}(x - 0.54\, At)$$

$$M = e^{0,18\frac{\pi A t}{D}} + e^{-0,18\frac{\pi A t}{D}}, \quad P = e^{0,18\frac{\pi A t}{D}} - e^{-0,18\frac{\pi A t}{D}} \tag{36}$$

The factor $e^{0,18\frac{\pi A t}{D}}$ is doubled in the time $T = 1.3\frac{D}{A}$. At the times T and $2T$ the solutions are

$$\Phi^\star(T) = B\left\{1.98 \sin\left(\frac{\pi}{D}x + 1.64\right) \sin \frac{\pi y}{D} + 0,38 \sin\left(\frac{\pi}{D}x - 0.46\right) \sin \frac{3\pi y}{D} + 0.06 \sin\left(\frac{\pi}{D}x + \gamma_T^{(1)}\right) \sin \frac{5\pi y}{D} + 0.02 \sin\left(\frac{\pi}{D}x + \gamma_T^{(2)}\right) \sin \frac{7\pi y}{D}\right\}$$

$$\Phi^\star(2T) = B\left\{5.04 \sin\left(\frac{\pi}{D}x + 2.97\right) \sin \frac{\pi y}{D} + 0.98 \sin\left(\frac{\pi}{D}x + 0.79\right) \sin \frac{3\pi y}{D} + 0.18 \sin\left(\frac{\pi}{D}x + \gamma_{2T}^{(1)}\right) \sin \frac{5\pi y}{D} + 0.10 \sin\left(\frac{\pi}{D}x + \gamma_{2T}^{(2)}\right) \sin \frac{7\pi y}{D}\right\}$$

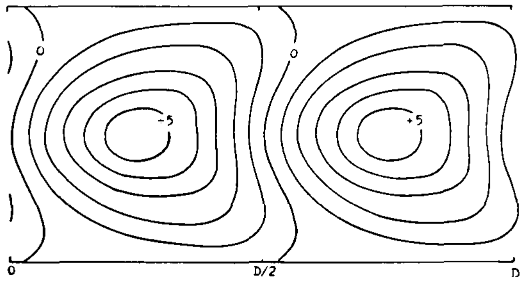

Fig. 4. The isolines of $\frac{\Phi^\star(2T)}{B}$ according to (36), drawn for every whole number.

$\gamma^{(1)}$ and $\gamma^{(2)}$ denote certain phase-angles. The isolines $\Phi^\star(x, y, 2T) = \text{constant}$ are shown in fig. 4. It is seen that the amplitude of the perturbation at the time $2T$ is increased to more than $5B$.

The function R, which according to (21) acts as a source for the error ψ, has in this case the simple form

$$R = 2\frac{\pi^3}{D^3}AB\Big[-0.46\,M\cos\frac{\pi}{D}(x+0.34\,At) + 0.16\,P\sin\frac{\pi}{D}(x+0.34\,At) + 0.96\cos\frac{\pi}{D}(x+0.36\,At) - 0.04\sin\frac{\pi}{D}(x-0.54\,At)\Big]\sin\frac{9\pi y}{D} \tag{37}$$

For the integrated effect we have

$$\int_0^T R dt = 0.26\cdot 2\frac{\pi^2}{D^2}B\sin\left(\frac{\pi}{D}x+\delta_T\right)\sin\frac{9\pi y}{D}$$

$$\int_0^{2T} R dt = 2.61\cdot 2\frac{\pi^2}{D^2}B\sin\left(\frac{\pi}{D}x+\delta_{2T}\right)\sin\frac{9\pi y}{D}$$

where δ is a certain phase-angle. This induced vorticity is for t up to nearly $2T$ small compared with the vorticity $\nabla^2\Phi^\star$ of the approximate field. But certainly the error can be further increased by the interaction with the basic flow U.

As essential features of the large-scale atmospheric motion can be described by a two-dimensional barotropic model, the question arises whether the generation of the large-scale disturbances can be explained, fully or partly, as an instability of the kind considered above. This problem has been investigated most thoroughly by KUO (1949). The numerical method used in this paper can give in a very simple manner rather detailed results. The limitation that the numerical solutions are only approximations in a certain interval of time does not seem serious as physical factors have been excluded, which will make any solution of little value after a certain time.

The zonal flow at middle latitudes is often nearly symmetrical with a pronounced maximum. As a typical case we can consider a linear flow given by (32) superimposed on a constant zonal flow. It has been shown above that this flow will be unstable for perturbations of certain wavelengths, thereby doubling their amplitudes in the time $T = 1.3\,\frac{D}{A}$ for the wavelength of maximum instability $L = 2\,D$. For the large-scale atmospheric motion we may quite well have conditions making $\frac{D}{A}$ of the order of magnitude 1 day.

The above results cannot be applied directly to the atmospheric motion as the curvature and rotation of the earth has not been taken into account. Further boundary conditions like (3) are not correct in a strict sense for the atmosphere. Strictly we should therefore base the numerical method on an harmonic analysis on a spherical surface.

A well-known effect from the including of the variation with latitude of the Coriolis parameter is that of adding a velocity towards the west to the velocity of propagation c. Also the instability of the linear flow is influenced by the varying Coriolis parameter. If we include the Coriolis parameter f in equation (1) in the usual way, we get instead of equation (3)

$$(U-c)\,\varphi'' - \{(U-c)\,\mu^2 + U'' - \beta\}\,\varphi = 0 \tag{38}$$

Considering $\beta = \frac{df}{dy}$ as a constant in the considered interval and using the approximation (14) with $N=4$ we get for a symmetrical perturbation the following equation for c

$$c = \frac{1}{4}\frac{\mu^2 D^2 - 3\pi^2}{\mu^2 D^2 + \pi^2} A -$$

$$- \frac{\mu^2 D^2 + 5\pi^2}{(\mu^2 D^2 + \pi^2)(\mu^2 D^2 + 9\pi^2)} D^2 \beta \pm$$

$$\pm \frac{A}{4(\mu^2 D^2 + \pi^2)(\mu^2 D^2 + 9\pi^2)} \cdot$$

$$\cdot \sqrt{(\mu^2 D^2 - 3\pi^2)(\mu^2 D^2 + 9\pi^2)} \cdot$$

$$\left[\left(5\mu^4 D^4 + 30\,\pi^2 \mu^2 D^2 - 7\pi^4 - 32\,\pi^2 \frac{D^2}{A}\beta\right) + \right.$$

$$\left. + 256\,\pi^4 \frac{D^4}{A^2}\beta^2.\right]^{1/2} \qquad (39)$$

(39) compared with (35) shows clearly the stabilizing effect of the variation of the Coriolis parameter. It is seen that the influence of β depends upon the magnitude of $\frac{A}{D^2}$. The larger $\frac{A}{D^2}$ is, the smaller is the stabilizing effect of β.

The imaginary part of c in the case $\frac{A}{D^2} = \beta$ is shown as function of L in fig. 3.

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