The Gravimetric Method for Determination of the Form of the Geoid

By R. A. HIRVONEN, Institute of Technology, Helsinki

(Manuscript received October 5, 1953)

The gravimetric method for the investigation of geoid is based on a theorem of STOKES (1849): If a surface is given, which encloses all attracting masses and at which the potential of the gravity —i.e. the sum of attracting and centrifugal forces—is constant, the gravity at this surface and everywhere outside it is given too. Here we are interested in the gravity at the geoid only; this surface will be compared with a regularized surface of reference, preferably with the ellipsoid that has been used for the trigonometric measurements. The deviations of the ellipsoid from a sphere are of order 10 km = $1/600 = p$, those of the geoid from the ellipsoid, according to the comprehensive investigation of TANNI (1948), of order 60 m = $= 1/100{,}000 = q$. The differences between geoid and ellipsoid can be expressed by infinite sums of spherical harmonics:

$$\triangle r = R\Sigma_2^{\infty} u_n$$

$$\triangle g = G\Sigma_2^{\infty} v_n$$

and Stokes' theorem by the formula

$$v_n = (n-1)u_n$$

In this original direct form the theorem has no practical significance, owing to the fact, that the local disturbances are represented by terms with great values of n, wherefore the computation from u to v, i.e. from the form of geoid to the gravity looses in accuracy. But it is more advantageous when applied to the inverse problem: given the gravity at the geoid, to be computed the form of the latter. For this purpose Stokes eliminated the spherical harmonics from the formulas above, obtaining that famous integral formula known by his name. Nevertheless, this formula requires, that the gravity is known everywhere around the world, before the deviation of the geoid from the ellipsoid can be computed at a single point.

Hence even this formula had no practical significance before VENING MEINESZ (1927, 1932) had developped the device for measuring the gravity at deep sea. Vening Meinesz accomplished a valuable improvement to the theory too: performing an horizontal differentiation of the Stokes' formula he obtained another integral formula for the computation of the tilt of the geoid, or the deflection of the vertical, immediately from the anomalies of gravity.

Before entering the practical applications of these formulae, there are some points in the theory that must be discussed. After Stokes his formula has been developped also without using the spherical harmonics, but one particularity remains: the development of the gravity anomalies must not contain any terms of first order. In other words: if the anomalies are represented by a mass coating, the gravity center of this coating must coincide with the centre of the Earth. This balance of anomalies prescribes three equations of condition which should be taken into consideration by a proper adjustment of the gravity field observed. This adjustment has always been neglected in the applications hitherto performed. No error in the computations, however, has been caused, because the integral formulae of Stokes and Vening Meinesz contain the adequate correcting terms, which were unnecessary for an adjusted gravity field.

The original treatise of Stokes neglected all terms of order p^2 or smaller, but, q being of the order p^2, this was not satisfactory. Later it has been shown, that the theoretical errors of Stokes' formula anyway are of order $pq = 10$ cm. Moreover, as ZAGREBIN (1944) has shown, the spherical harmonics can be replaced by Lamé's functions and the Stokes' formula duly corrected, in which case a considerable—at present quite unnecessary—improvement can be ascertained.

As mentioned above, the theory presumes, that there are no masses outside the geoid, which requirement by no means is met in the practice. Nowadays it has been shown by MALKIN (1933) and others, that the masses protruding outside the geoid can be allowed for by applying corrections to both sides of the Stokes' formula. Stokes' own idea was, that these masses should—in imagination—be condensed to the geoid, in which case slight corrections both to the anomalies and to the resulting elevations of geoid must be computed. In practice both these methods thus are qualitatively similar. But there are many other methods of reduction, and a closer comparison of them deserves now our attention.

The purpose of the reductions is threefold. At first the theory demands, that the values of gravity which have been observed at different altitudes, refer to the same surface of equilibrium. Then the original formula of Stokes presumes, that no masses are left outside the geoid. Finally the reduced anomalies must vary so smoothly, that the gravity field can be completed by interpolation using as small a number of original observations as possible.

The two last mentioned requirements mean, that the gravity field used is not the actual one. The actual anomalies could be obtained by Prey's method of reduction, in which case the Stokes' formula should be replaced by the corrected formula of Malkin. But these Prey's anomalies vary very irregularly requiring a great number of observations and an intricate integration. Therefore Malkin's formula or the other similar modifications hardly have any practical advantages.

The most simple method of reduction is that of Faye, usually known by the name free-air reduction. The value of the observed gravity is transferred to sea level as through empty air. If the terrain is level, this reduction comprises the entire Stokes' condensation, because an even infinite layer can be moved or condensed in a vertical direction without any changes in the gravity outside it. Only for a rough terrain a topographic correction of gravity is necessary. This correction can be evaluated with the aid of topographical maps, and it is not necessary to consider very wide surroundings of the station.

The transfer of the attracting masses causes, that the potential of gravity changes too, and thus Stokes' formula does not give the actual geoid but a deformed cogeoid, which in general lies slightly above the actual one. The difference has been estimated to be at most 2 m; and its indirect effect back to the gravity can be neglected. The only drawback of the method is, that these free-air anomalies even with terrain corrections still show too rough variations.

If the masses are transferred still further downwards below the sea level, the potential, which was increased by the condensation, begins to decrease again, and when the masses have reached the mirror points of their original positions, the potential has its true value again. This is Rudzki's method of reduction, which leaves the geoid unaltered; the gravity, however, needs almost a double topographic correction. The labour involved by the reduction is augmented, and the resulting anomalies still show rough variations.

The isostatic hypotheses upon the structure of earth's crust suggest, that the topographic irregularities are compensated by opposite irregularities deep below the sea level. Consequently, if we continue the transfer of the masses past the mirror points of Rudzki down to the proper depth of isostatic compensation, we have a possibility to get rid of the rough and irregular variations of anomalies. The experience shows, that in most cases this is true enough. Considering the third of our requirements the isostatic reduction seems to be ideal. But this method has its drawbacks too: the transfer of masses is so great, that the labour involved by the computations reaches its maximum. Not only the topographical maps of the nearest surroundings must be consulted, but this reduction must be extended around the entire globe. The contribution of the remotest zones, which varies very slowly from point to point, has once for all been computed by the Isostatic Institute in Helsinki, and that of the inner zones surely in the future will be computed by the aid of electronic machines. The drawback, however, remains that the deformation of the geoid and the indirect effect of the isostatic reductions are great.

Now DE GRAAFF HUNTER (1935) has suggested a new kind of reduction that combines the advantage of smooth anomalies with the minimal transfer of masses. It can be called regional Stokes' condensation: the protruding topographic features will be condensed to sea level, only not just vertically downwards, but spreading them around on an adequate region. This method being rather new, nobody has tried it in actual studies of the geoid, but I believe it will prove to be very useful.

The reduction methods that give smooth anomalies imply transfer of masses and require a special computation of the deformation of the true geoid. If different reductions, or even different depths of isostatic compensation have been used in different regions of the earth, this computation can be as intricate as the integration of Stokes' formula with unsmoothened anomalies. Therefore it is advisable to use only the free air anomalies as a standard. It will be quite satisfactory in most regions where the terrain is not too rough. In mountainous or otherwise disturbed areas the isostatic or the regional Stokes' condensation can be used as a device for interpolation. The topographic reduction will be applied at first to all observed gravity stations, rendering a smooth field of gravity anomalies. These anomalies can be interpolated for any desidered number of intermediary stations,and the topographic reductions for these stations applied backwards, so that very reliable free air anomalies can be obtained with a minimum amount of actual observations.

There is another point concerning the free-air reduction that may deserve consideration. The coefficient implied by this reduction consists of the curvature of the geoid, which in unfavourable cases can differ by 10 % from that of the ellipsoid. For the first determinations of the geoid only the latter can be used, and the results may need a correction, when the true form of the geoid is approximately known. No experimental information about the magnitude of this correction is so far available.

With the field of anomalies provided, the integration by the aid of Stokes' or Vening Meinesz' function can begin. There are now two alternative ways to be chosen:

1) If the number of points, where the elevation of the geoid or the deflection of the vertical is wanted, is not very great, a special template can be prepared, dividing the surface of the earth into concentric zones and compartments. This template will be attached to each point to be computed, and the mean anomalies in the compartments evaluated separately in each case. Because the outer zones can be very broad, the actual integration, i.e. the multiplication by a factor tabulated and the summing up of the products, can be performed very easily in the usual simple way.

2) If the number of points to be computed is great, or if the computation must be renewed often on the ground of additional observations, it is better to divide the earth once for all into fixed compartments, square degrees or $10^\circ \times 10^\circ$ squares. For each compartment a punch card will be prepared, and the integration can be performed by electronic machines. It is always possible to use smaller squares for the nearest surroundings and larger ones for the more distant zones.

The second method has not yet been experimented, but it will be applied to the investigations that at present are going on in Columbus, Ohio, under the direction of W. A. Heiskanen.

Now we shall consider the question, which accuracy can be obtained in computing undulations of the geoid and deflections of the vertical from gravity anomalies.

At first we must estimate the *probable error of representation*, the uncertainty arising from the fact that it is impossible to measure the gravity at each point on the earth. An observed anomaly in general must be taken as representative of the mean anomaly on a greater area, extending halfways towards the neighbouring stations. The error of representation is the difference of the observed and the real mean value, and taken statistically, it increases, when the size of the area to be represented increases. For very small areas, where an interpolation between the observed stations is reliable, the error of representation can be taken as equal to the mean error of the observation itself, for modern observations practically speaking zero.

The mean anomaly for a square or compartment, on the other hand, begins with a maximal value for smallest squares and decreases, statistically taken, towards zero, when the squares increase. The maximal value of error of representation for the largest squares is equal to the maximal mean anomaly for smallest squares. The curve expressing the increasing of error of representation with the size of square leaves slowly its minimal value (zero), then rises more rapidly, and at last tends again slowly towards its maximal value. But no mathematical function can properly be prescribed for this curve; it can be established only empirically on the basis of numerous different samples. I have made an attempt to determine this curve, in order to get a basis for the investigation of the accuracy of the gravimetric computations of the geoid. My results concerning the precision obtainable for the elevations of geoid and for the deflections of vertical can be stated as follows.

The height of the geoid above the ellipsoid can be computed with a probable error of $\pm$ 7 m, if there are 1,728 observed stations properly distributed around the earth. The distribution used consists of 248 stations up to 40° from the point to be computed with a decreasing density; and of a general world survey with a constant spacing of 5° for distances from 40° to 180°. This is almost the same result which was obtained by de Graaff Hunter.

The deflection of the vertical can be computed with a probable error of $\pm 0''.8$ ($\pm 0''.56$ for either component), if there are 1975 stations in all. The distribution consists of 100 stations up to 0°.5, 137 stations from 0.5 to 5°, 258 stations from 5° to 40° and 1,580 stations from 40° to 180° with a constant spacing of 5°. This result differs unfavourably from that of de Graaff Hunter, who obtained $\pm 0''.5$ with 1,654 stations. As shown elsewhere, the disagreement is due to a questionable detail in the discussions of de Graaff Hunter.

These results are based on the prerequisite of a complete gravimetric survey over the entire world that by no means has been accomplished at present. The most urgent questions right now are: 1) which is the precision obtainable on the basis of the present observations for the points where the gravimetric elevations of geoid or the deflections of the vertical are desirable, and 2) where are the most harmful gaps of the gravimetric survey that in the first place should be filled up. The second question is one of the main concerns of the Columbus-program, namely to give suggestions to observers, where additional observations are wanted. Here in Scandinavia we know, that submarine observations in the Baltic Sea, North Sea and Northern Atlantic are our first goals.

As a summary about the present situation can be stated: There are about 12,000 gravity observations with pendulum apparatus of Sterneck, Lejay or Vening Meinesz. Most of these have been used by Tanni in 1949 for his extensive studies on the continental undulations of the geoid. The additional stations are situated in Europe, South Africa, on the Pacific and around the American continents. Then there are the new observations with Worden-gravimeter at the national reference stations and at about 50,000 field stations around the earth. The main bulk of gravimeter observations with millions of stations has been furnished by the oil companies. These stations are concentrated with an extraordinary density on the areas where oil or ore is to be expected, but, unfortunately, they cover extensive and conherent regions only in Canada, United States and around the Gulf of Mexico.

It is obvious, that the best results for the gravimetric deflections of the vertical can be obtained in the central parts of regions that have a general gravimetric survey. The immediate surroundings of the points to be computed need a detailled survey with a density of stations which makes the interpolation between the observed points possible. In addition to this it must be pointed out that there should be no exceptional disturbed anomalies within these regions. The mountainous regions are not favourable, even though the triangulations often preferably follow the mountain chains. Considering the moderate accuracy of the deflections of the vertical, it is not necessary, that the astronomical points should be first order trigonometric points. They can be choosen from the most favourable parts of the anomaly map and connected by triangulations of lower order or by traverses to the main net. The number of these super control points, which render the absolute orientation of triangulations possible, is just a few for each continent, and therefore it is probable, that the adequate completion of the present gravity survey can be accomplished in a very near future.

REFERENCES

DE GRAAFF-HUNTER, J., 1935: The Figure of the Earth from Gravity Observations and the Precision Obtainable. *Phil. Trans. R. S.* London.

HEISKANEN, W., 1953: Isostatic Reductions of Gravity Anomalies by the Aid of High-speed Computing Machines, *Ann. Ac. Sci. Fennicae.*

MALKIN, N., 1933, LAMBERT, W. D., 1937: *Gerlands Betiräge zur Geophysik.*

STOKES, G. G., 1849: On the Variation of the Gravity at the Surface of the Earth. *Transact, Cambridge Phil. Soc.*

TANNI, L., 1948: On the Continental Undulations of the Geoid as Determined from the Present Gravity Material. *Annales Ac. Sci. Fennicae.*

VENING MEINESZ, F. A., 1927: A formula expressing the deflection of the plum-line in the gravity anomalies and some formulae for the gravity field and the gravity potential outside the geoid. *Koninkl. Akad. van Wetensch. te Amsterdam.*

— —, 1932: Gravity Expeditions at Sea. *Netherl. Geod. Comm.*

ZAGREBIN, D. V., 1944: Stokes Formula for the Case of an Ellipsoidal Level Surface (in russian). *Bull. Inst. Astr. Ac. Sci. URSS.*